On open 3-manifolds

School on Geometric Evolution Problems

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Outline

The question

Post-Perelman Question : what could be a good statement for the classification (geometrization) of open 3-manifolds ?

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Involves the structure at infinity of non compact surfaces.

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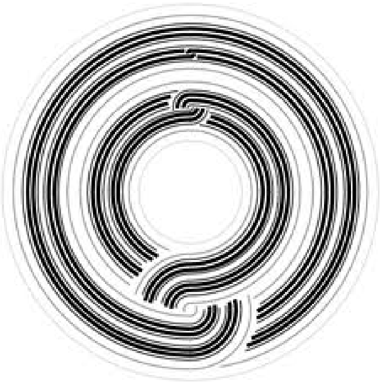

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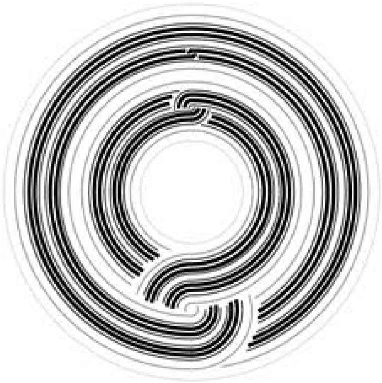

On the picture $T_{i+2} \subset T_{i+1} \subset T_i \subset T_{i-1}$.

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Theorem (J.H.C. Whitehead)

X is contractible and not homeomorphic to $\mathbf{R}^3$.

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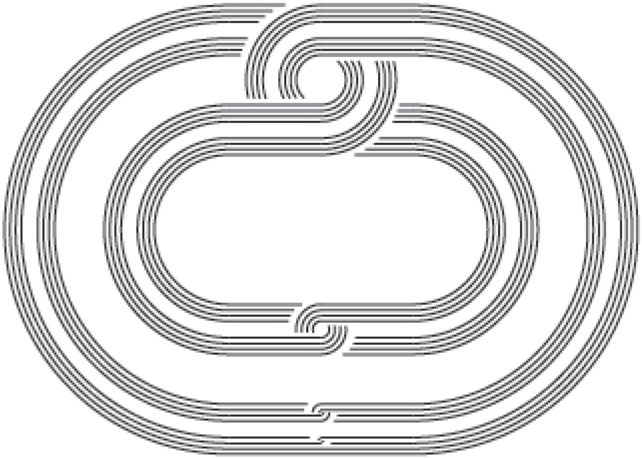

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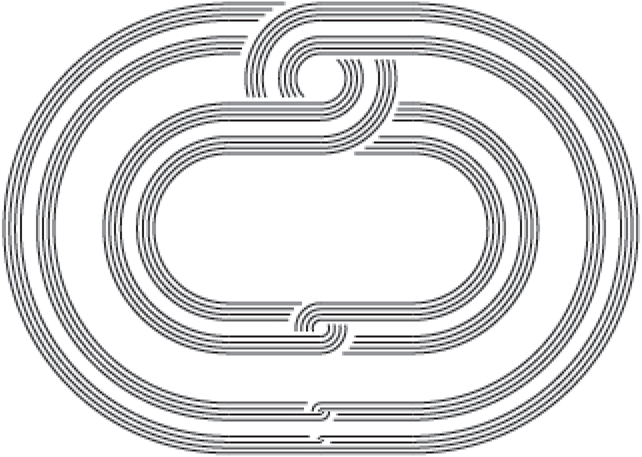

Hausdorff dimension > 1.

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- X is a limit of solid tori.

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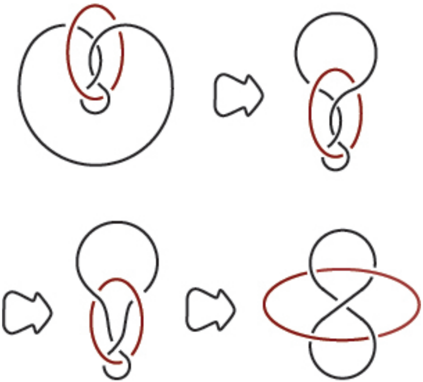

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- A more general construction.

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Definition

X topological space is simply connected at infinity if $\forall C$ compact, $\exists D$ compact, $D \supset C$ s.t. any loop in $X \setminus D$ is null homotopic in $X \setminus C$.

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- Examples that cannot cover non-trivially any manifold (Myers).

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- **Question 2** : what is the evolution of the Ricci flow (with surgery) ?

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Theorem (Gang Liu)

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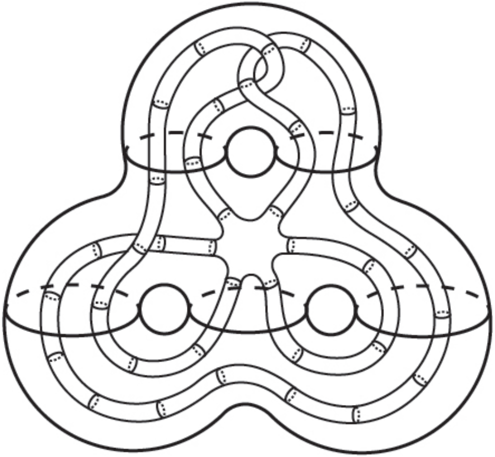
Higher genus Whitehead manifolds



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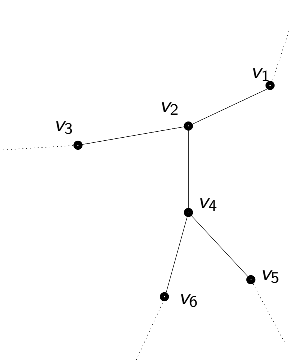

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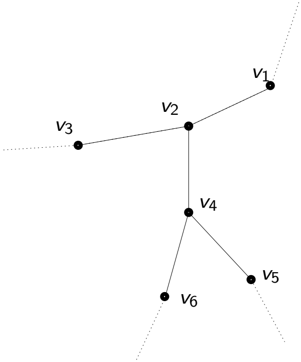

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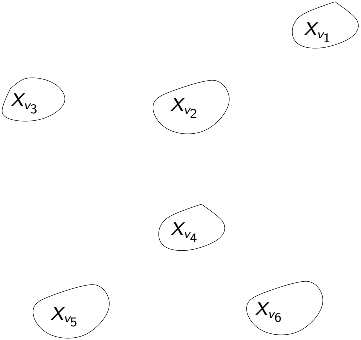

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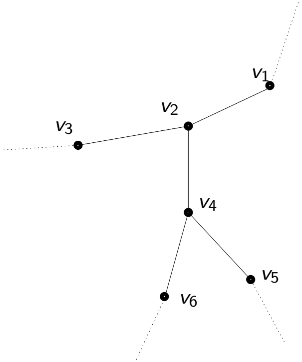

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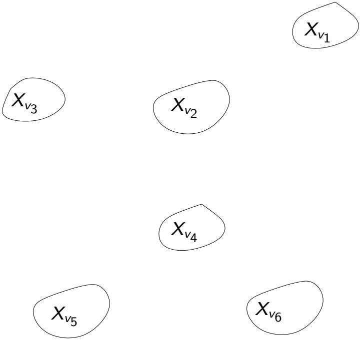

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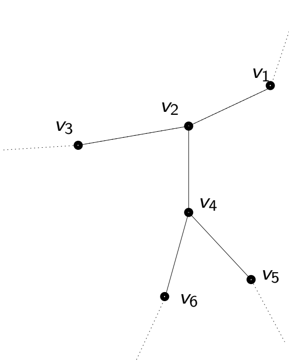

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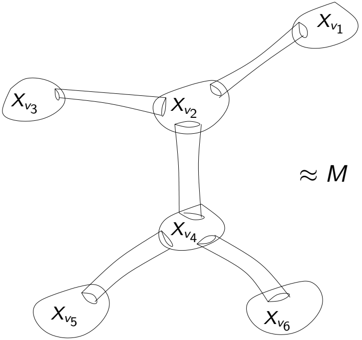

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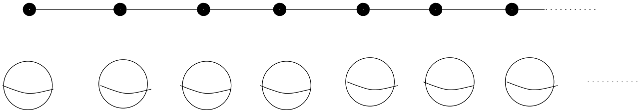

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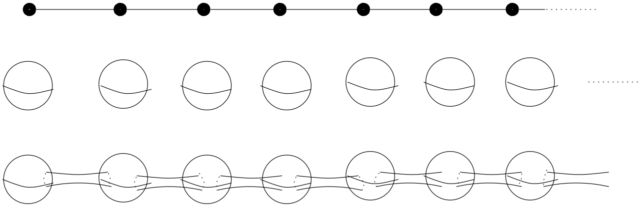

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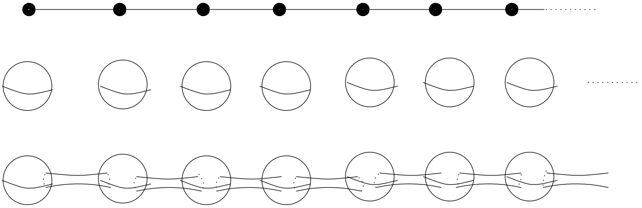

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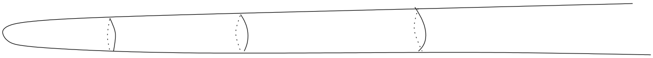

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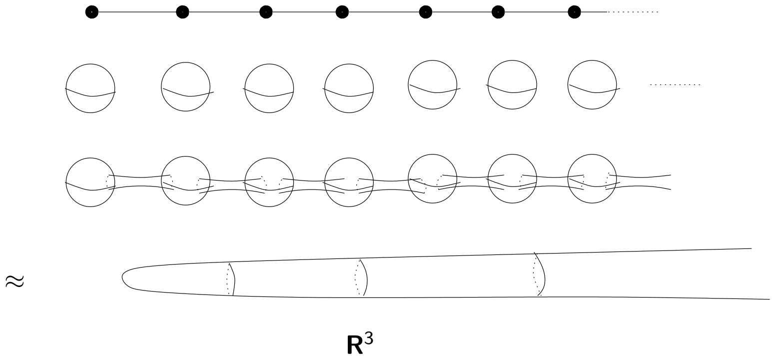

- $\mathcal{X} = \{S^3\}$, $T =$ the line $\rightsquigarrow S^2 \times \mathbf{R}$

Graphs versus trees

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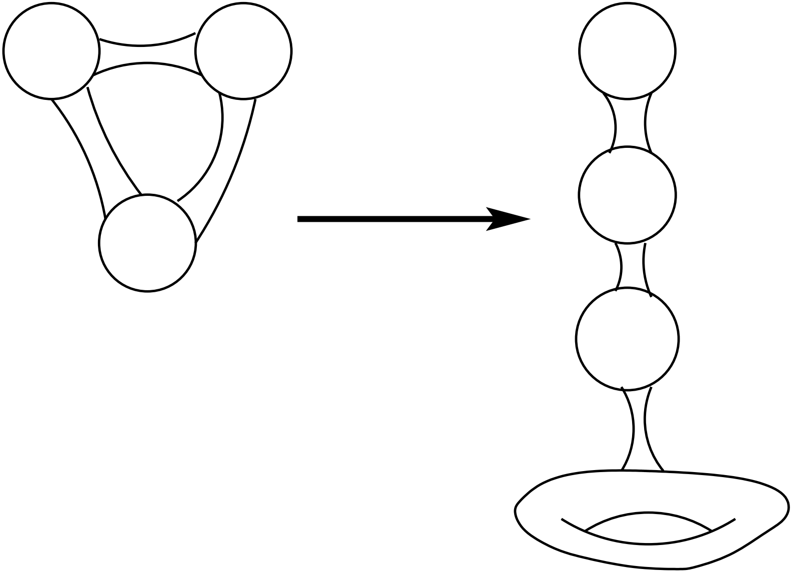

One result

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Theorem (Bessières-B.-Maillot)

M has a complete metric of bounded geometry and $\operatorname{Scal} \geqslant 1$ iff there is a finite collection $\mathcal{F}$ of spherical manifolds such that M is a (maybe infinite) connected sum of copies of $S^2 \times S^1$ and members of $\mathcal{F}$.

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- Uses a version of the Ricci flow with surgery for non compact manifolds.
- What if we relax the assumptions ?

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The compact case is a result of F. Codá Marques.

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Let $E \subset \mathbf{S}^3$ *(closed set), what is the "best" complete metric on* $\mathbf{S}^3 \setminus E$ *?*

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- What if E is a Cantor set or a fractal ?

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Theorem (Souto-Stover)

There is a cantor set $E \subset \mathbf{S}^3$ such that the complement $\mathbf{S}^3 \setminus E$ admits a complete hyperbolic metric.

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- $\mathcal{C}$ is a Bessel capacity.

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- Satisfied for $E =$ Whitehead continuum.

Conical singularities

Question : Constant scalar curvature metrics with conical singularities on E ?

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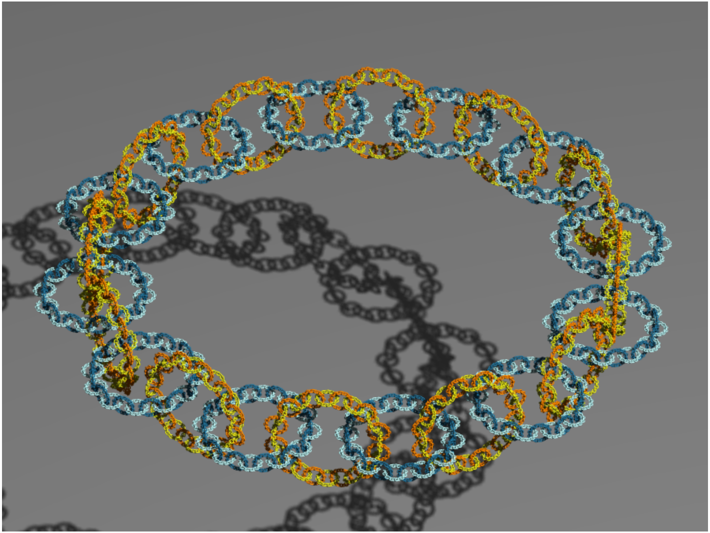
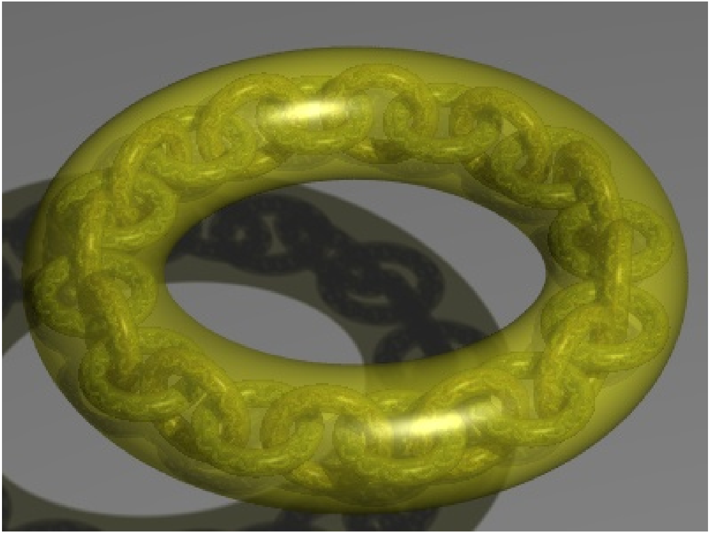
- What does that mean ?
- Start with $n = 2$.

Antoine's necklace



Antoine's necklace birth



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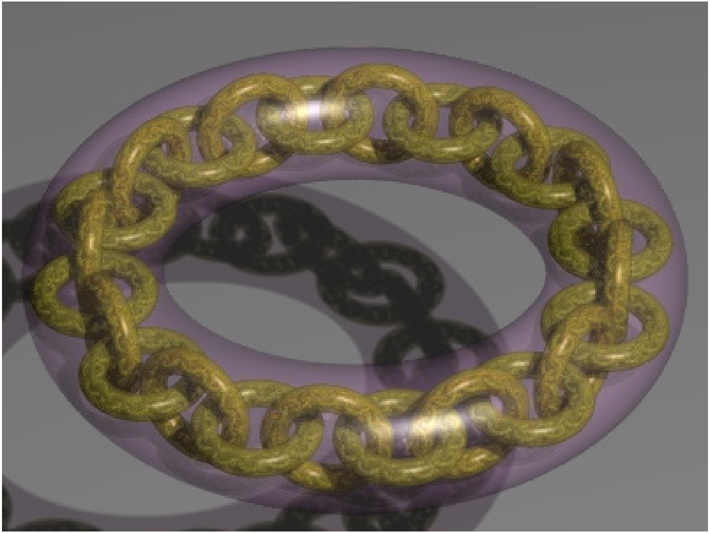

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