ERC Workshop on Geometric Measure Theory, Analysis in Metric Spaces and Real Analysis

Centro De Giorgi, Scuola Normale Superiore

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Titles and abstracts

William K. Allard (Univ. Duke University). *Some useful techniques for dealing with multiple valued functions.*

Abstract. In recent the recent work *Multiple Valued Functions and Integral Currents*, by DeLellis and Spadaro, the authors need to manipulate Almgren's multiple valued functions in various ways for purposes of carrying out the three subsequent papers, yet to be published, *Regularity of Area Minimizing* with the subtitles *I. Gradient L^p Estimates*, *II. Center Manifold* and *III. Blow-Up*. Together these work with provide a proof Almgren's so called Big Regularity Theorem. It appears that DeLellis and Spadaro, while adhering to the broad strategy of Almgren's proof, have introduced many new analytical ideas with did not exist when Almgren did his original work over fourty years ago.

In this note I will present an alternative treatment to the work of Spadaro and DeLellis in the first paper mentioned above about theory of push forward of multiple valued maps as well as the reparameterization of multiple valued graphs.

I hope as well to present a simple treatment of the near retraction used to regularize the embeddings of multiple valued maps in Euclidean space; this is an essential step for the construction of comparisons in *I. Gradient L^p Estimates*.

Costante Bellettini (Princeton University). *Tangent cones to integral and normal calibrated currents: different behaviour.*

Abstract. Calibrated currents are a very important family of mass-minimizers and appear in several major problems in geometry, e.g. Yang-Mills fields, geometric invariants, holonomy groups. One of the long standing question in the theory of calibrated currents is whether tangent cones are unique. We address the case of pseudo holomorphic currents and introduce a pseudo algebraic blow up technique, which mimics the classical blow up of points in algebraic geometry. This technique leads to a very direct and geometric proof of the uniqueness of tangent cones for integral cycles and gives a deep insight on how this uniqueness can fail in the case of normal cycles.

Vladimir Bogachev (Moscow State University). *Extensions of Sobolev functions and BV functions in infinite dimensions.*

Abstract. Some recent positive and negative results and open problems related to extensions of Sobolev functions and BV functions on infinite dimensional domains will be discussed.

Zoltan Buczolich (Eötvös Loránd University). *Measures and functions with prescribed homogeneous multifractal spectrum.*

Abstract. This is a joint work with Stéphane Seuret. We construct measures supported in $[0,1]$ with prescribed multifractal spectrum. Moreover, these measures are homogeneously multifractal (HM, for short), in the sense that their restriction on any subinterval of $[0,1]$ has the same multifractal spectrum as the whole measure. The spectra f that we are able to prescribe are suprema of a countable set of step functions supported by subintervals of $[0,1]$ and satisfy $f(h) \leq h$ for all $h \in [0,1]$. We also find a surprising constraint on the multifractal spectrum of a HM measure: the support of its spectrum within $[0,1]$ must be an interval. This result is a sort of Darboux theorem for multifractal spectra of measures. This result is optimal, since we construct a HM measure with spectrum supported by $[0,1] \cup \{2\}$.

Guy David (Université de Paris Sud). *A variant of the Alt-Caffarelli-Friedman problem for the localization of eigenfunctions.*

Abstract. This may be subject to change. The localization problem concerns eigenfunctions for an elliptic operator (for instance, Laplacian on a complicated domain, or Schrödinger on a simple one). We (with M. Filoche, D. Jerison, S. Mayboroda) propose a free boundary problem to find subregions where the eigenfunctions are likely to be localized, and study regularity properties of the minimizers.

Silvano Delladio (Università di Trento). *Superdensity and infinitesimal Euclideanity of sets of finite perimeter.*

Abstract. The rough idea that finite perimeter sets are "closer" to open sets than generic measurable sets with positive measure is confirmed by many facts (e.g. the decomposition properties, the constancy theorem, the path-connectedness). In this talk we furtherly develop such an idea by exposing some recent results concerning superdensity and infinitesimal Euclideanity of finite perimeter sets.

Guido De Philippis (Bonn University). *The sharp quantitative Faber-Krahn inequality.*

Abstract. The classical Faber-Krahn inequality asserts that balls (uniquely) minimize the first eigenvalue of the Dirichlet-Laplacian among sets with given volume. I will show a sharp quantitative enhancement of this result, confirming a conjecture by Nadirashvili and Bhattacharya-Weitsman:

$$\lambda_1(\Omega) - \lambda_1(B_1) \geq c_N \mathcal{A}(\Omega)^2 \qquad \text{for all } \Omega \subset \mathbb{R}^N \text{ such that } |\Omega| = |B_1|,$$

where $\mathcal{A}(\Omega)$ is the Frankel asymmetry of a set:

$$\mathcal{A}(\Omega) = \inf_{x_0 \in \mathbb{R}^N} |\Omega \Delta B_1(x_0)|.$$

More generally, the result applies to every optimal Poincaré-Sobolev constant for the embeddings $W_0^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$. (Joint work with L. Brasco and B. Velichkov).

Francesco Ghiraldin (Zurich University). *Flat currents in metric spaces and variational applications.*

Abstract. After introducing, in the metric space framework, the definition of size for a flat current, I will present a rectifiability result for currents of finite size. I will then outline the application of these objects to the distributional jacobian Ju of a vector valued Sobolev map u, which can we viewed as a current. In particular one can prove an existence results for the minimization of a new kind of Mumford-Shah type functional, featuring the size of singular set of u.

Bernd Kirchheim (Leipzig University). *Contractions and isometries.*

Abstract. We discuss the class of strongly isometric maps, i.e. maps who preserve the length of every curve, rectifiable or not. It is shown that these maps are prevalent (in the topological sense) in the class of all contractions on Euclidean space. When considering a similar result on subset of $\mathbf{R}^n$, the openness of the restriction operator with respect to the supremum norm becomes crucial - several positive but also negative results concerning this question are presented.

Jan Kolář (Czech Academy of Sciences). *A stationary varifold with non-unique tangent.*

Abstract. An example of a rectifiable stationary 2-varifold in $\mathbb{R}^4$ with non-unique (conical and/or non-conical) tangent varifolds at a point. This answers a question of L. Simon (Lectures on geometric measure theory, 1983, p. 243) and provides a new example for a related question of W. K. Allard (On the first variation of a varifold, Ann. of Math., 1972, p. 460).

Alexander Kolesnikov (Higher School of Economics, Moscow). *Riemannian approach to convexity inequalities.*

Abstract. We study Poincaré- and Sobolev-type inequalities on manifolds with densities. Our approach strongly relies on instruments of Riemannian geometry. Among of them are estimates for the Bakry–Émery tensor, integration by parts on manifolds with boundary, and the Bochner–Weitzenböck–Lichnerowicz formula. It is well known that by dualizing the Bochner–Weitzenböck–Lichnerowicz formula, one obtains the Lichnerowicz and Brascamp-Lieb Poincaré-type inequalities on Riemannian manifolds equipped with a density. When the manifold has a boundary, the Reilly formula and its generalizations may be used instead. By systematically dualizing this formula for various combinations of boundary conditions of the domain (convex, mean-convex) and the function (Neumann, Dirichlet), we obtain new Poincaré-type inequalities on the manifold and on its boundary. As an application, we prove Sobolev-type inequalities for log-concave distributions equipping the space with various Riemannian metrics.

Antoine Lemenant (Université Paris Diderot). *Regularity of one-dimensional almost minimal sets in Banach spaces.*

Abstract. I will talk about a recent work with Thierry De Pauw (Paris 7, IMJ) and Vin-

cent Millot (Paris 7, LJLL), which contains some regularity results for one-dimensional almost minimal sets in Banach spaces. An almost minimal set is a closed and connected set that locally minimizes the Hausdorff measure $\mathcal{H}^1$, up to an excess controlled by $\xi(r)$ in a ball of radius r, where ξ is a gauge function. It is well known that such sets are C^1 regular almost everywhere in euclidean spaces, provided that the gauge is Dini integrable. In this talk I will present a condition on the ambient norm for which this result still holds true.

Sean Li (Chigago University). *Coarse differentiation of Lipschitz functions.*
Abstract. Bates, Johnson, Lindenstrauss, Preiss, and Schechtman introduced a large-scale notion of differentiation for Lipschitz maps between Banach spaces. We review this result and discuss a recent extension of this to the nonabelian setting of Carnot groups.

Xiangyu Liang (Warwick University). *Minimal sets and classification of singularities.*
Abstract. A minimal set is a closed set (in an Euclidean space) whose Hausdorff measure cannot be decreased by any compactly supported Lipschitz deformation. This notion was invented by Almgren to give a reasonable model for Plateau's problem, which aims at understanding the behavior of physical objects that admit certain minimizing property, such as soap films. We shall introduce some basic definitions, examples and facts about minimal sets and cones, as well as some results and open problems.

Philippe Logaritsch (Max Planck Institut, Leipzig). *Sobolev functions with values in a metric space.*
Abstract. I will discuss two different approaches to weakly differentiable functions with values in a metric space proposed by Korevaar-Schoen and Ambrosio-Reshetnyak. Although they essentially lead to the same set of functions, their notions of p-energy do not coincide. I will present joint work with Emanuele Spadaro where we give an explicit representation formula for the energies defined by Korevaar and Schoen, using only the framework of the second approach and thus further relating the two notions. Time permitting, I will also consider the generalized Dirichlet problem.

Olga Maleva (University of Birmingham). *Null universal differentiability sets.*
Abstract. Given a space X, we are looking for its subsets S as small as possible with the universal differentiability property, i.e. that every Lipschitz function on X has a point of differentiability in S. We show that every finite dimensional space contains universal differentiability sets of Minkowski dimension 1. We discuss possible generalisations and show that this result is optimal.

Andrea Marchese (MPI, Leipzig). *Differentiability of Lipschitz functions with respect to measures.*
Abstract. Rademacher theorem states that Lipschitz functions on the Euclidean space are Lebesgue a.e. differentiable. In this talk I will show an extension of this theorem, when the Lebesgue measure is replaced by an arbitrary measure μ. I will explain how to

associate to μ a bundle of vector spaces such that every Lipschitz function is differentiable at μ-a.e. point along the corresponding vector space. It turns out that this bundle is related to the decompositions of the measure in terms of rectifiable one-dimensional measures. I will also show that this bundle is optimal, exhibiting a Lipschitz function which is μ-a.e. non differentiable along any direction which does not lie in the bundle. This is joint work with Giovanni Alberti.

András Mathé (University of Warwick). *Disintegrating measures onto Lipschitz curves and surfaces.*

Abstract. Let us say that a measure in a Euclidean space has a k-dimenisonal Alberti representation if we can disintegrate it onto measures on k-dimensional Lipschitz surfaces which are absolutely continuous with respect to the k-dimensional Hausdorff measure. We present a measure in $\mathbf{R}^3$ which has two independent 1-dimensional Alberti representations (where independent means that the two families of Lipschitz curves go in separate directions) yet it is supported by a 2-purely unrectifiable set.

Pertti Mattila (University of Helsinki). *Singular integrals, rectifiability and fractality.*

Abstract. I shall survey the theme: singular integrals behave nicely on rectifiable sets and badly on fractal sets.

Ulrich Menne (AEI/Potsdam University). *Connectedness properties of varifolds.*

Abstract. Consider varifolds in an open subset of Euclidean space with density bounded from below whose mean curvature satisfies a critial integrability condition. Amongst the aspects of their structure which will be covered in the talk are:

(1) Connectedness of support vs. indecomposability.

(2) Relative isoperimetric inequality.

(3) Poincaré type embedding results.

(4) Local connectedness (by paths of finite length) of the support.

Emanuele Paolini (Università di Firenze). *Decomposition of one-dimensional currents.*

Abstract. We see how every one-dimensional normal current can be written as an integral over rectifiable curves and solenoids. The result, proven in the Euclidean setting by Stanislav Smirnov is extended, with a new proof, to metric currents. This is a joint work with Eugene Stepanov.

Mircea Petrache (ETH, Zurich). *Weak curvatures and the Yang-Mills Plateau problem.*

Abstract. I will present the Yang-Mills minimization problem as a non-integrable version of the classical Plateau problem for minimal surfaces, based on a joint project with Tristan Rivière.

The starting point are Uhlenbeck's results which provided the analytic foundations for Donaldson's study of Yang-Mills connections on bundles over 4-manifolds. The object of study in that case was the class of Sobolev connections on smooth bundles. I will highlight an analogy between these results and the parametric approach to the Plateau problem. In dimensions 5 and higher the space of Sobolev connections over smooth bundles does not allow to apply the direct methods of the Calculus of Variations to obtain Yang-Mills minimizers. The substitute is a space of weak connections over singular bundles, in which a weak closure result allows constructing Yang-Mills connections by direct minimization. A parallel will be made between the proof of this closure result and the proof by slicing of the Federer-Fleming closure theorem for rectifiable currents.
I will then present an optimal partial regularity result for Yang-Mills energy minimizers in dimension 5, which recovers results of Tao-Tian and Rivière-Meyer. If time allows, I will mention links to several other topics, including combinatorial and optimal transport problems as well as more geometric directions.

Andrea Pinamonti (Università di Padova). *BV Minimizers of the area functional in the Heisenberg group under the bounded slope condition.* **Abstract.** We consider the area functional for t-graphs in the sub-Riemannian Heisenberg group and study minimizers of the associated Dirichlet problem. We prove that, under a bounded slope condition on the boundary datum, there exists a unique minimizer and that this minimizer is Lipschitz continuous. We also provide an example showing that, in the first Heisenberg group, Lipschitz regularity is sharp even under the bounded slope condition.

Tristan Riviére (ETH, Zurich). *Some aspects of the analysis of Gauge theory in supercritical dimensions.*
Abstract.

Gareth Speight (Scuola Normale Superiore). *Porosity and differentiability.*
Abstract. A set is porous if arbitrarily close to each point of the set there are holes in the set of radius proportional to their distance away. Porous sets (and their countable unions) give a geometrically interesting class of small sets which have applications in other areas of mathematics. We discuss two recent applications of porous sets to differentiability problems - generalizations of Rademacher's theorem to Lipschitz functions on infinite dimensional Banach spaces and differentiability of Lipschitz functions inside Lebesgue null subsets of Euclidean spaces.

Thomas Schmidt (Erlangen University). *Extremality relations for BV-minimizers.*
Abstract. Minimization problems for variational integrals with linear growth can often be solved in a generalized way in the space BV of functions of bounded variation. The talk will focus on convex duality theory for such integrals and specifically on pointwise extremality relations for the BV-minimizers. These relations are useful in the study of uniqueness and regularity questions, and one possible proof is based on an analysis of singular measures via Anzellotti's pairing – in an up-to-the-boundary version – of gradient

measures and divergence-free bounded functions.

Luca Spolaor (Zurich University). *Uniqueness of the tangent cone to 2-d almost area minimizing currents.*

Abstract. We call almost minimizer those currents which satisfy a monotonicity formula and are area minimizing apart for an error which, in a ball of radius r, is of order r^a, for $a > 0$. We extend White's proof of uniqueness of the tangent cone to 2-d area minimizing current to these almost minimizers. We prove that area minimizing currents in Riemannian manifolds, semicalibrated currents and cross-section of 3-d area minimizing cones are almost minimizers, thus having unique tangent cone.

Yoshihiro Tonegawa (Hokkaido University). *Existence and regularity of mean curvature flow with transport term.*

Abstract. We consider the following problem. Suppose on R^n ($n \geq 2$) that we are given a vector field u and a bounded C^1 hypersurface M_0. Prove the existence and regularity of a family of hypersurfaces $\{M_t\}_{t>0}$ starting from M_0 such that the normal velocity of M_t is equal to $h + u^\perp$, where $u^\perp$ is the normal projection of u. When $u = 0$, it is the usual mean curvature flow (MCF). If $u \in L^q_{\rm loc}([0,\infty); W^{1,p}(\mathbf{R}^n))$ with $2 < q < \infty$ and $nq/2(q-1) < p < \infty$, ($4/3 \leq p$ in addition if $n = 2$), we prove that there exists a time-global weak solution of this problem. M_t remains C^1 for at least a short time, and is a.e. C^1 away from the higher multiplicity region. I discuss the definition of weak solution, the strategy of the existence proof, and how the partial regularity follows from our recent local regularity results on the MCF.

Tatiana Toro (University of Wahington). *Regularity for almost minimizers with free boundary.*

Abstract. In recent work with Guy David we introduce the notion of almost minimizer for a series of functionals previously studied by Alt-Caffarelli and Alt-Caffarelli-Friedman. We prove regularity results for these almost minimizers and explore the structure of the corresponding free boundary. A key ingredient in the study of the 2-phase problem is the existence of almost monotone quantities. The goal of this talk is to present these results in a self-contained manner, emphasizing both the similarities and differences between minimizers and almost minimizers.

Stefan Wenger (Fribourg University). *An upper gradient approach to weakly differentiable cochains.*

Abstract. The purpose of this talk is to discuss a recent extension of Whitney's theory of flat cochains in Euclidean space. Inspired by Heinonen-Koskela's concept of upper gradients for functions, we define a notion of weakly differentiable cochain in a metric measure space. We then prove a Hoelder continuity property with respect to the flat norm for weakly differentiable cochains with upper gradient in L^p for sufficiently high p, thus extending the Morrey-Sobolev inequality from the setting of Sobolev functions to that of higher-dimensional cochains. We moreover generalize a fundamental theorem of

Wolfe concerning the duality of flat cochains and flat forms to our setting. Namely, we show that our space of Sobolev cochains in R^n is isometrically isomorphic to the space of Sobolev differential forms. This is based on joint work with Kai Rajala and on joint work with Kai Rajala and Camille Petit.

Neshan Wickramasekera (Cambridge University). *Structure of branch sets of minimal submanifolds and multi-valued harmonic functions.*

Abstract. The talk will be based on results from an on-going project (joint with Brian Krummel) aimed at understanding the local structure of sets of branch point singularities of various classes of minimal varieties and multi-valued harmonic functions. For these classes, the work thus far establishes a set of new a priori estimates, valid near multiplicity 2 branch points, that are analogous to those established in L. Simon's pioneering work in the early 90's on the structure of singularities of minimal submanifolds in compact, multiplicity 1 classes satisfying a certain integrability hypothesis. In particular, our estimates combined with earlier (unpublished) work of the speaker imply rectifiability results for branch sets and uniqueness of blow-ups at generic branch points for (a) stable minimal hypersurfaces near density 2 branch points and (b) two-valued energy minimizing and $C^{1,\alpha}$ harmonic functions.