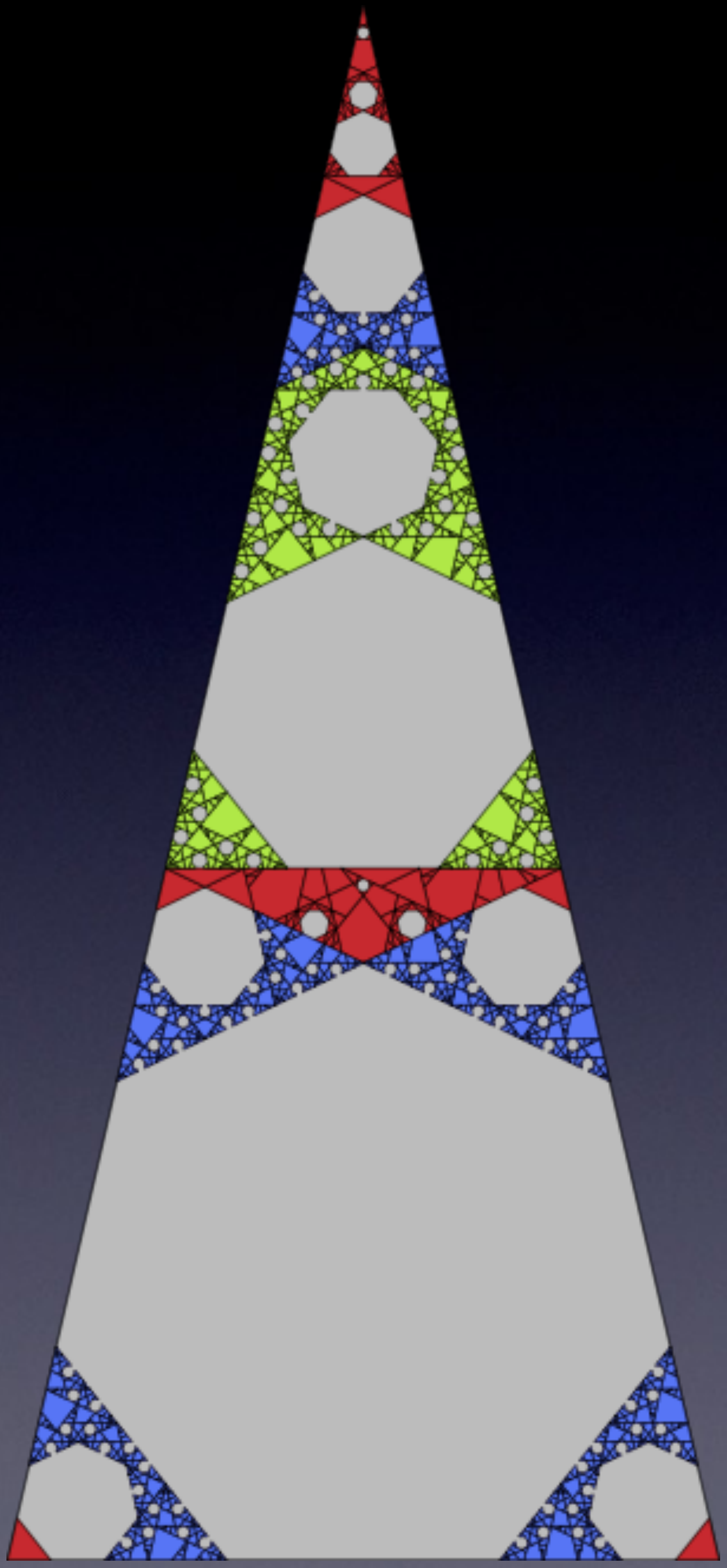
Renormalization

in parametrised families of polygon-exchange transformations

Franco Vivaldi
Queen Mary, University of London

with J H Lowenstein (New York)

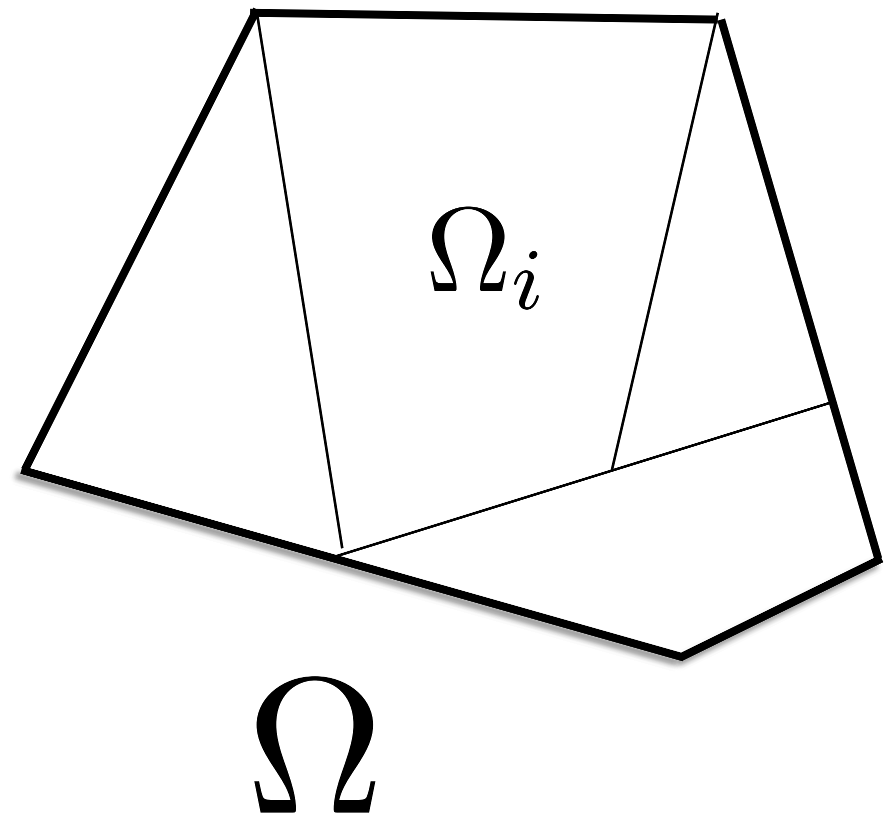


Piecewise isometries

the space:

$$\Omega \subset \mathbb{R}^n$$

$$\Omega = \overline{\bigcup \Omega_i}$$



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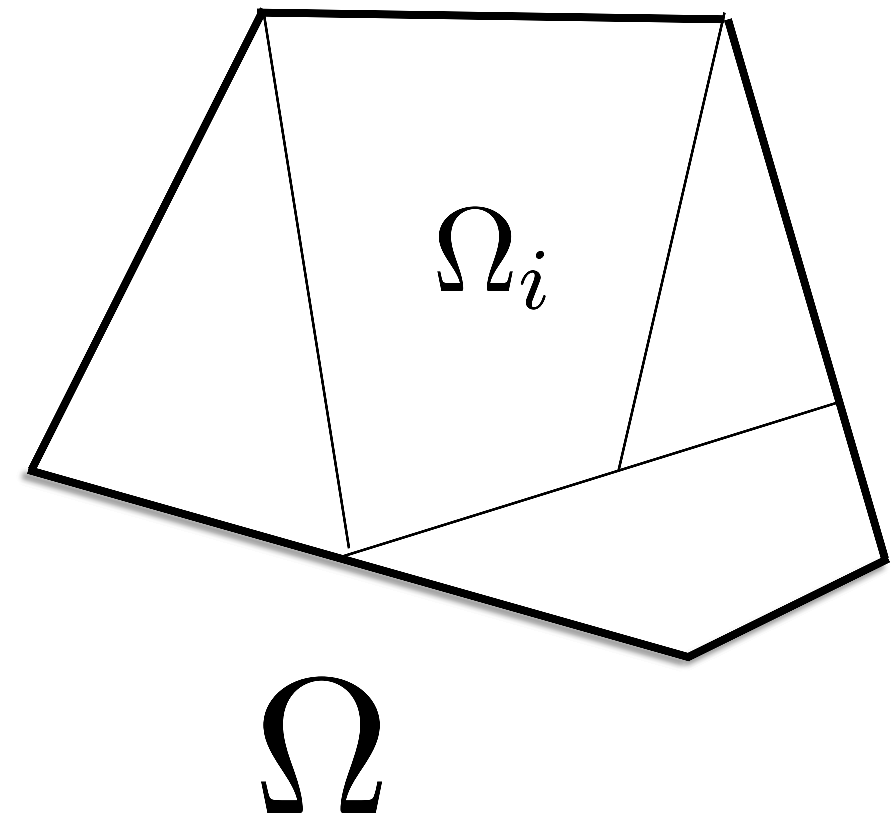

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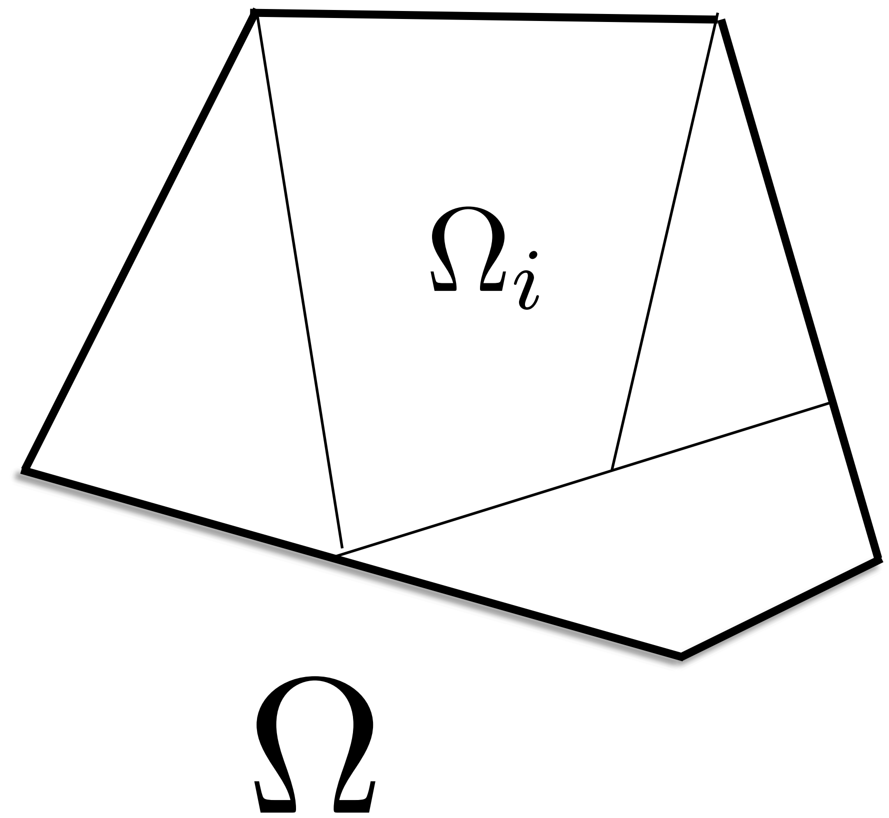

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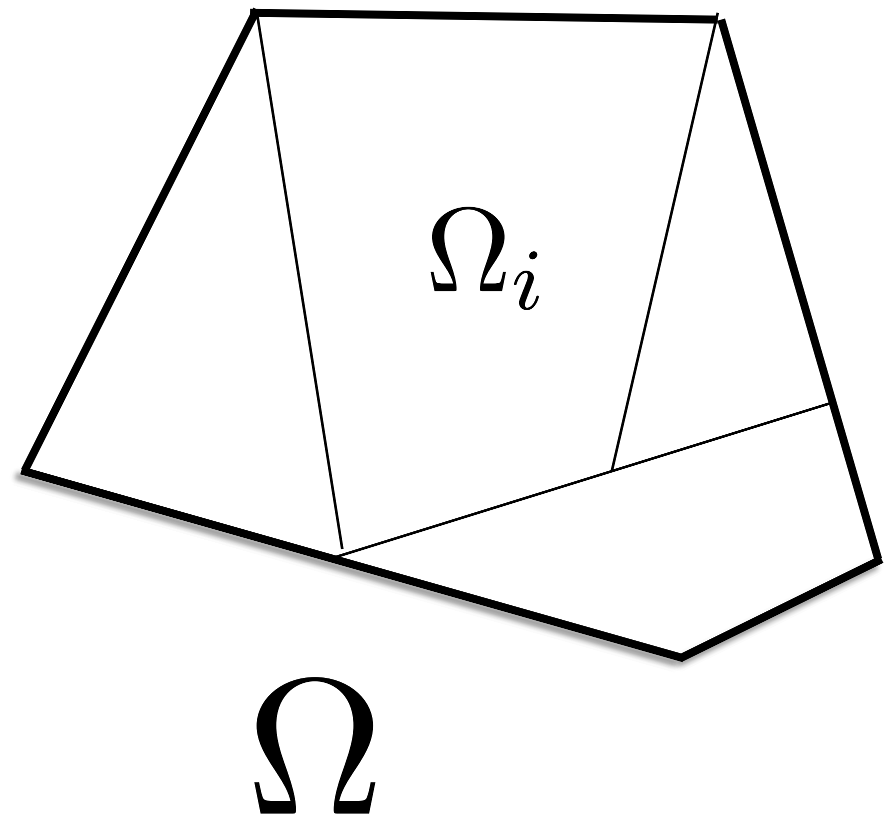

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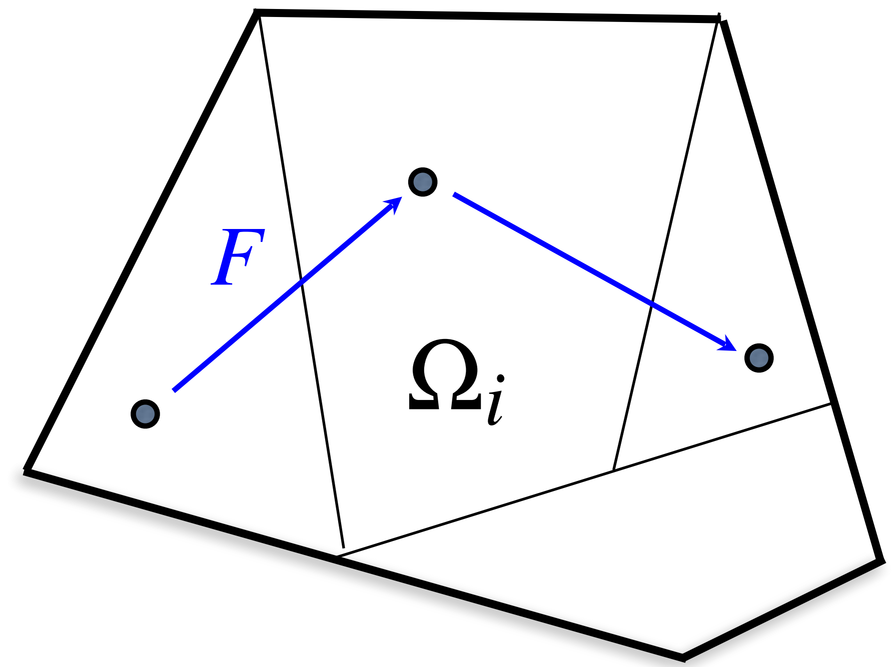
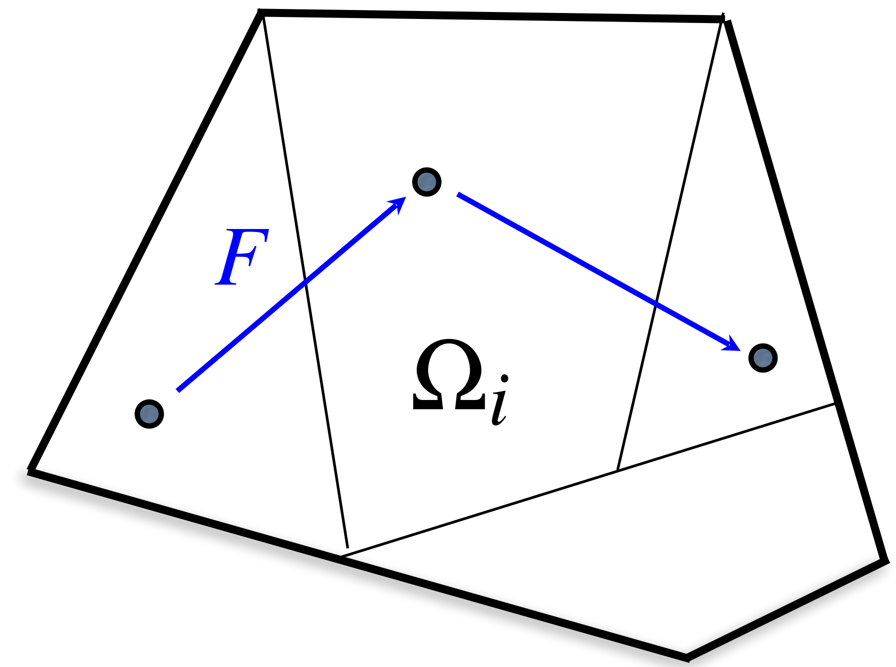
Theorem (Gutkin & Haydin 1997, Buzzi 2001)
The topological entropy of a piecewise isometry is zero.

Cells



Cells

The points of an orbit visit atoms in succession, defining a symbolic dynamics.



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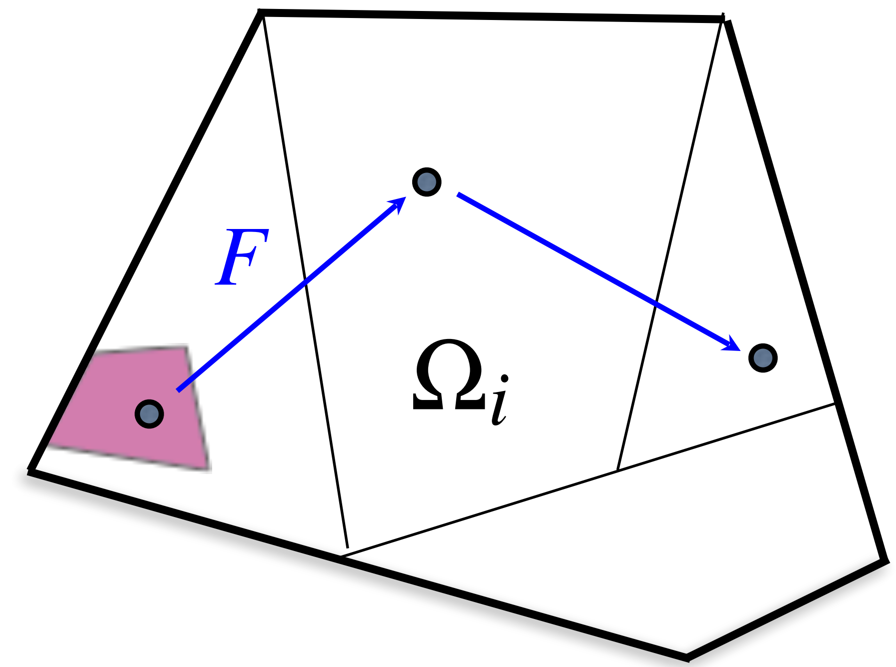
A cell is a set of points with the same symbolic dynamics; cells are convex sets.



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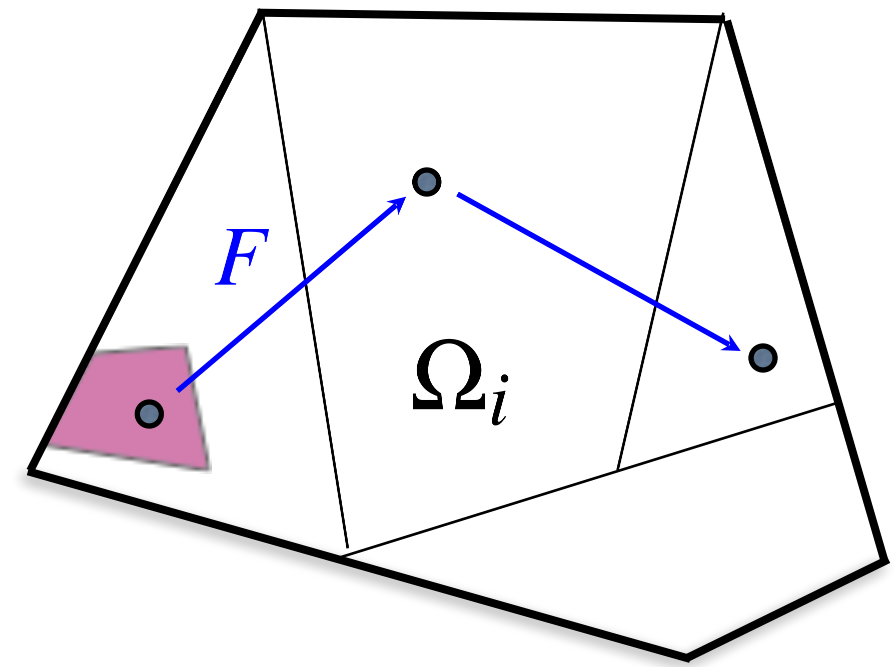
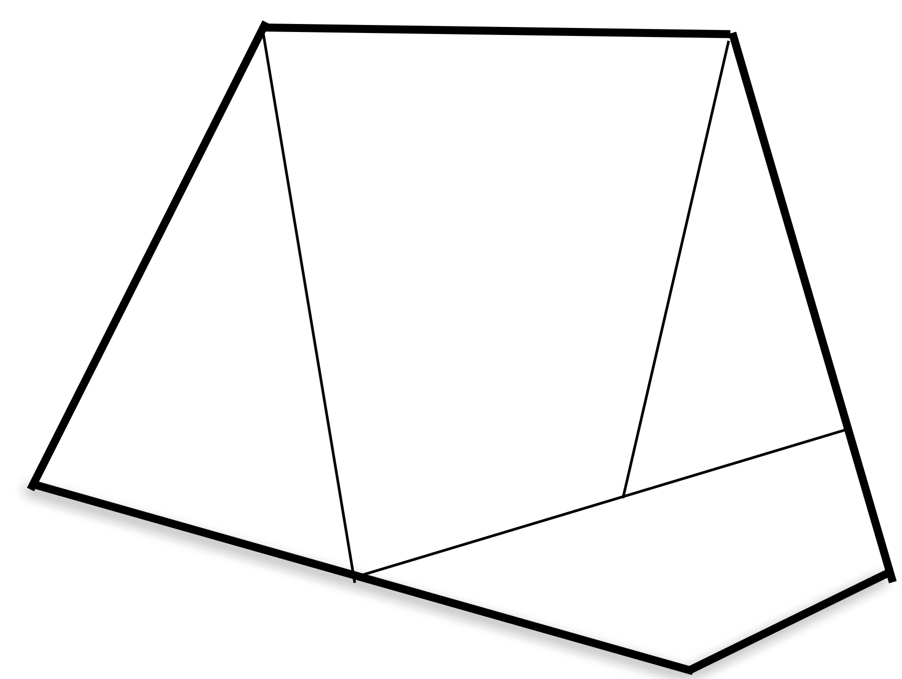


Induced maps: the basis of renormalization



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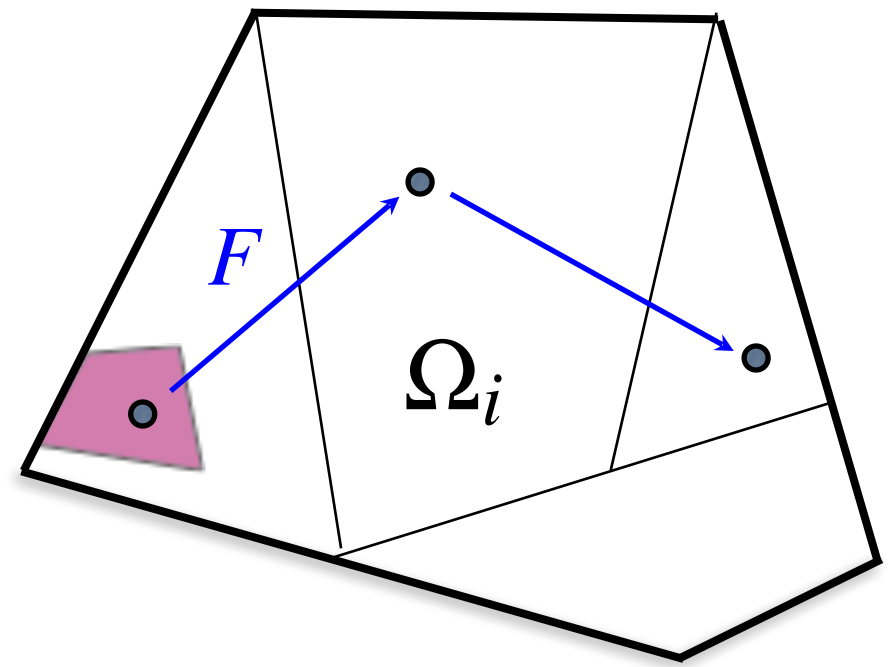

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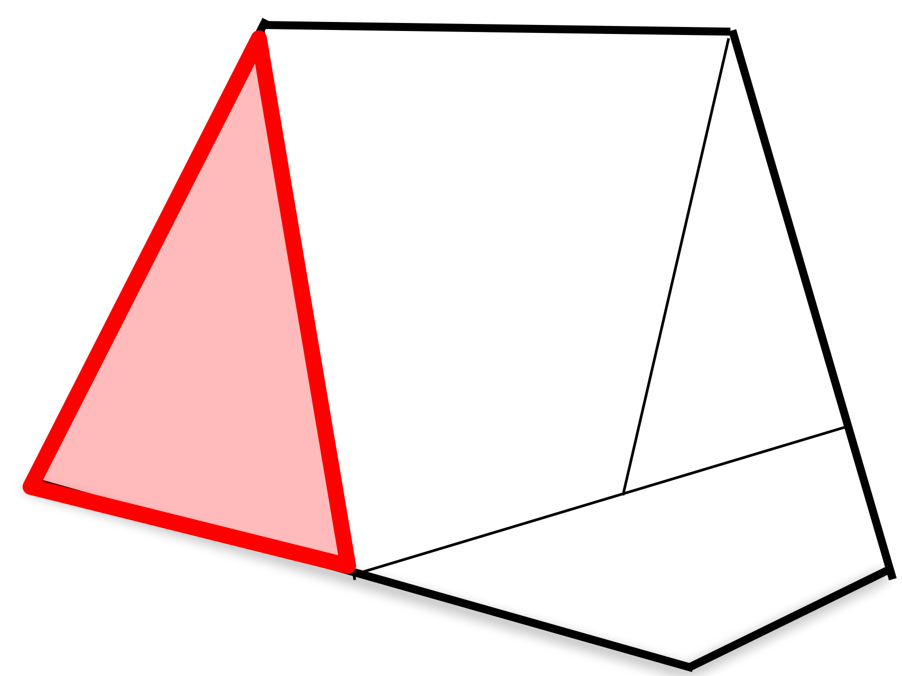

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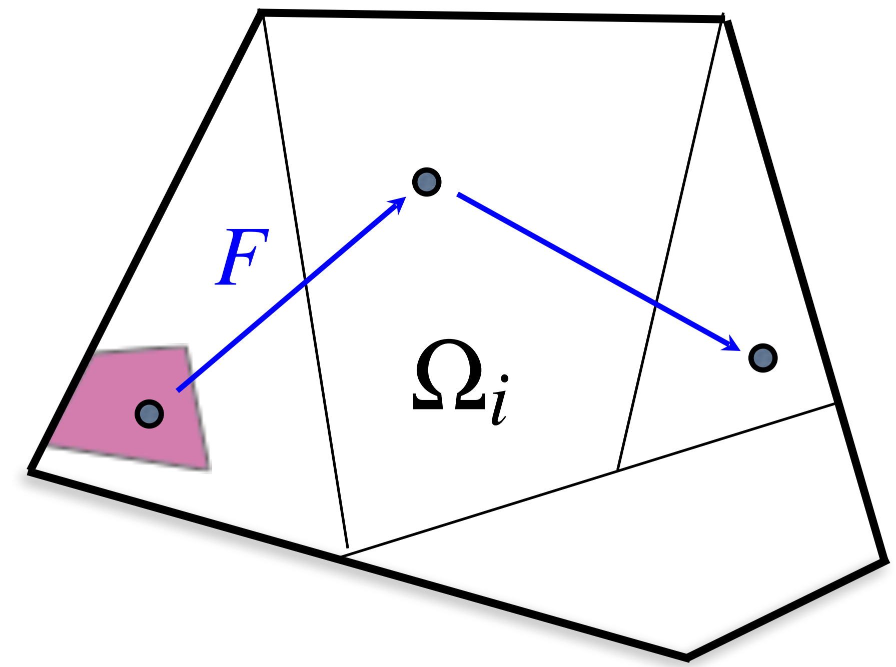

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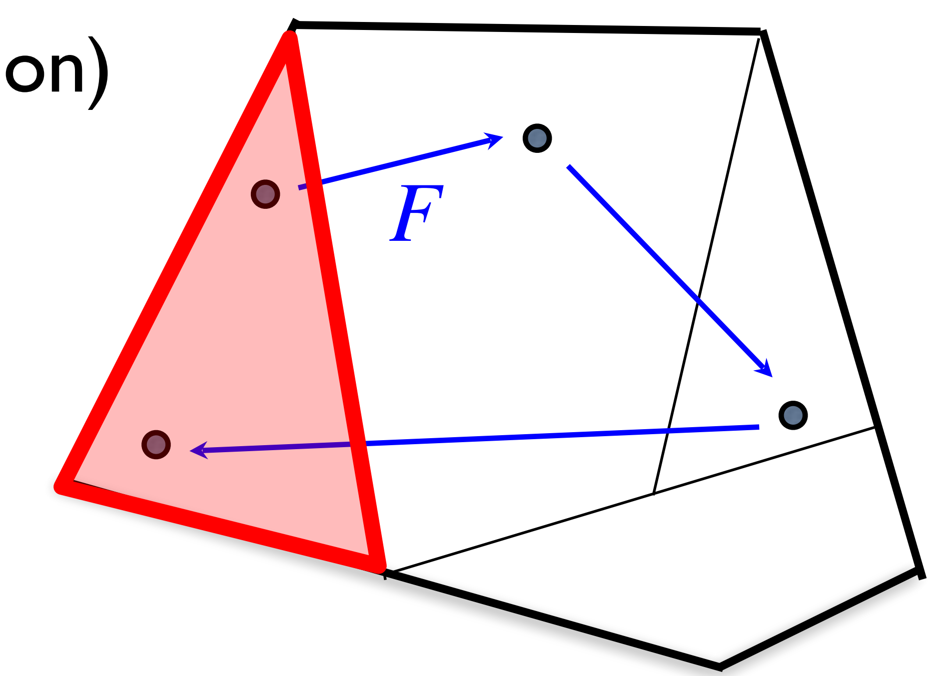

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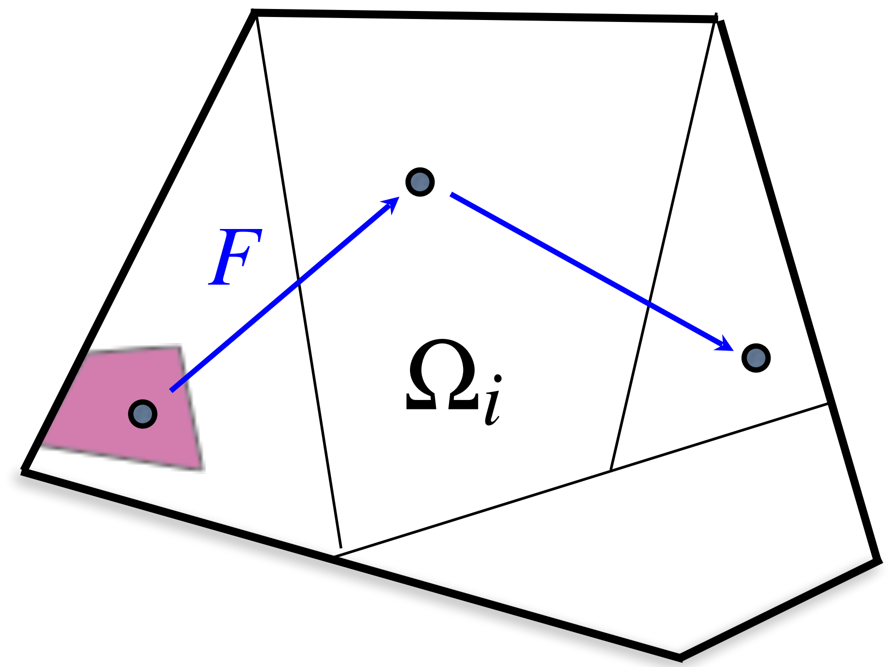

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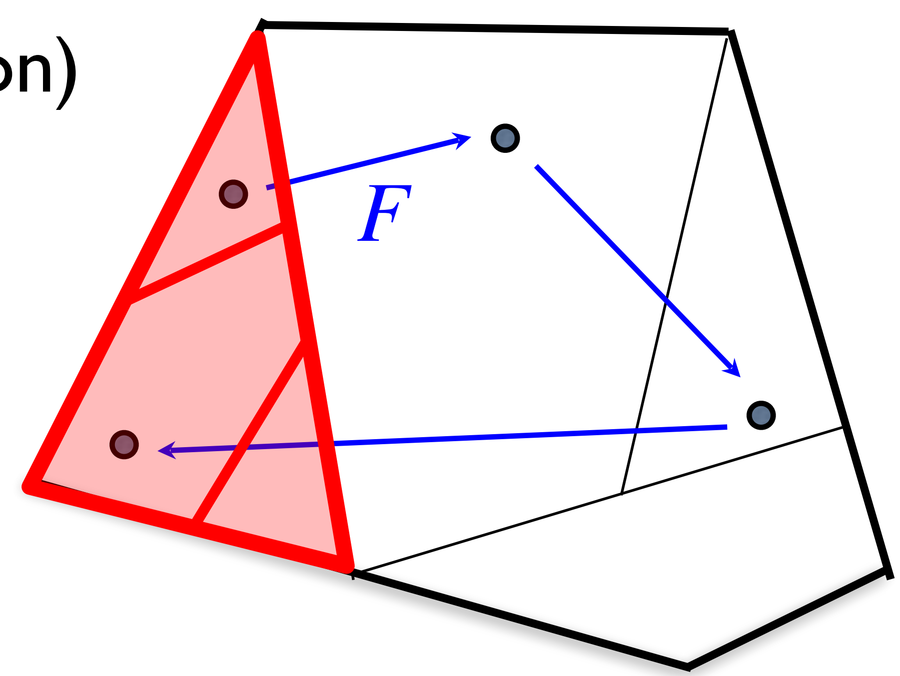

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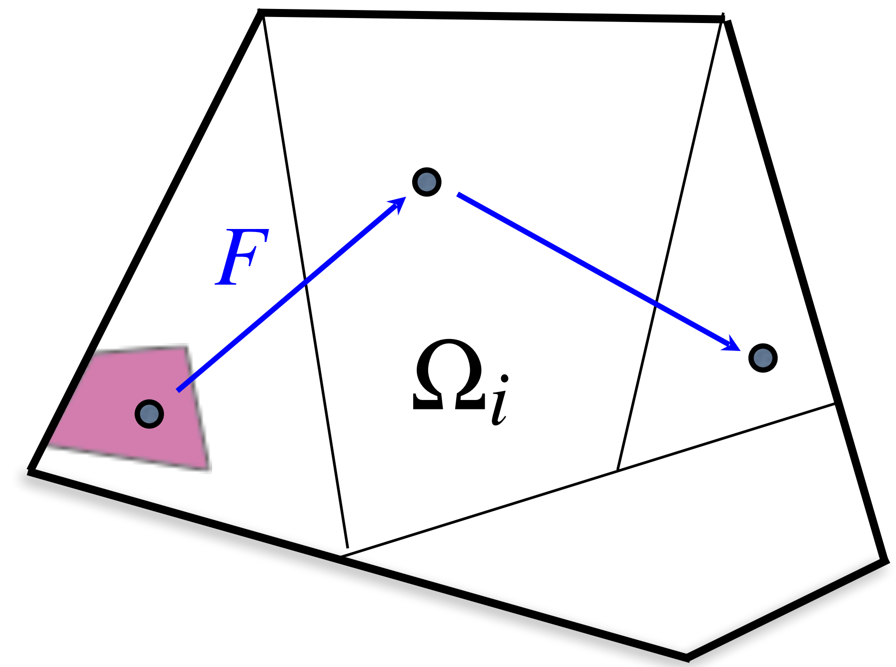

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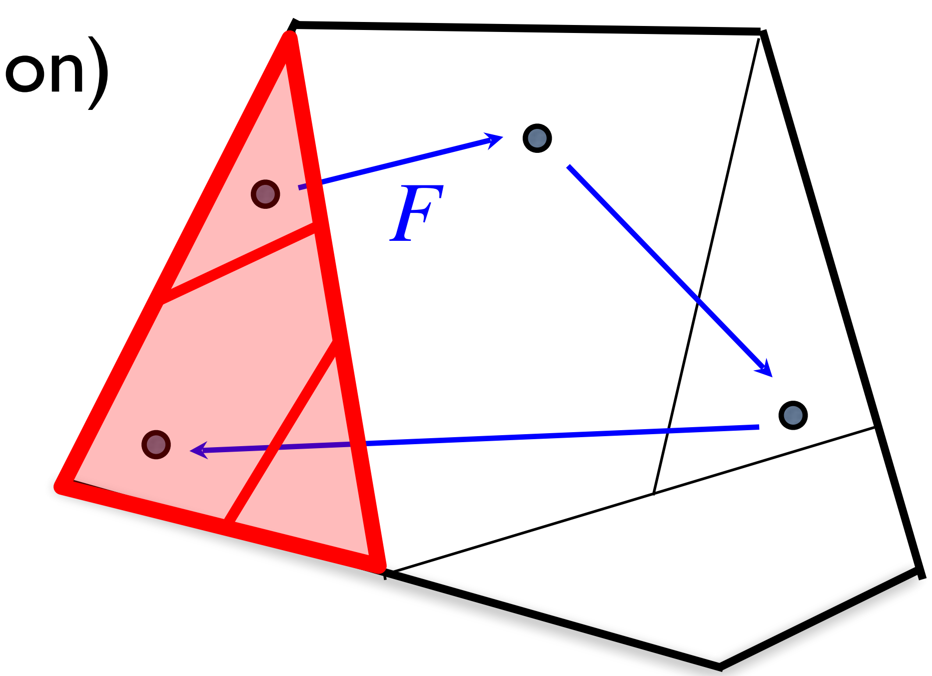
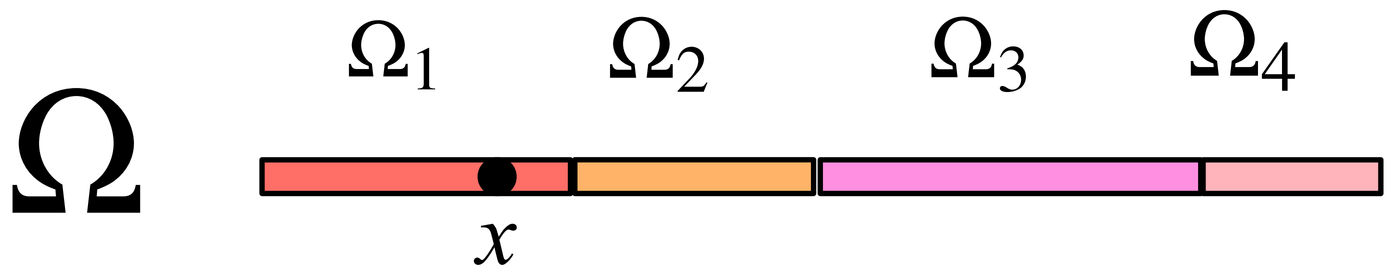
- Different return times in different atoms.



One dimension: interval-exchange transformations



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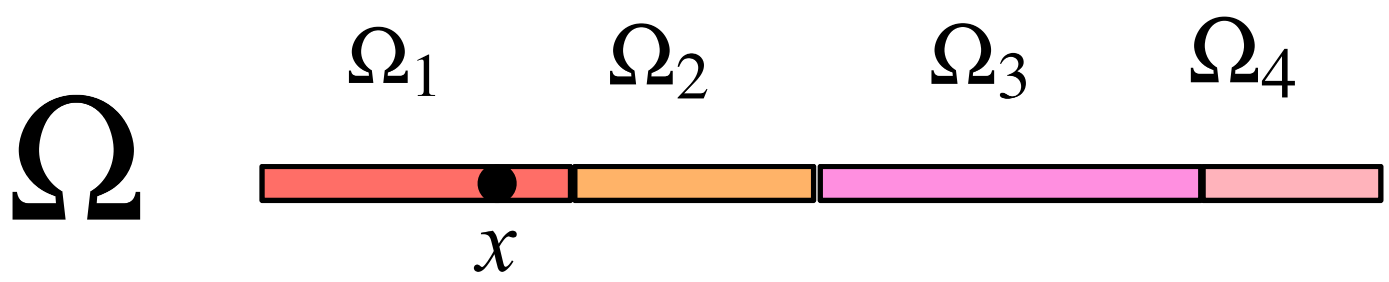

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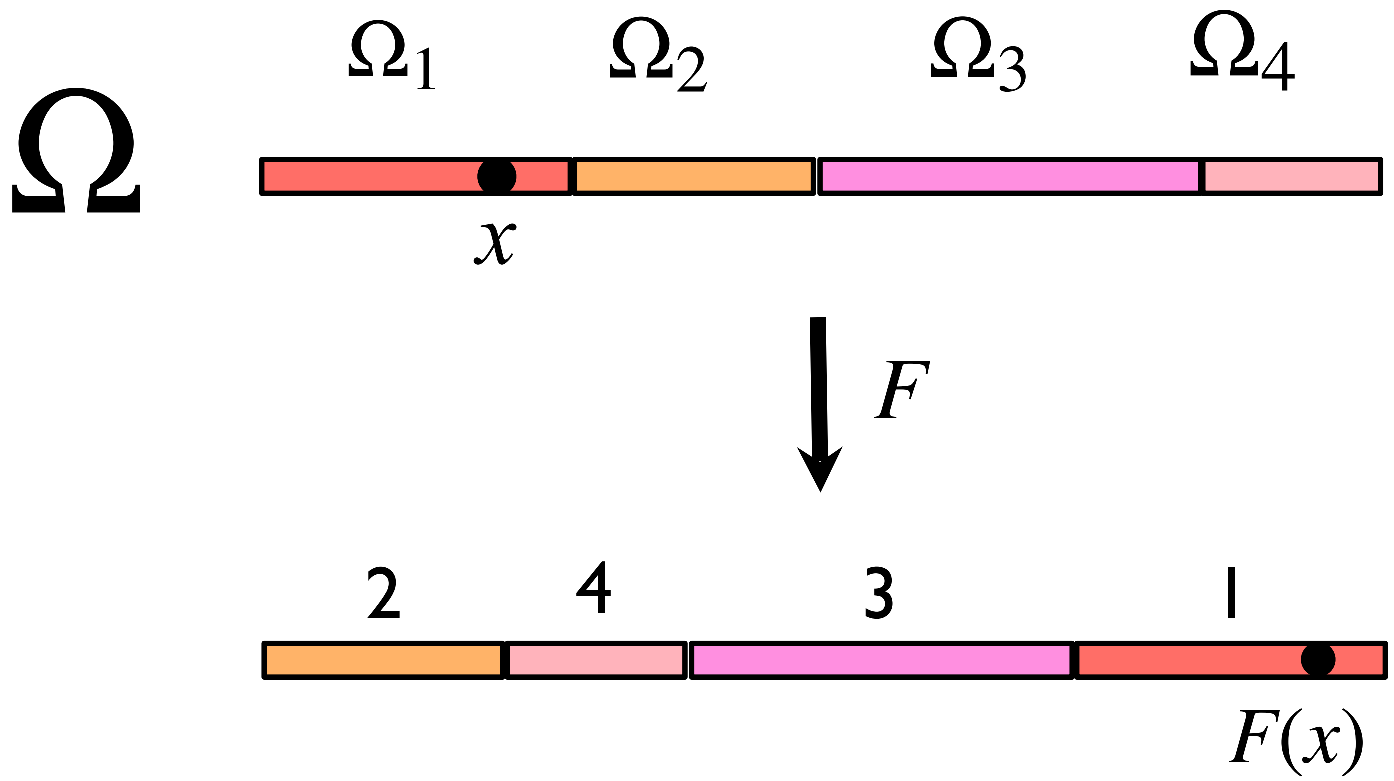

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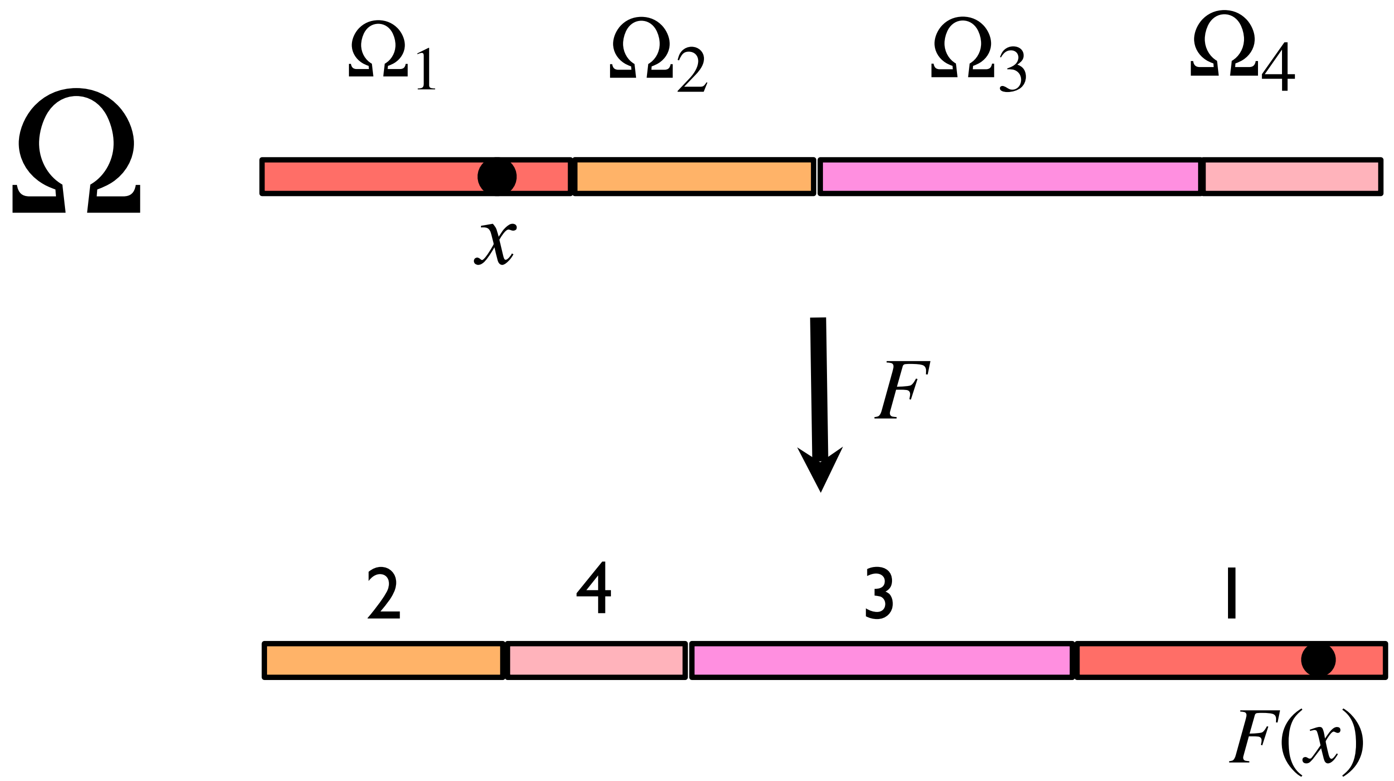

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Combinatorial data: a permutation $\quad (1234) \mapsto (2431)$

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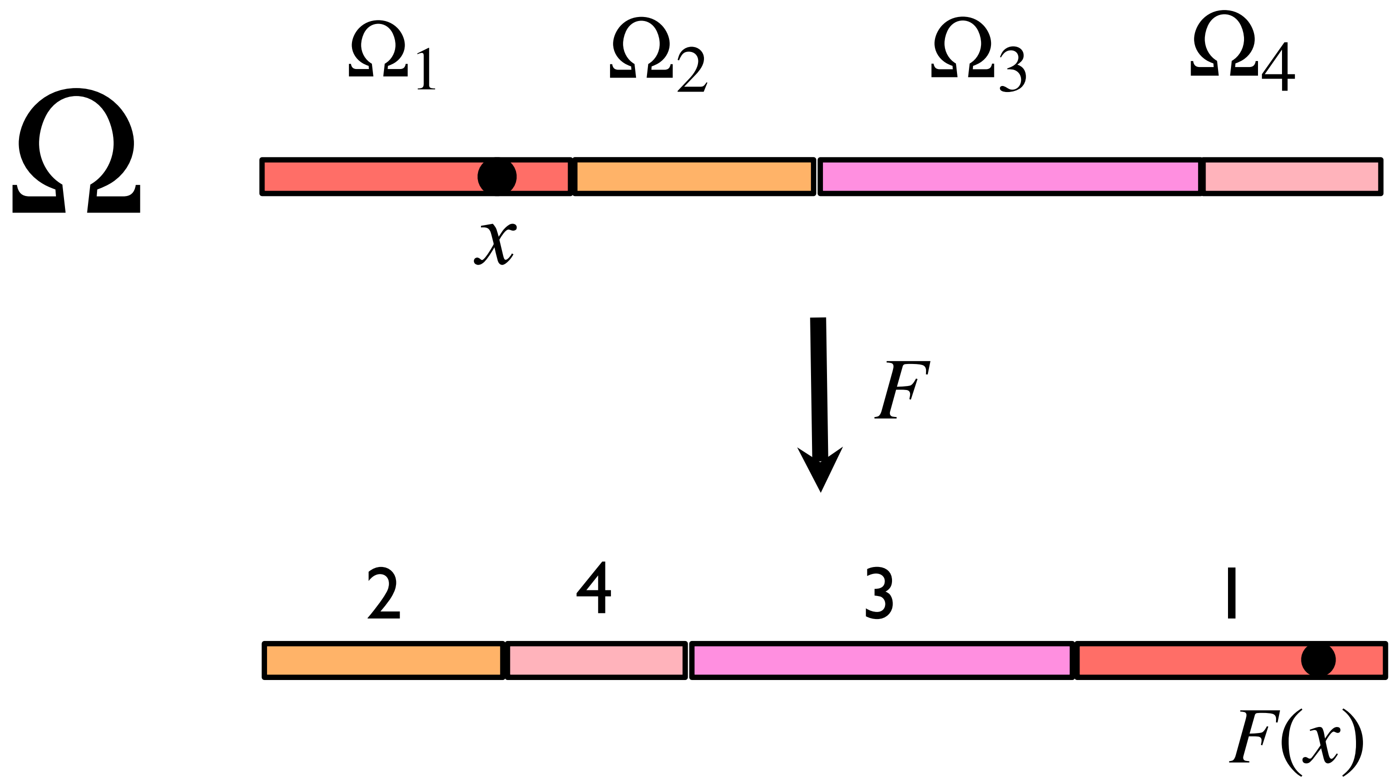

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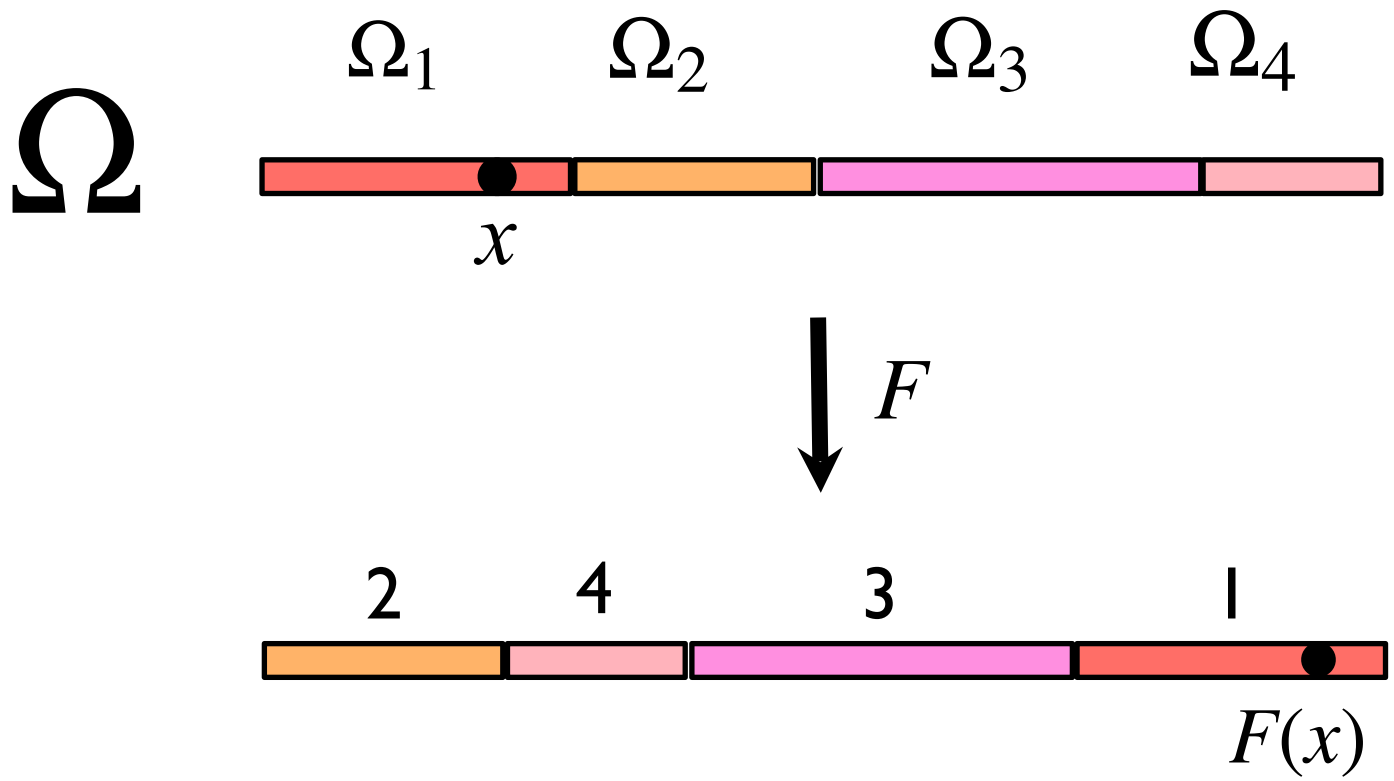

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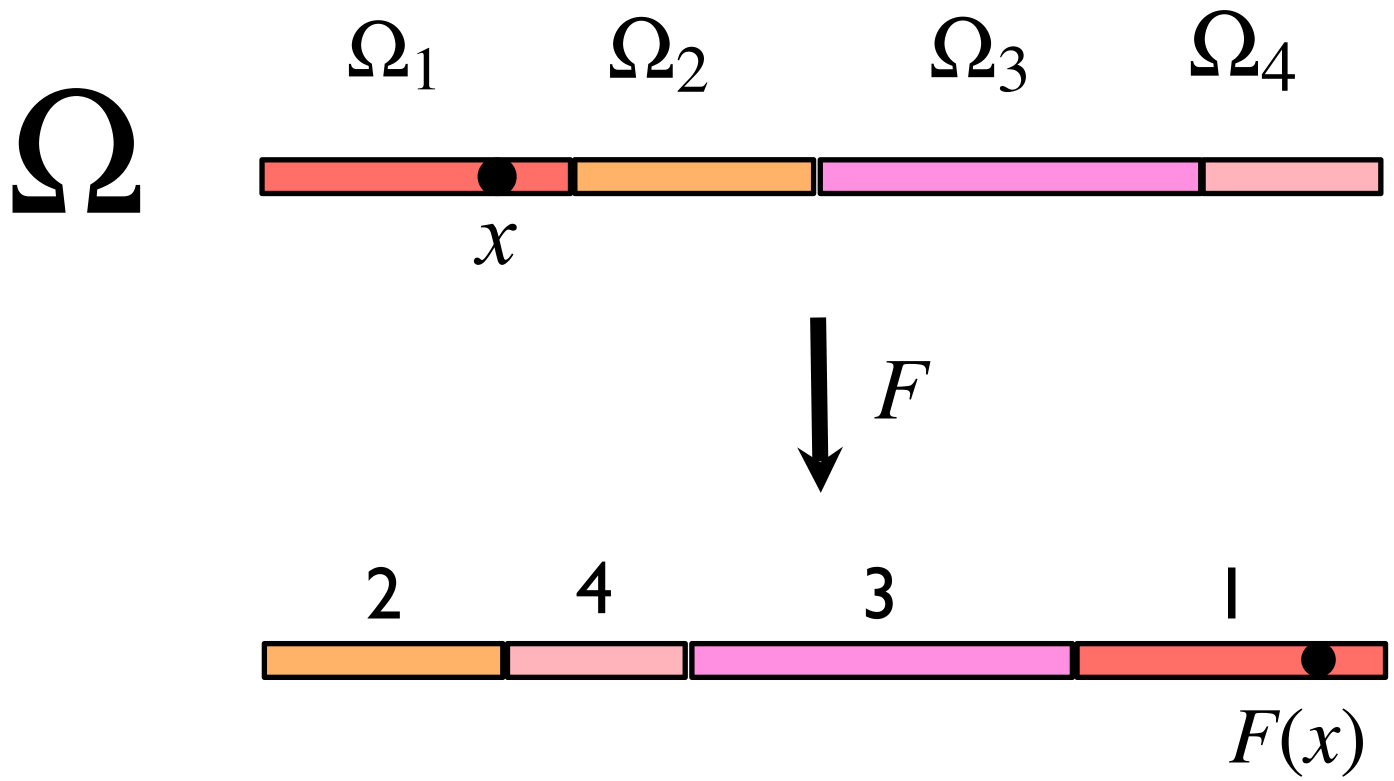

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$$K = \mathbb{Q}(|\Omega_1|, \ldots, |\Omega_n|)$$

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If an IET is defined over a quadratic field, then, up to scaling, the number of induced maps over sub-intervals is finite.

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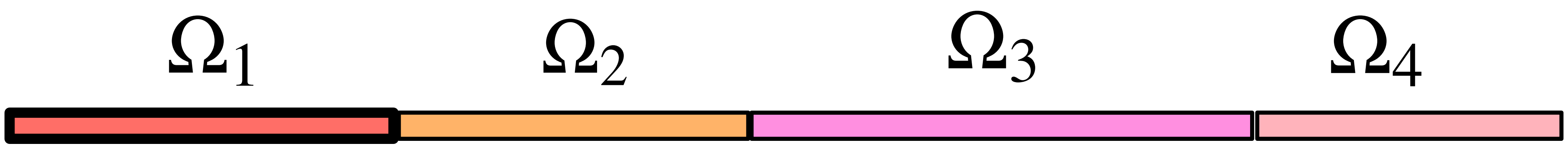

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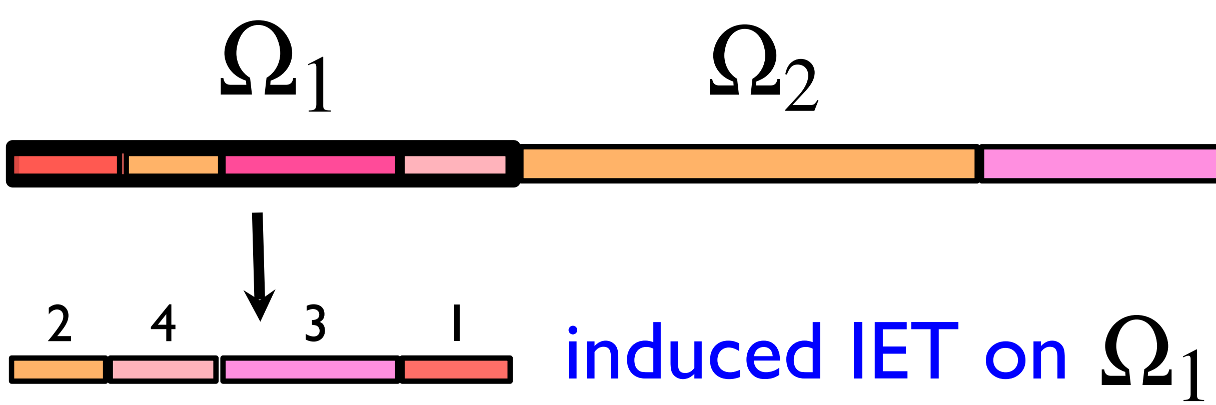

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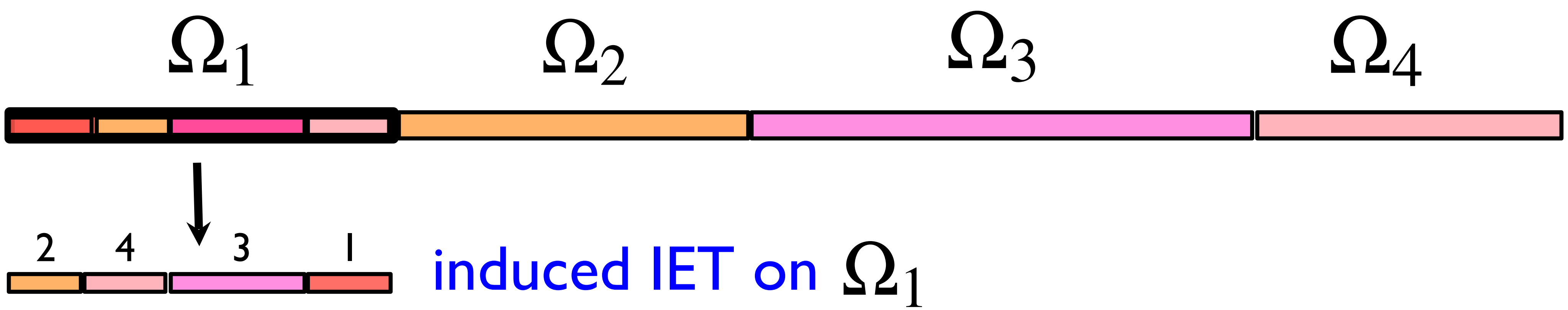

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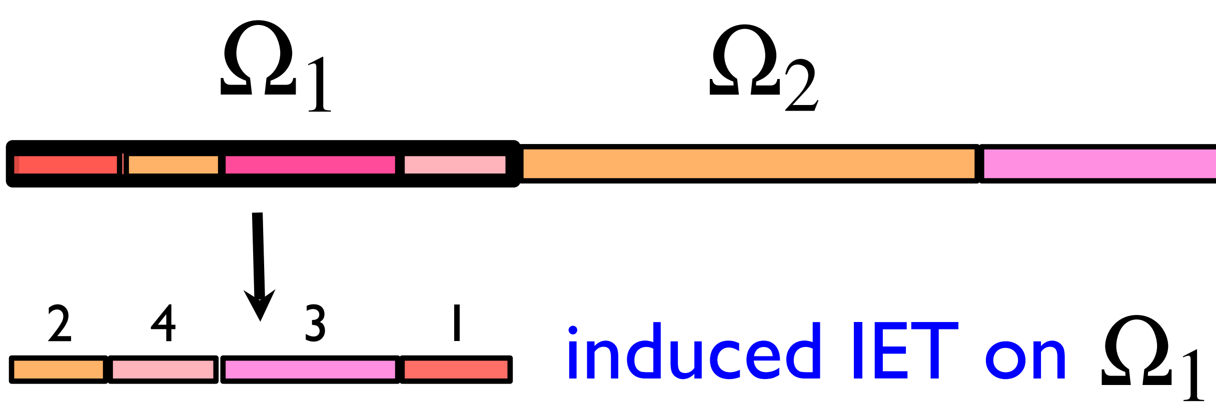

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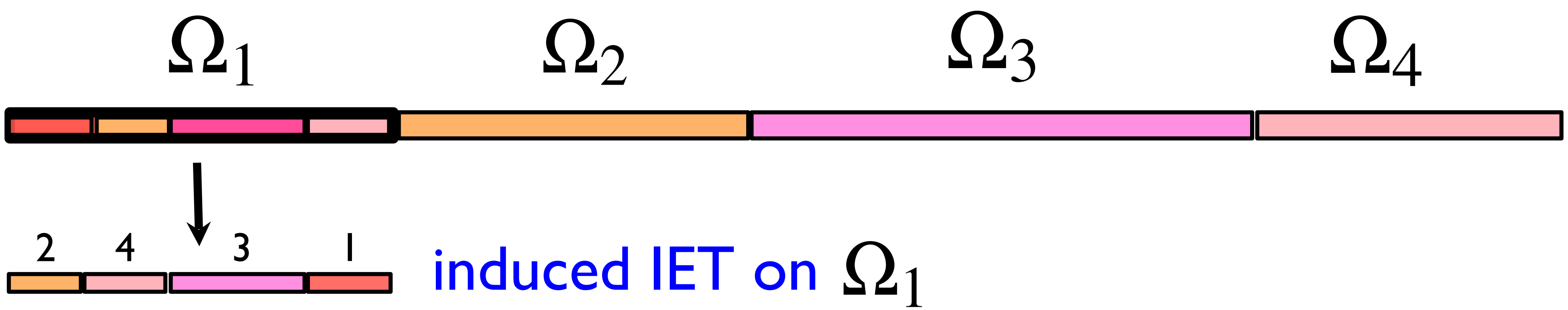

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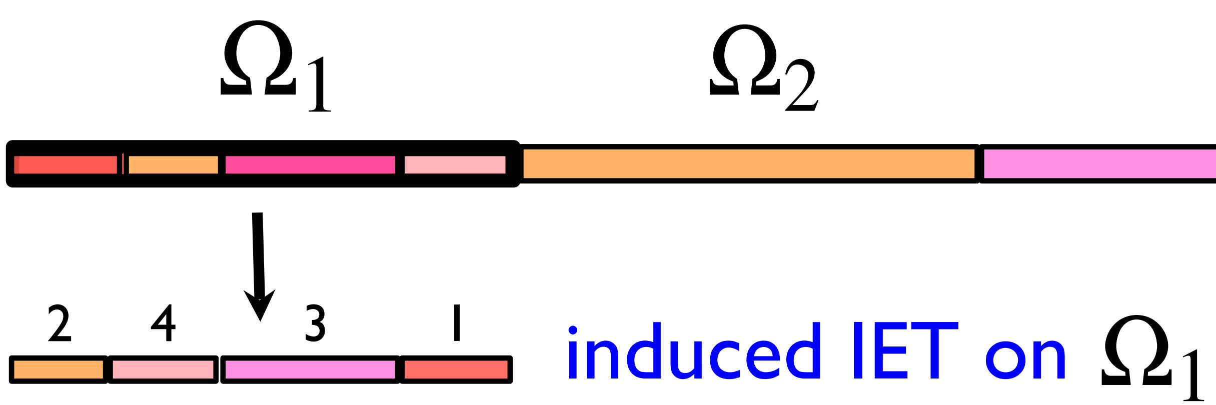

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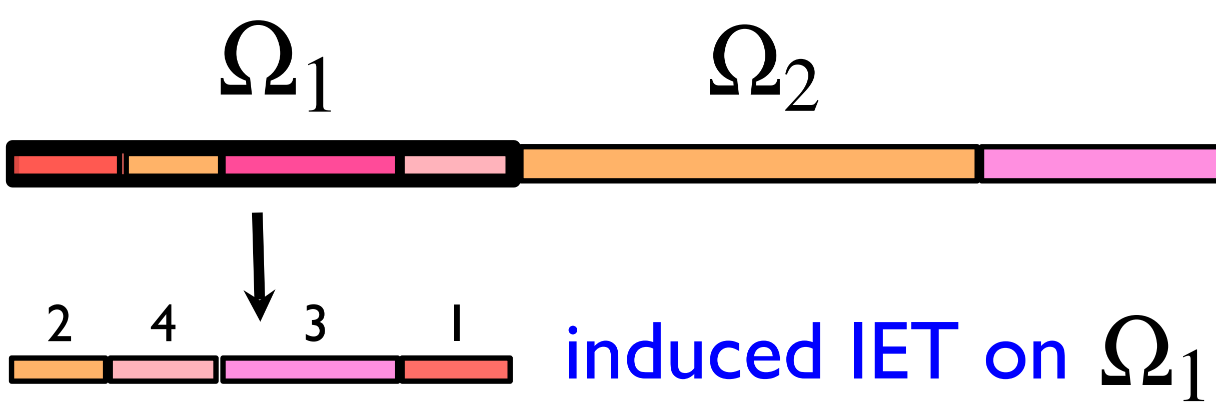

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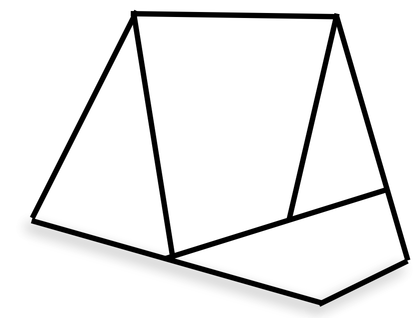
- With two parameters, we only find a degenerate form of renormalizability (one parameter is free).

Higher dimensions: topology



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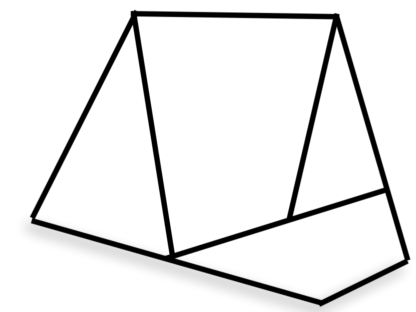

Iterate the boundary of the atoms: $\partial\Omega = \bigcup \partial\Omega_i$

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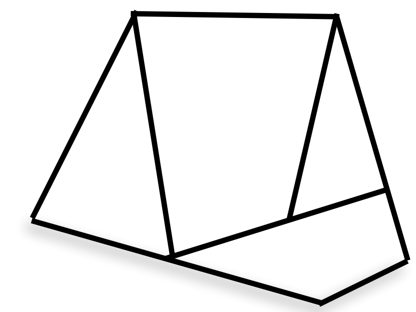

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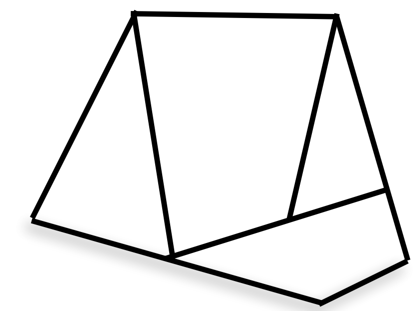

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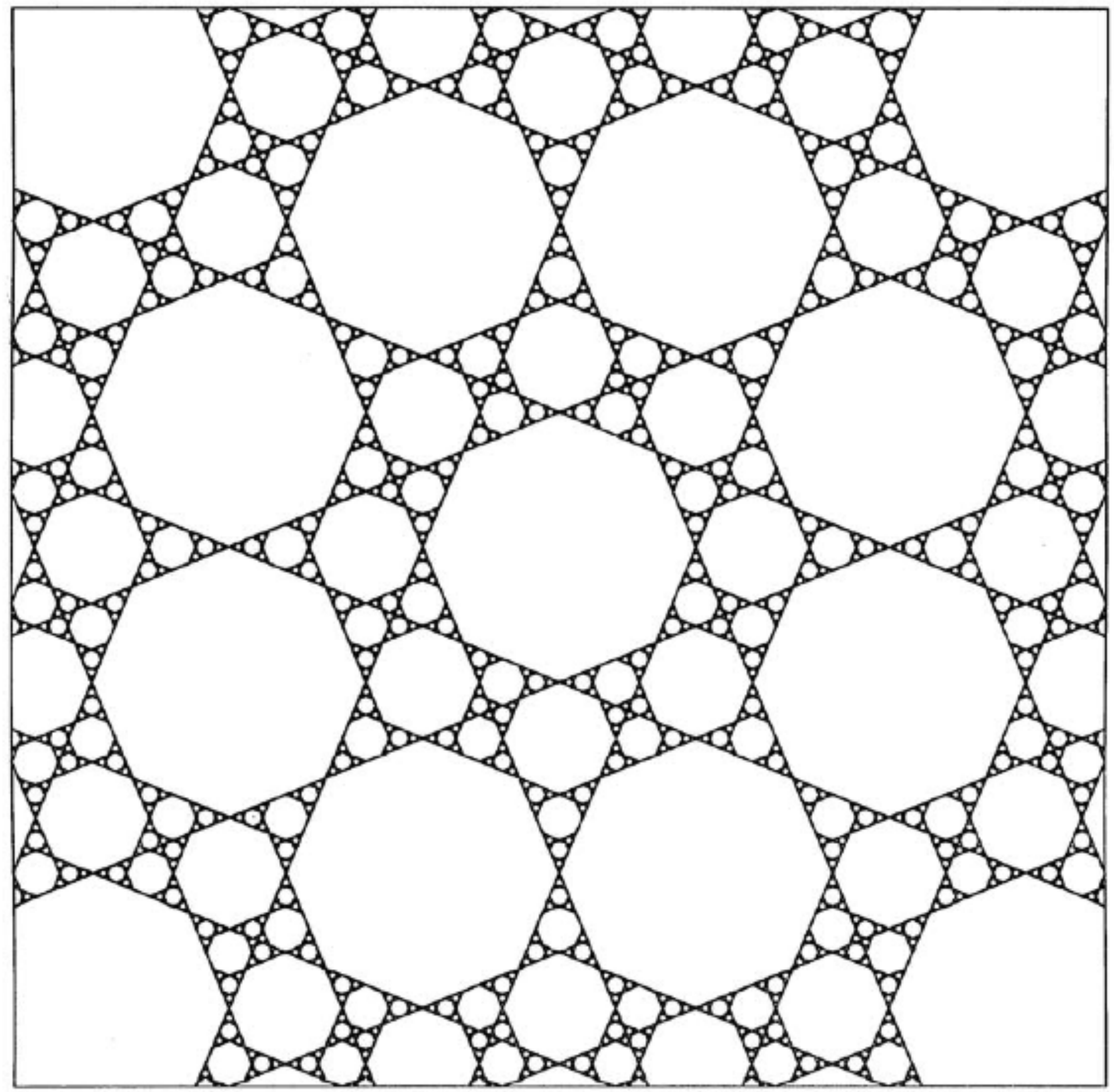

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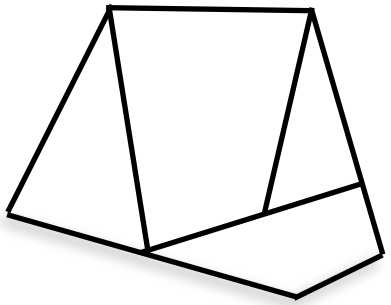

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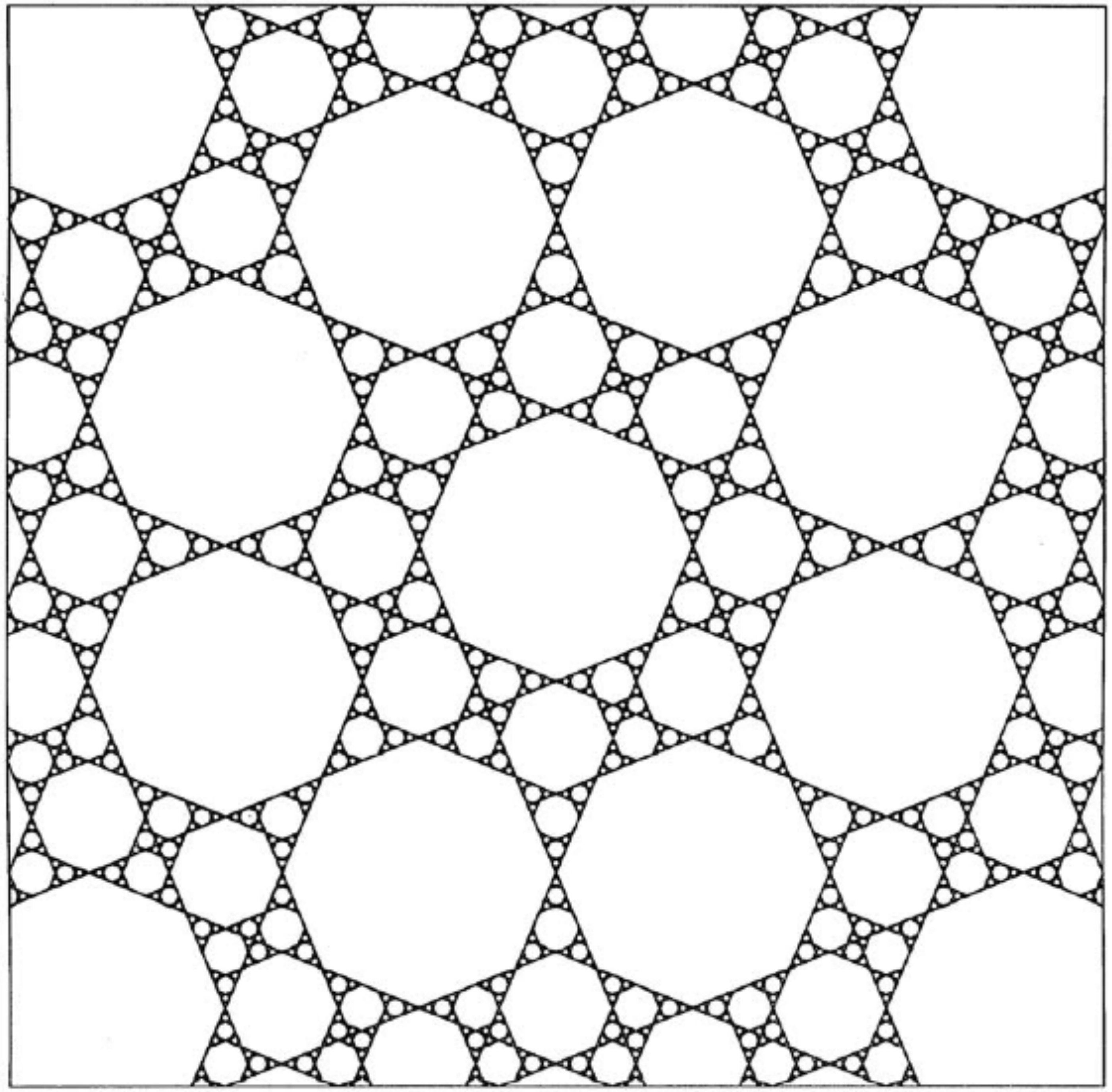

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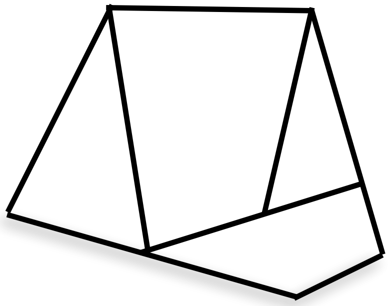

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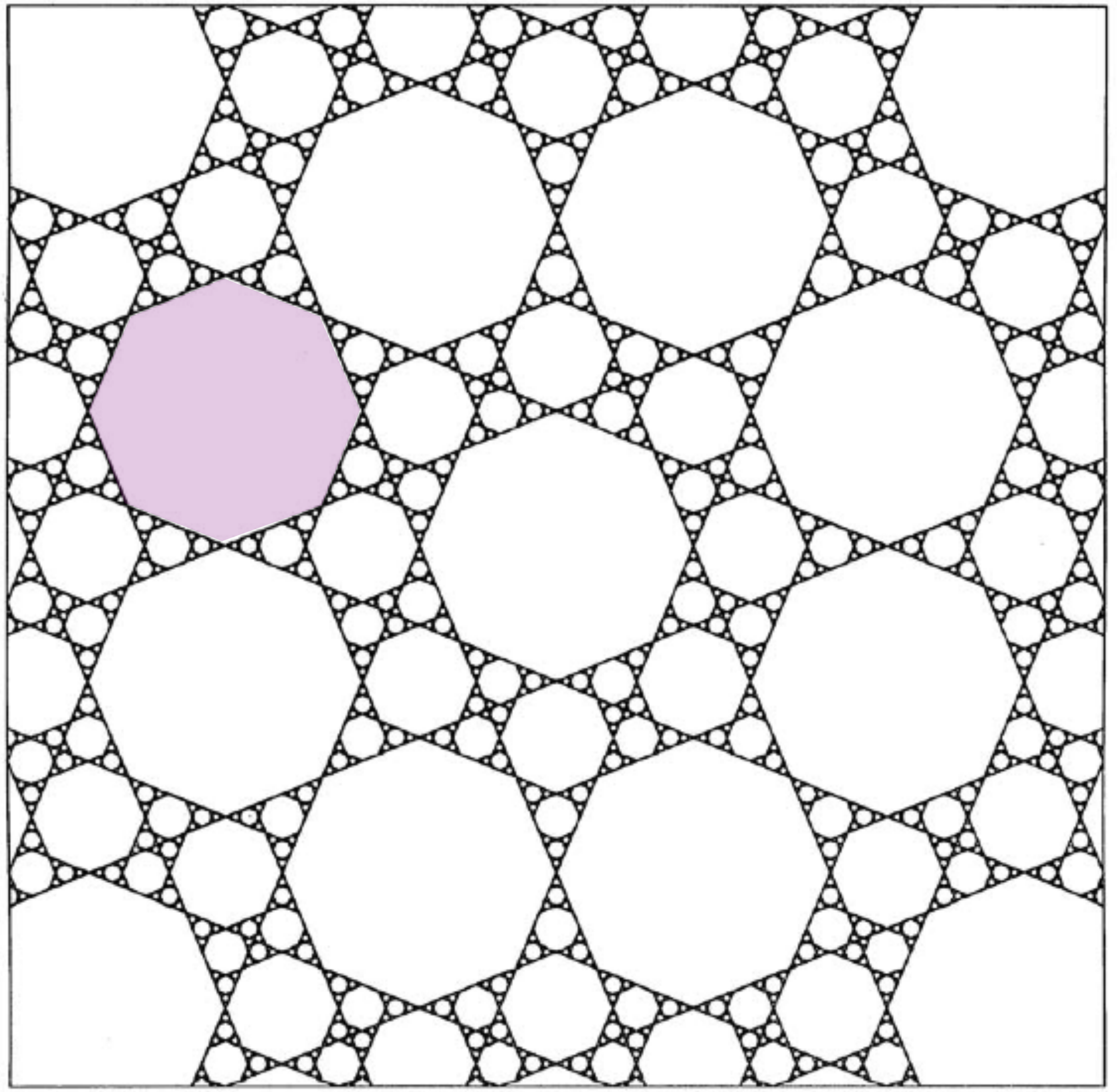

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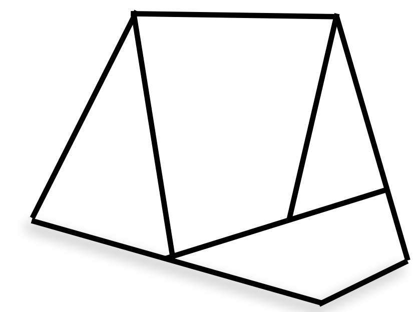

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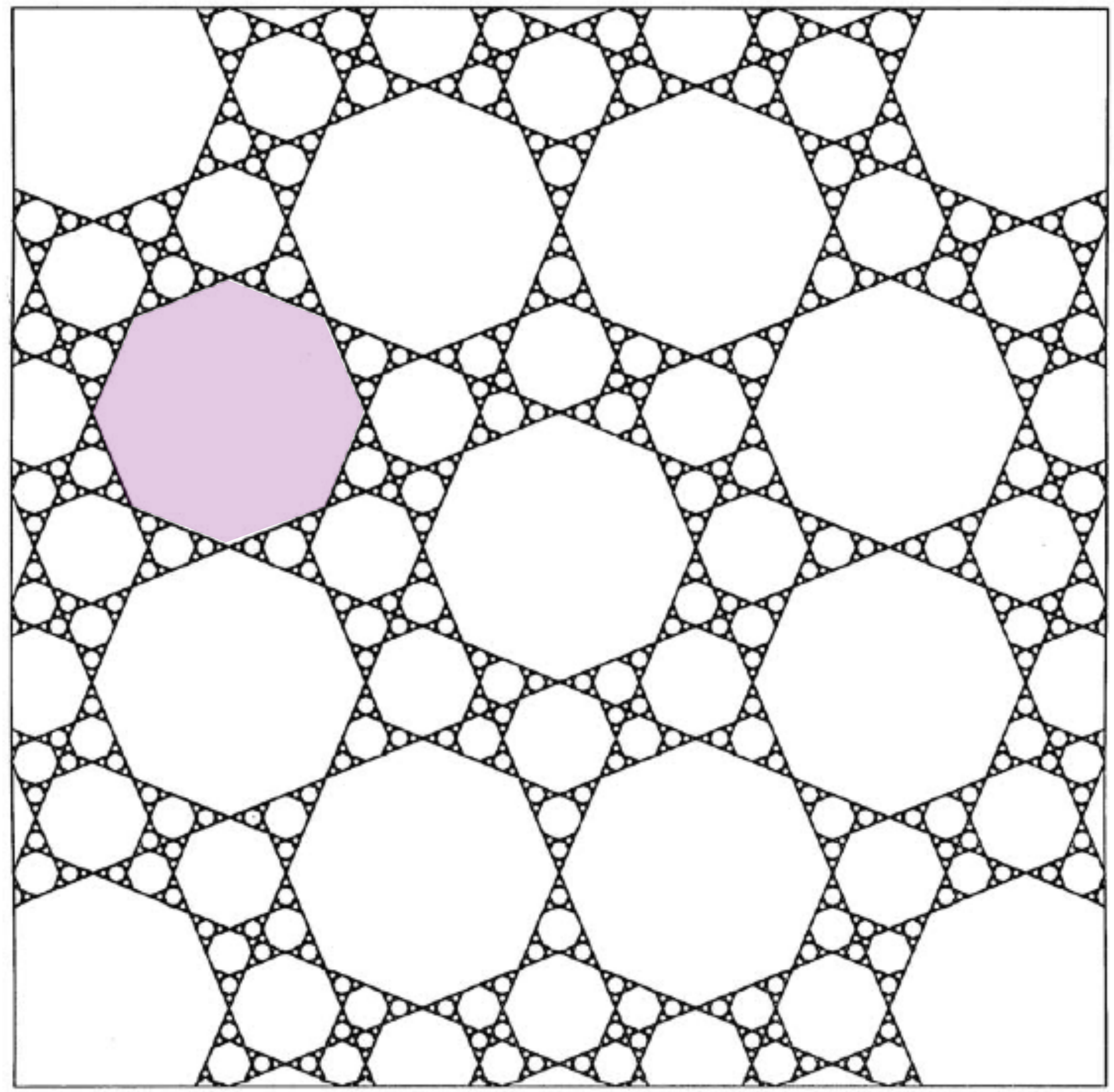

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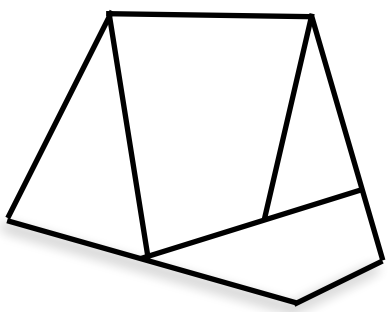

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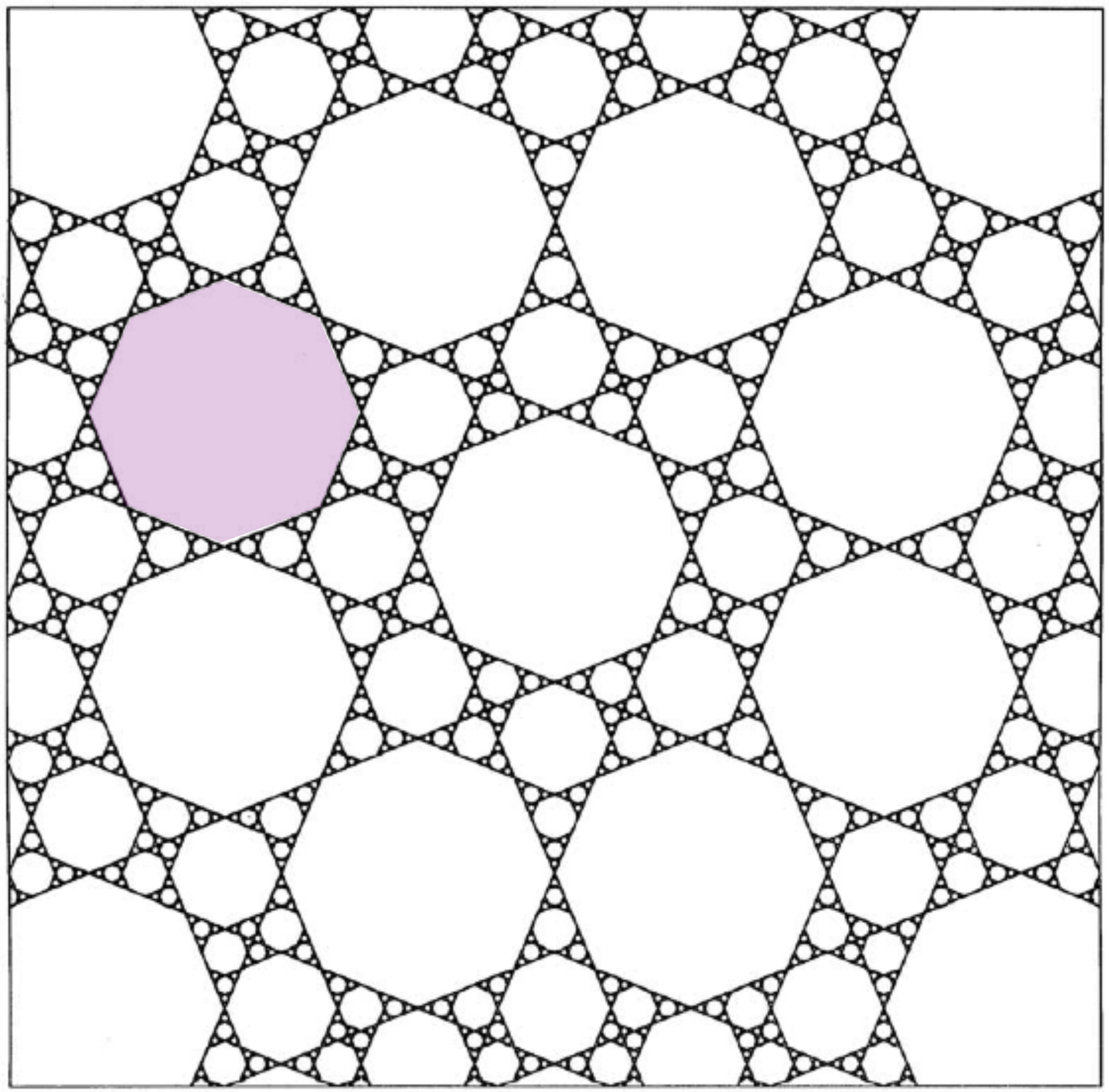

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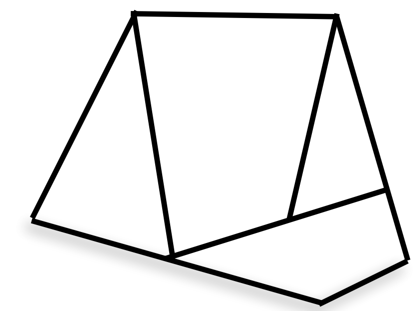

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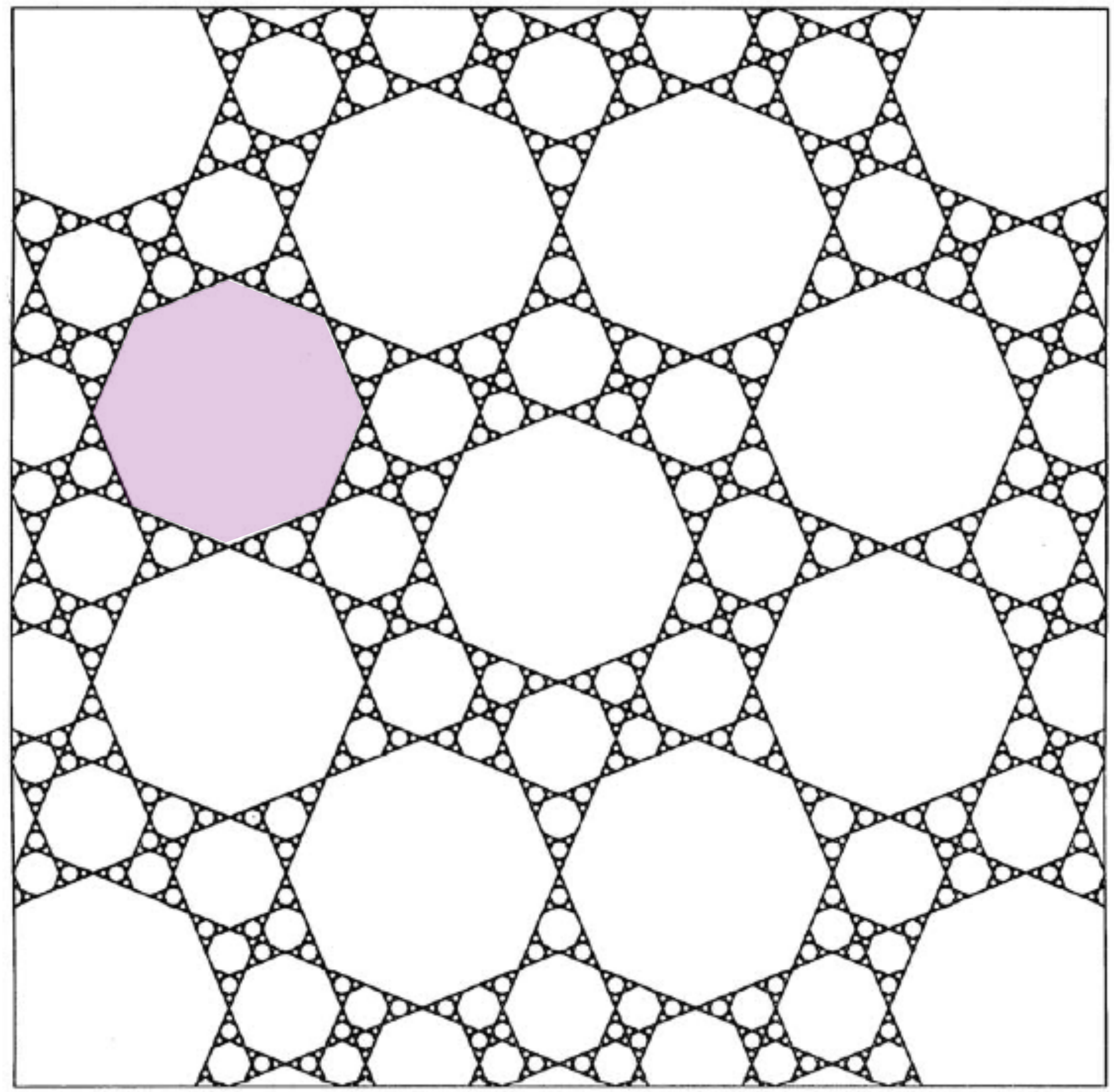

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$q = 5,8,10,12$: $\mathbb{Q}(\lambda)$ quadratic

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Simplest case: <u>rational</u> rotation number.

$$\lambda = 2\cos(2\pi p/q)$$

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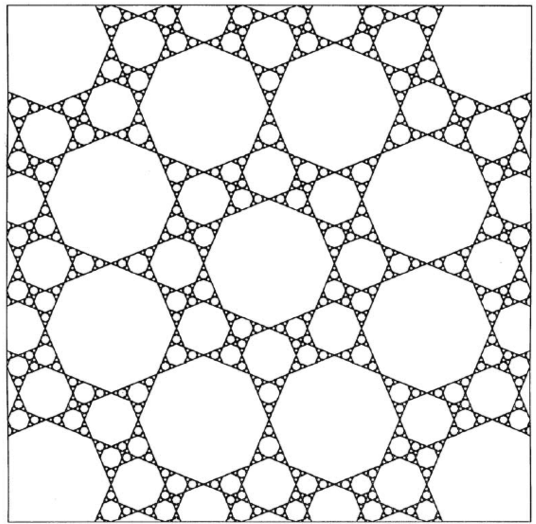
No general theory, but many examples of quadratic PWIs, where:



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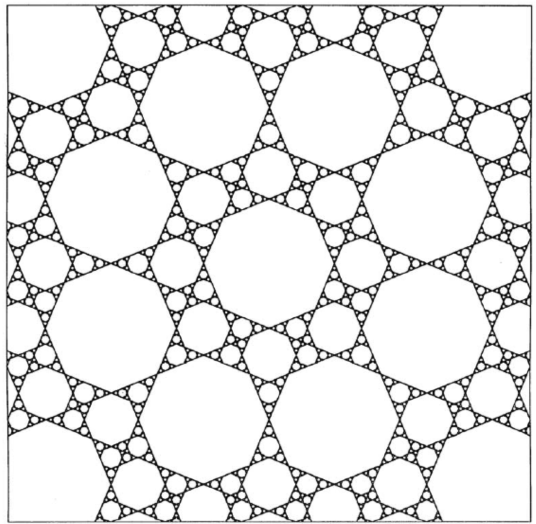

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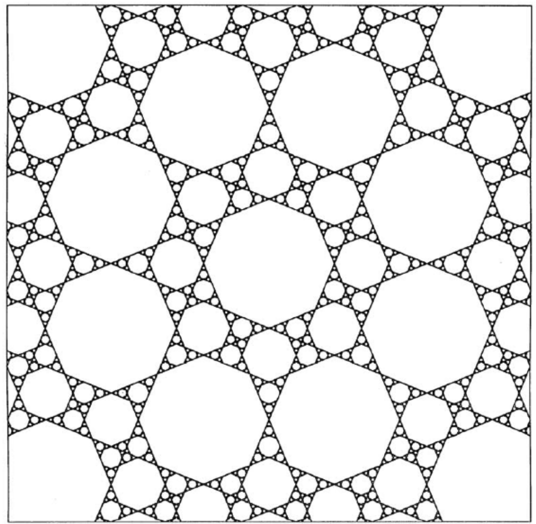

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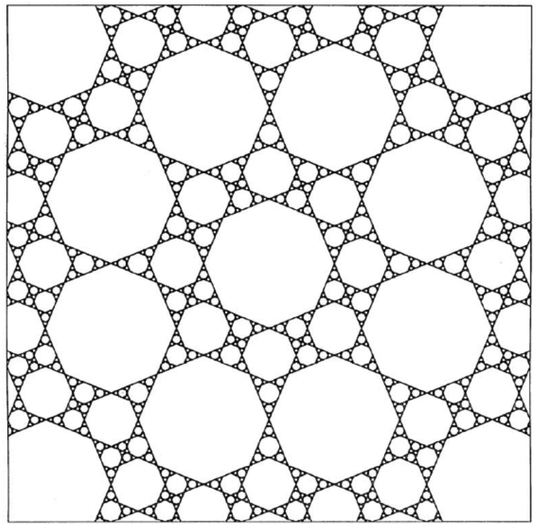
- scaling constants are units in the ring of integers of the relevant field.



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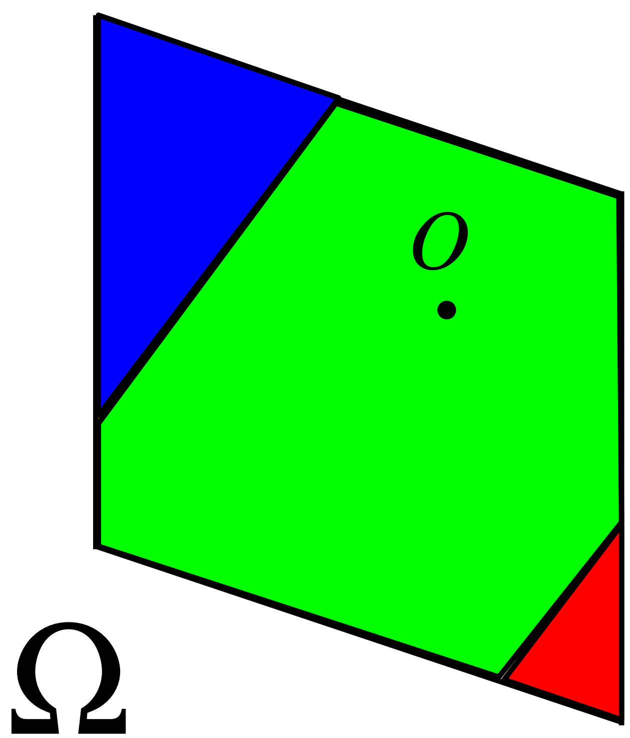

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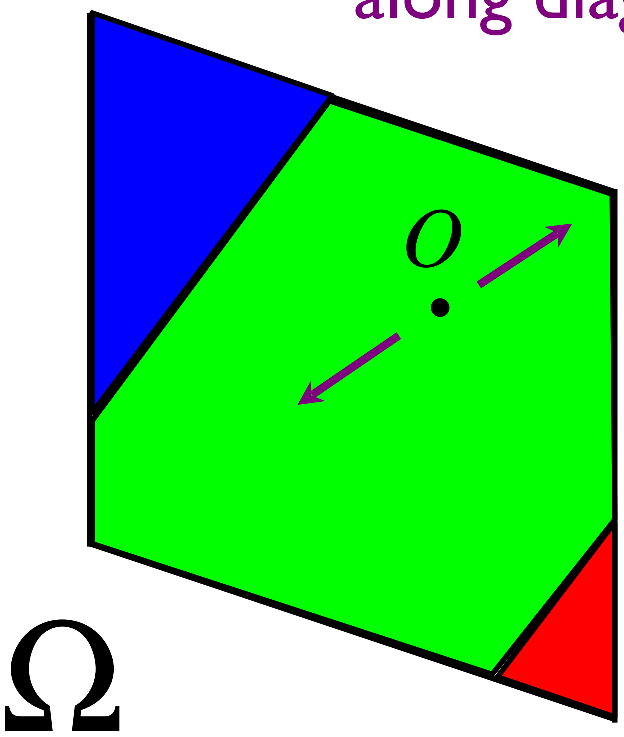

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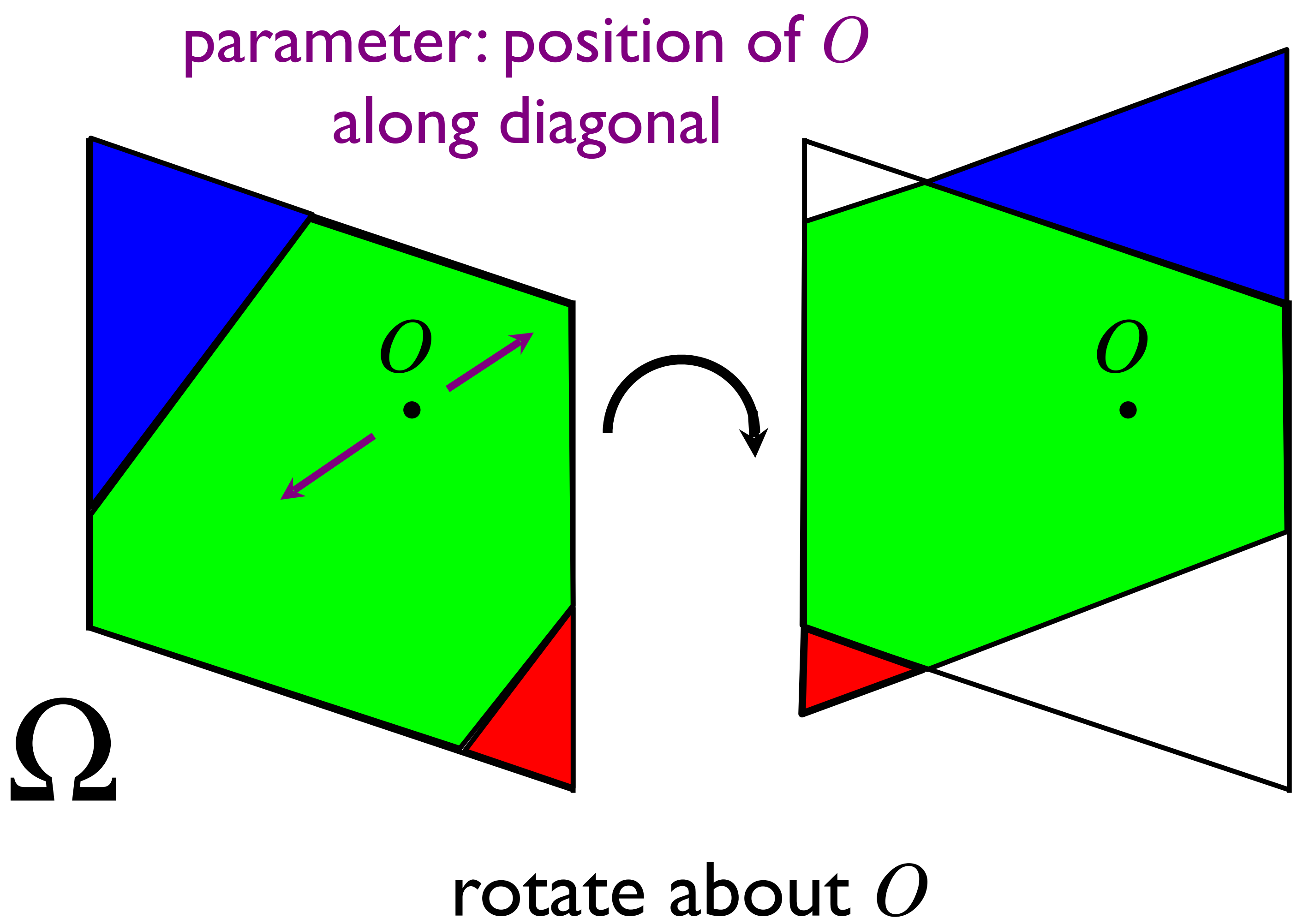

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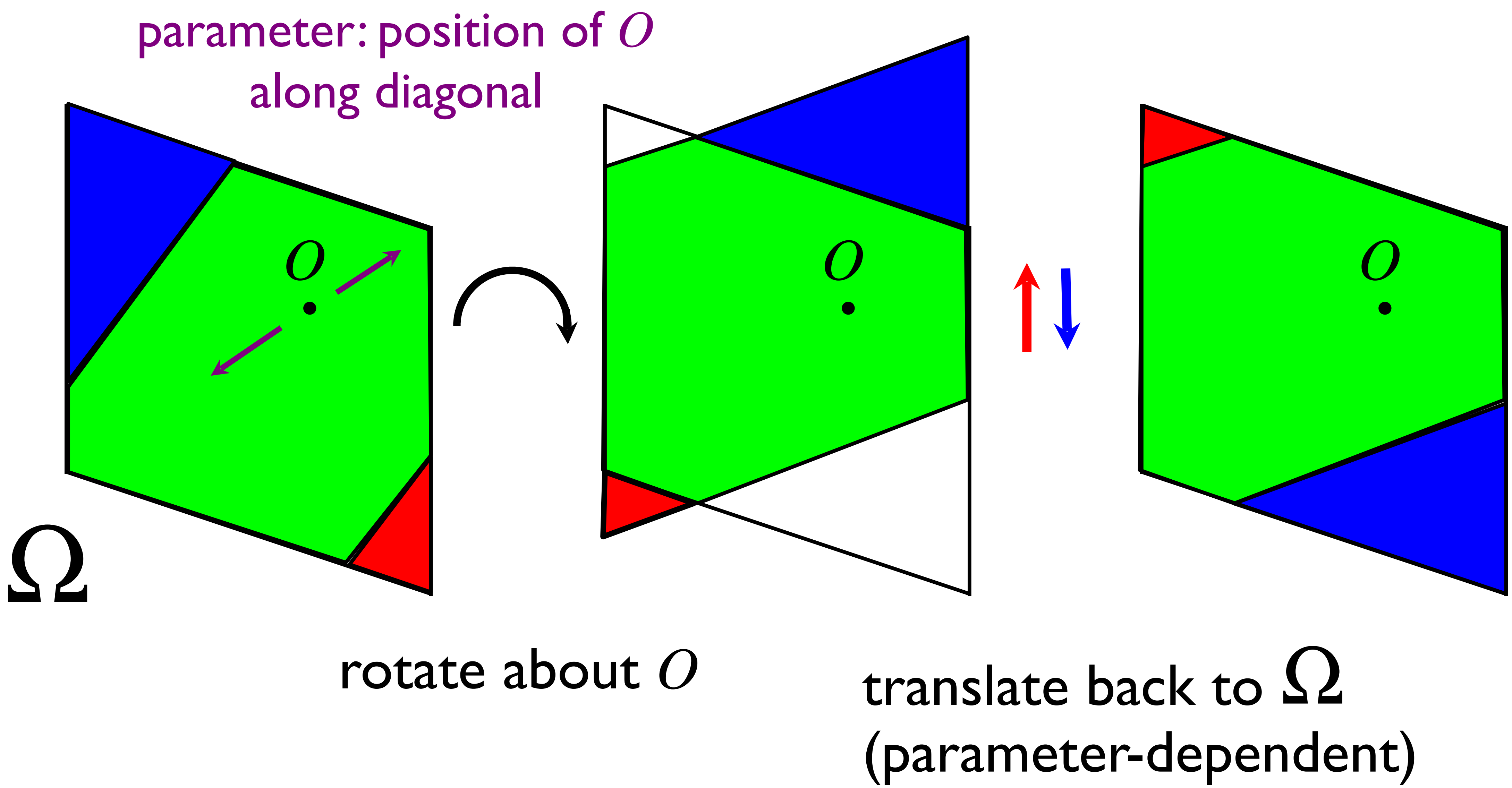

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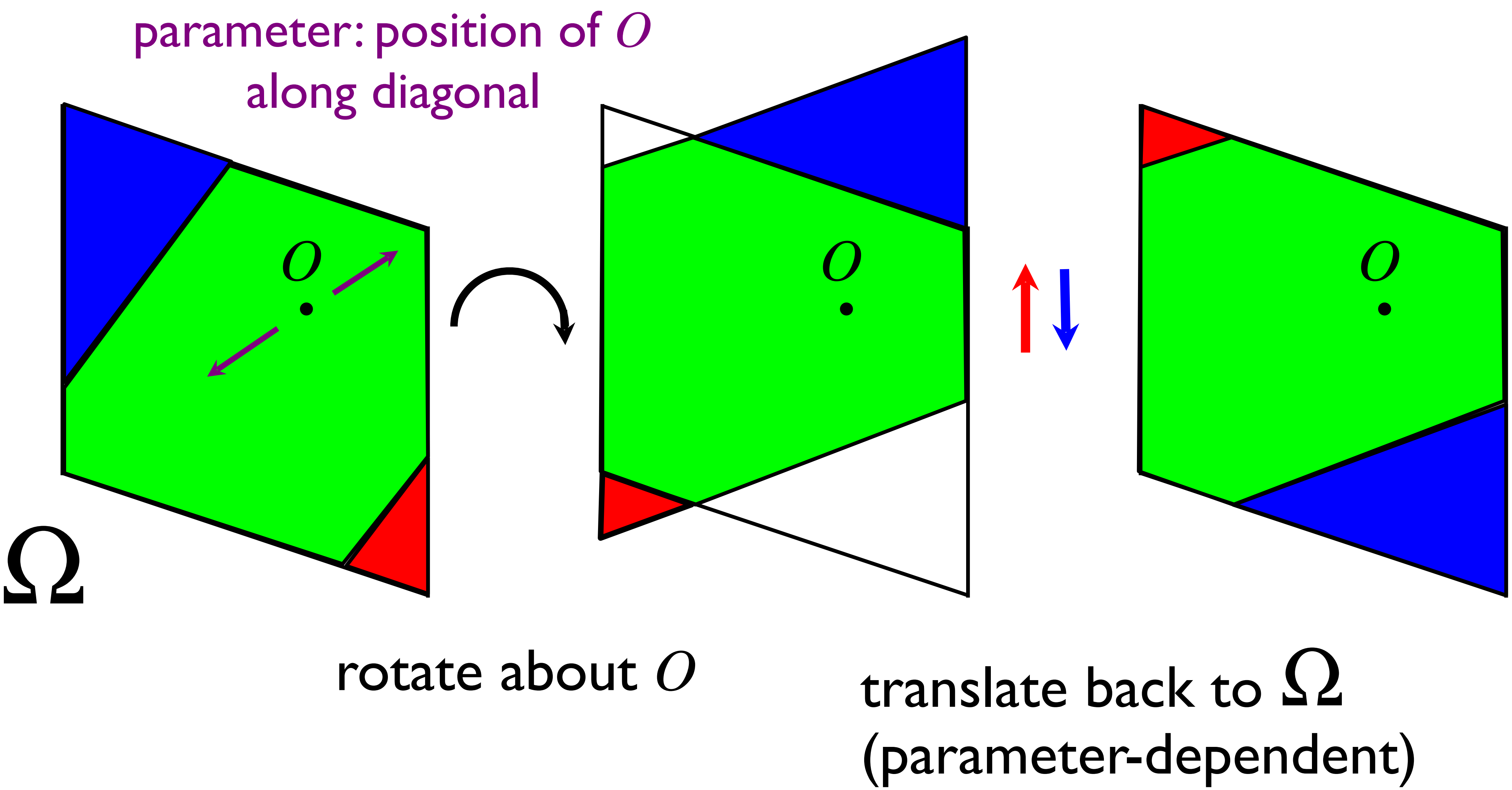

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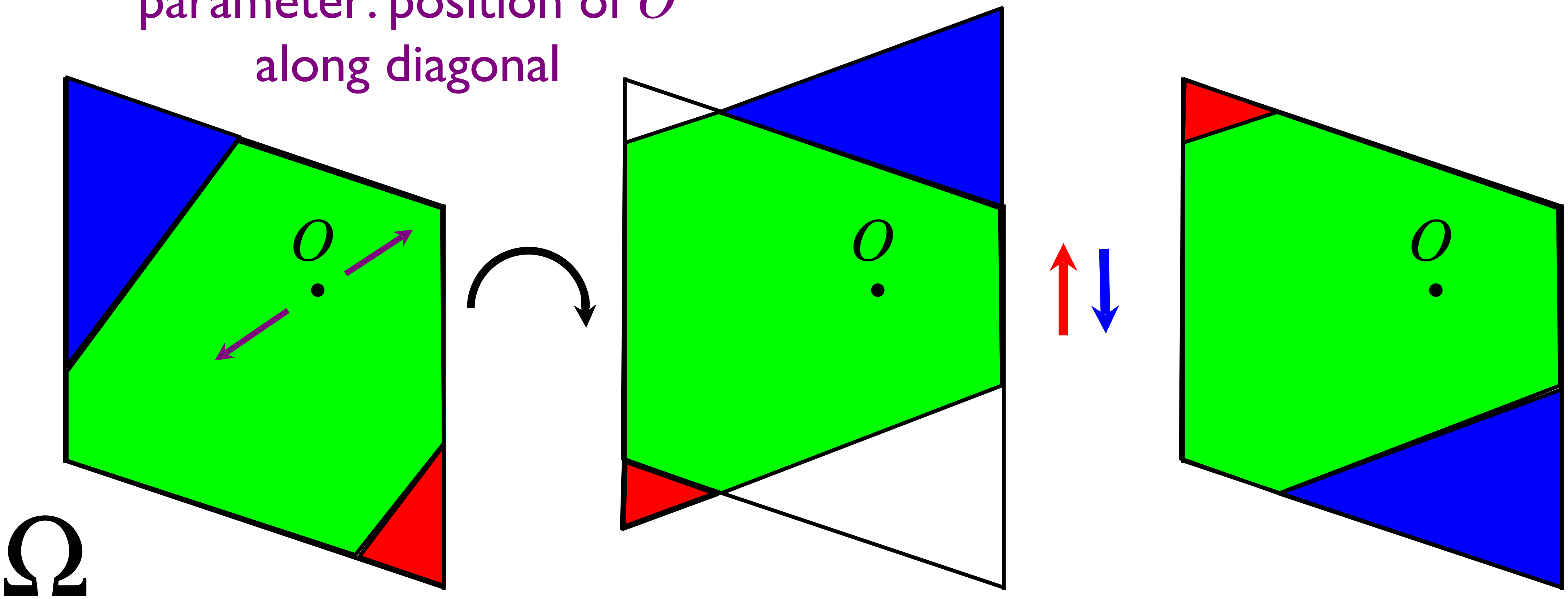


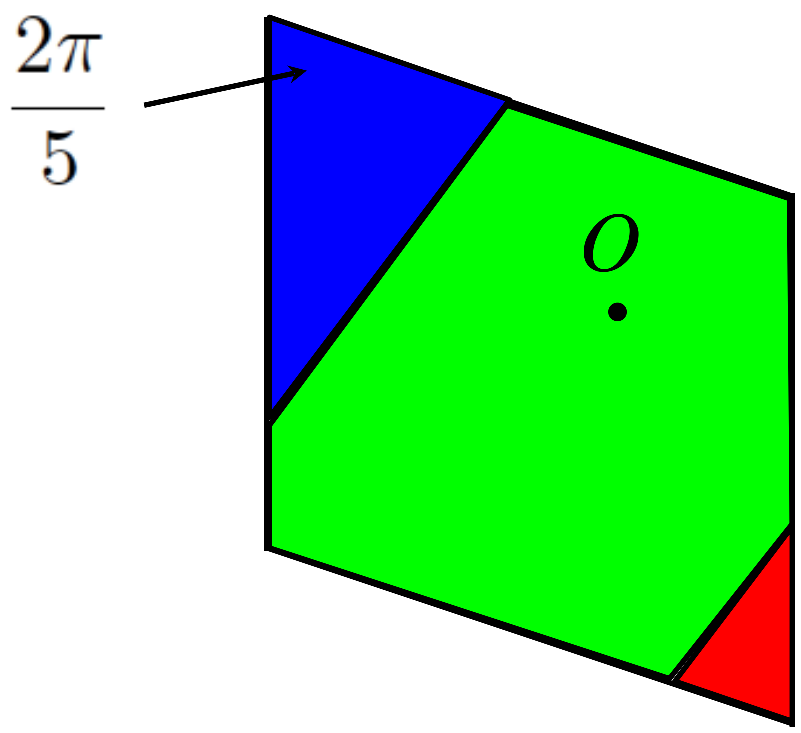
rotate about O

translate back to Ω (parameter-dependent)

quadratic rotation fields: $\mathbb{Q}(\lambda) = \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{5})$

translation module: $\mathbb{Q}(\lambda) + s\,\mathbb{Q}(\lambda)$ s: parameter

The pentagonal model (field $\mathbb{Q}(\sqrt{5})$)



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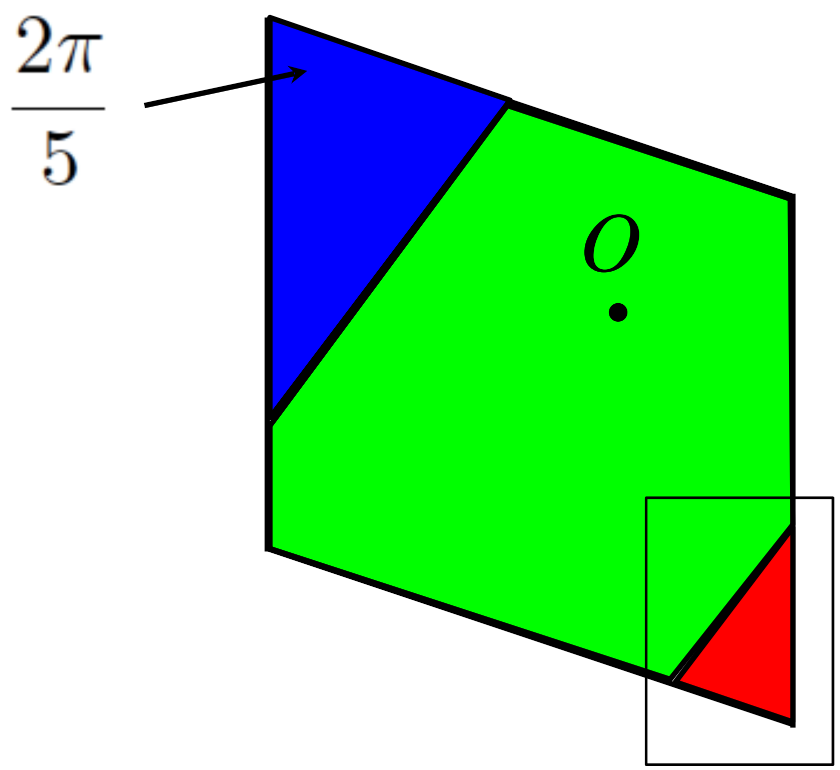

induction sequence

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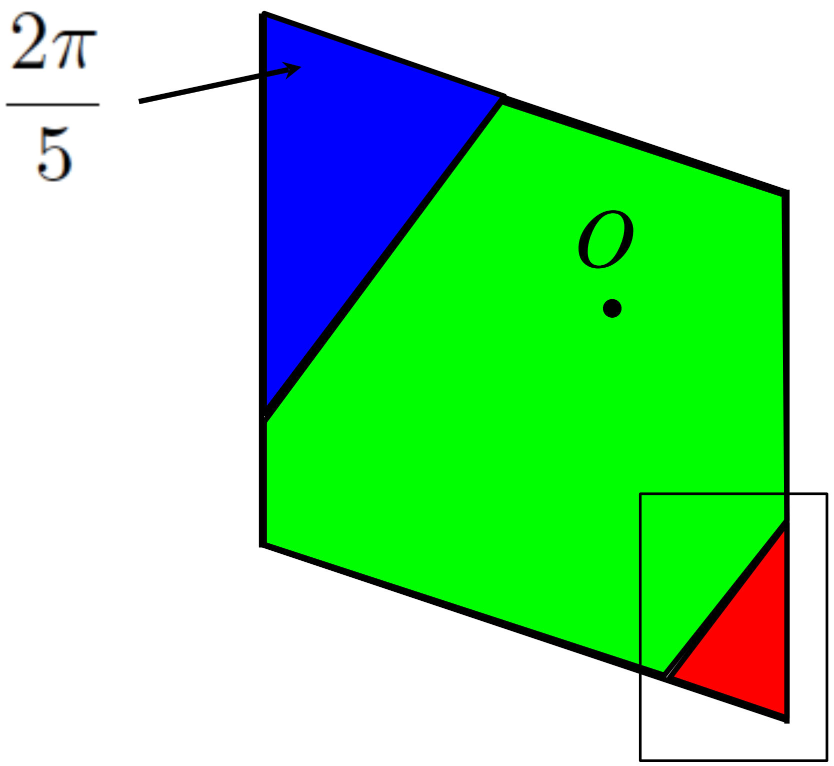

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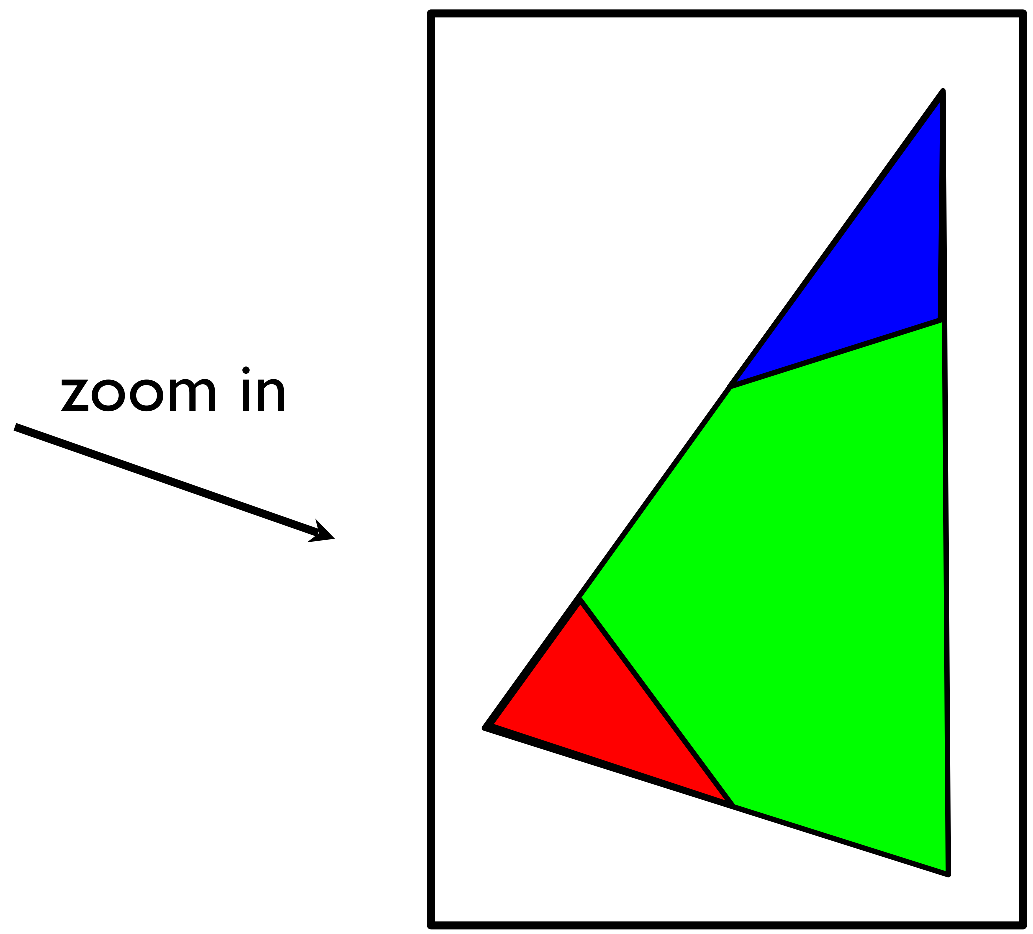

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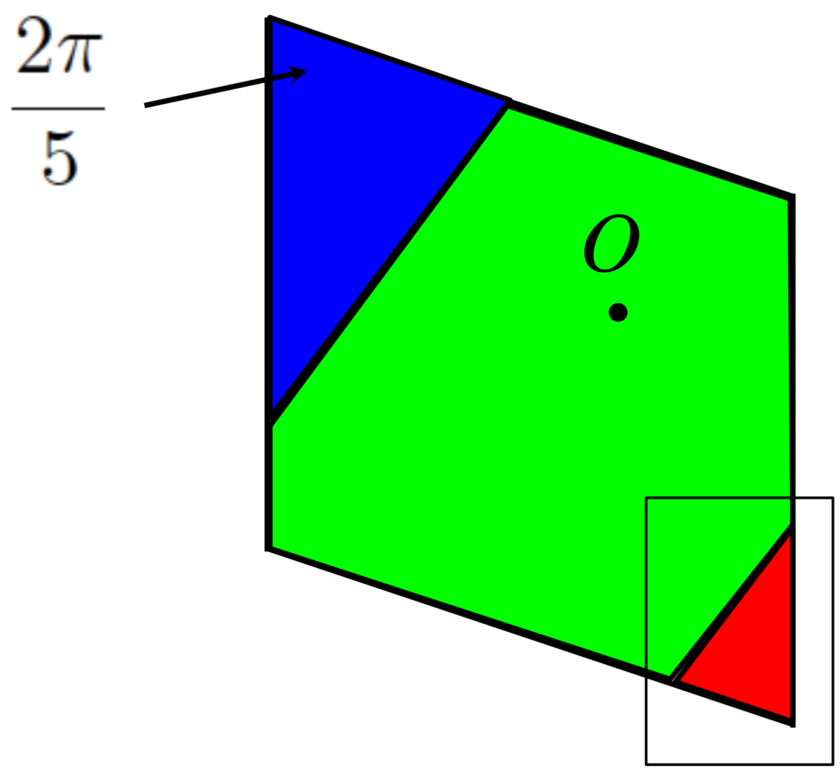

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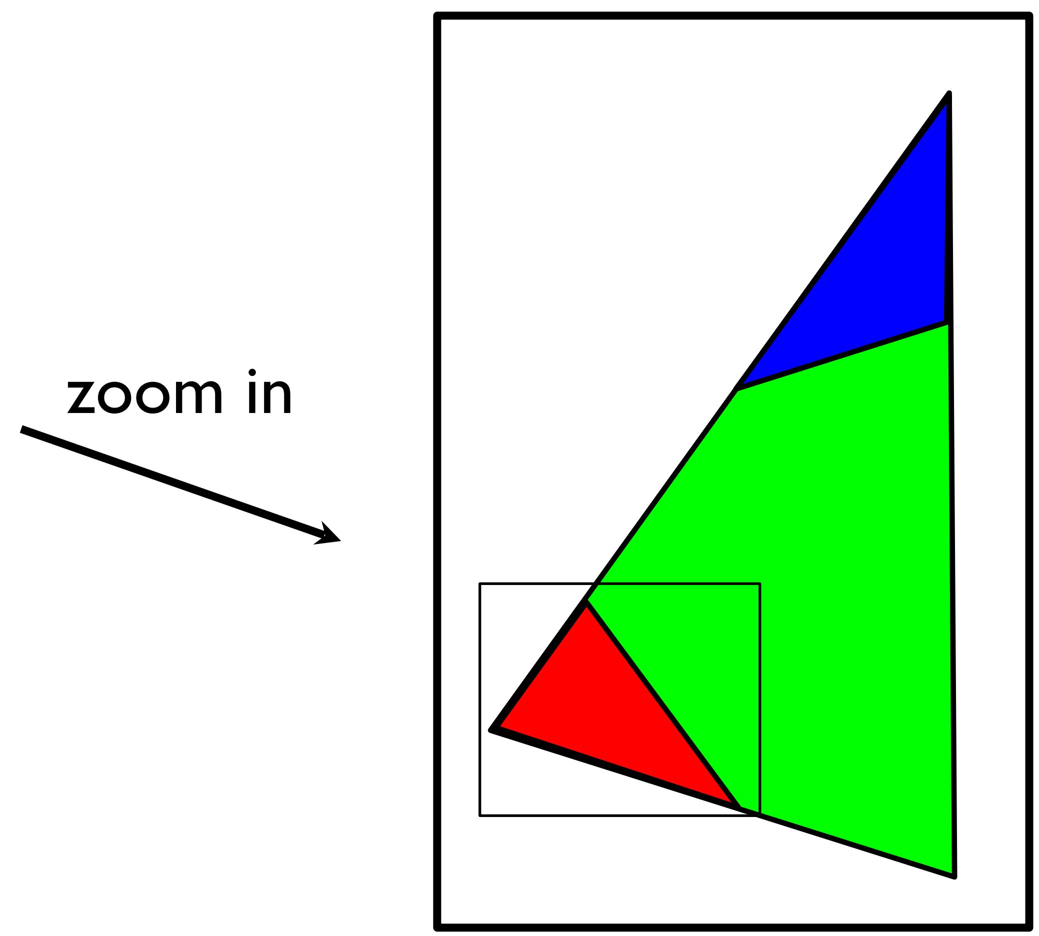

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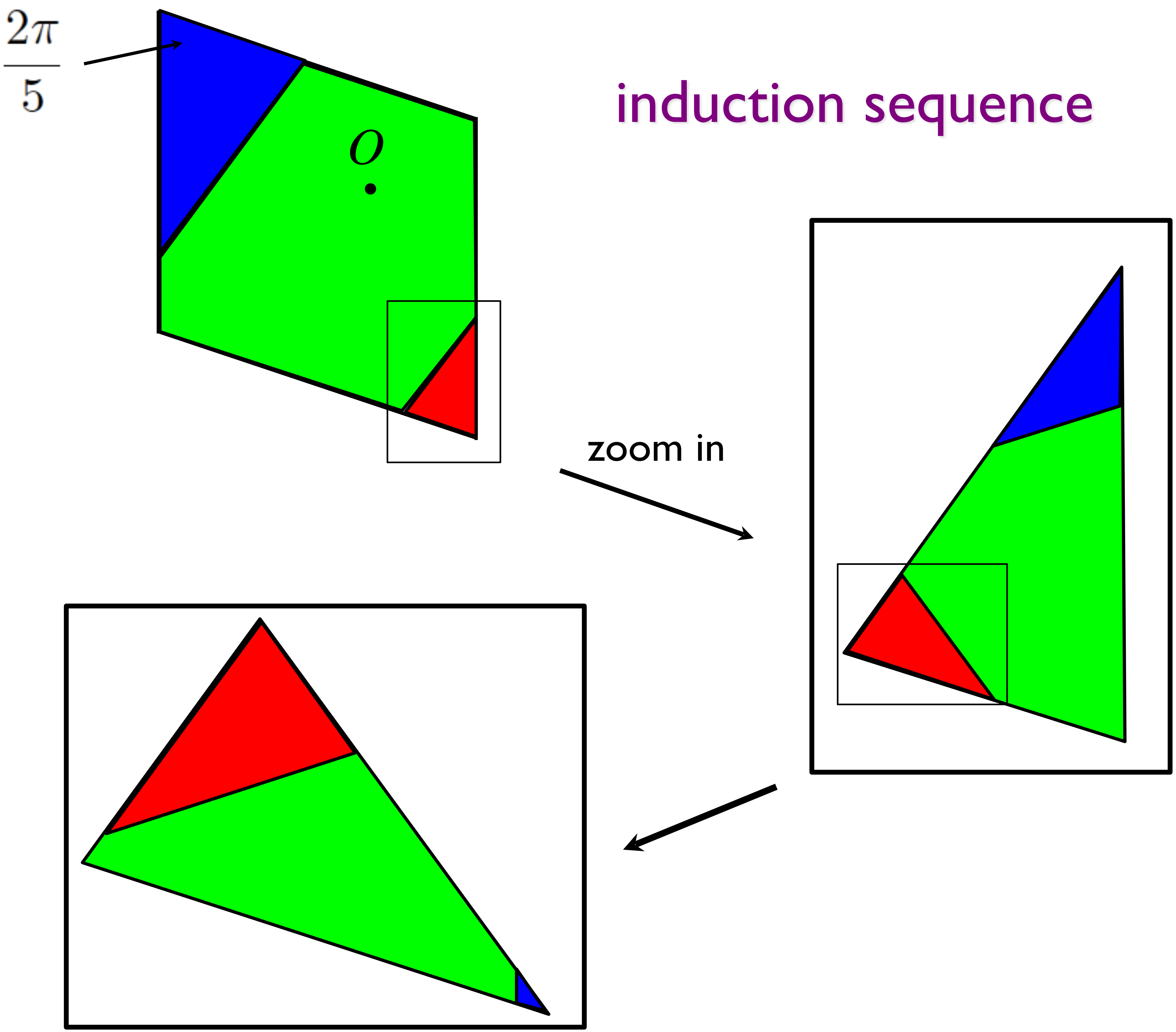

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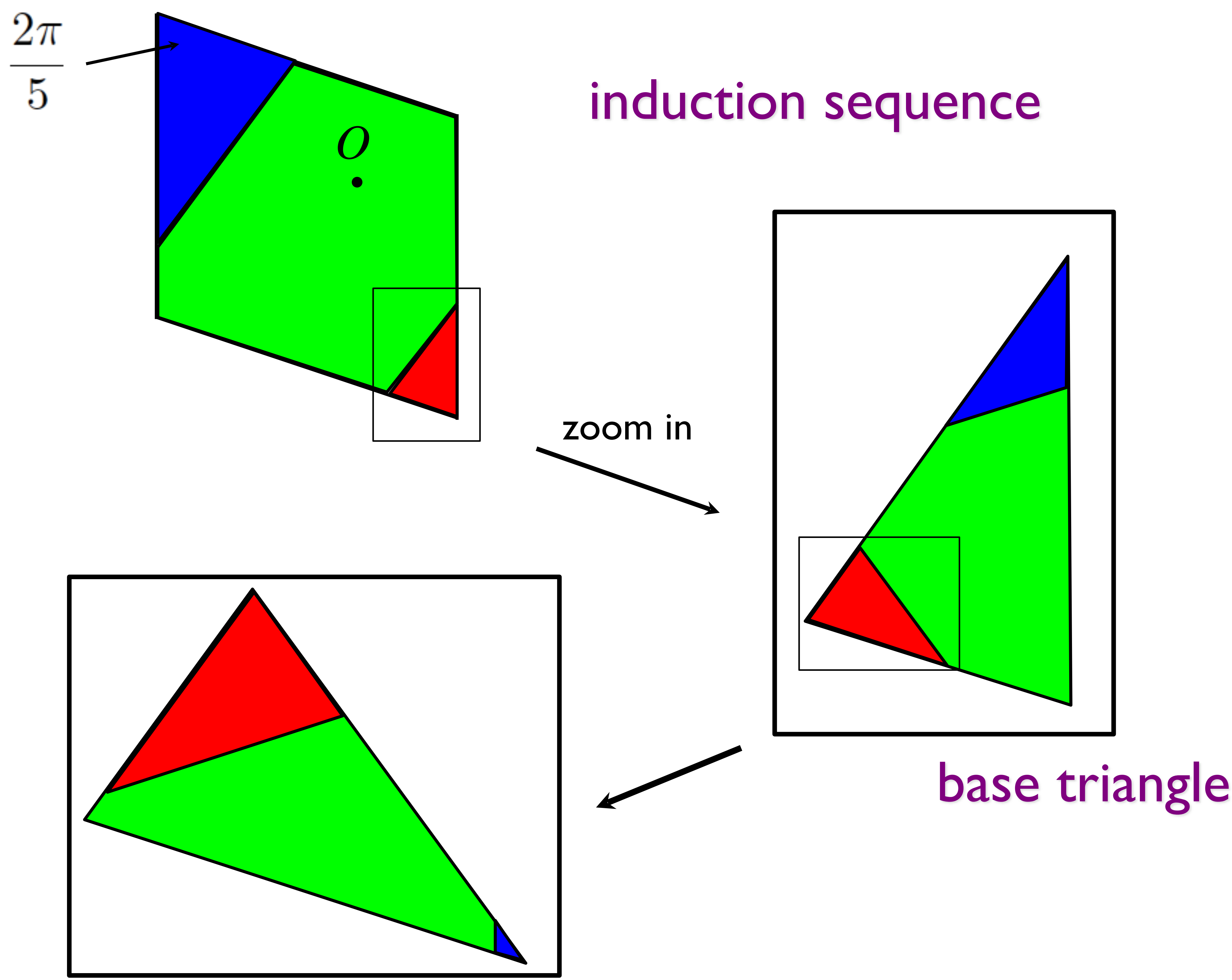
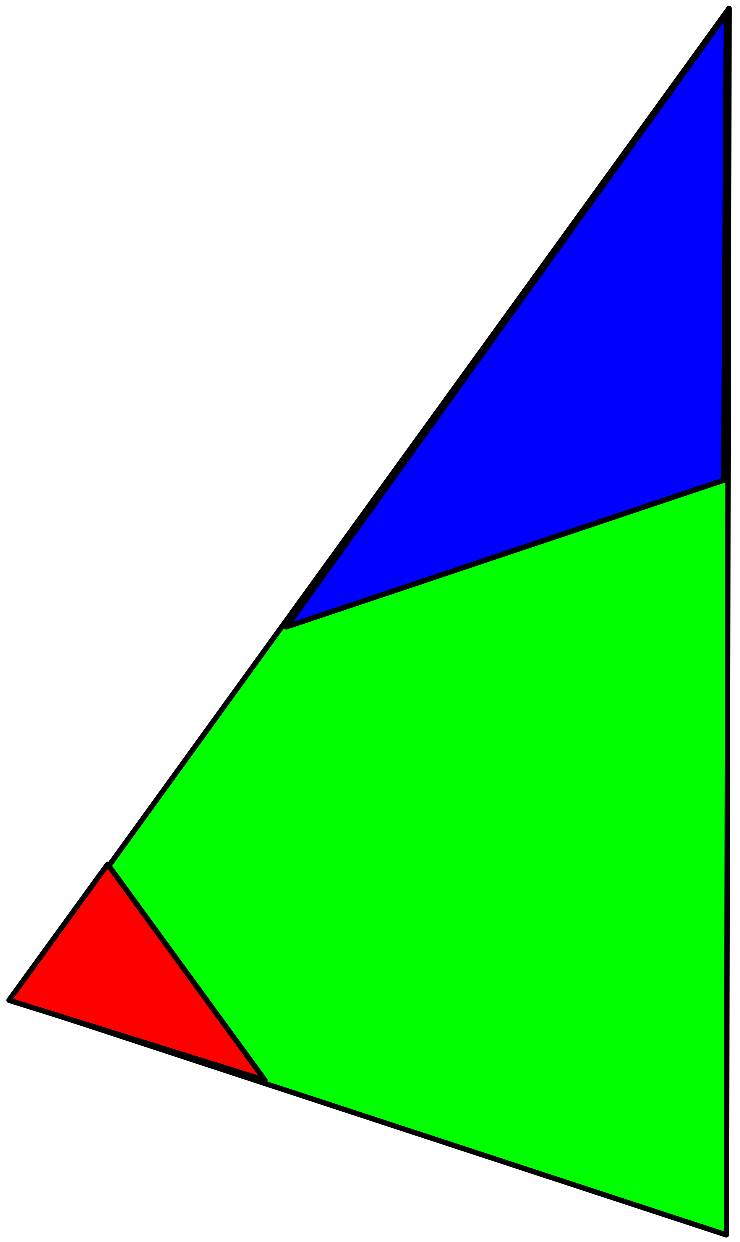


Base triangle: time-reversal symmetry



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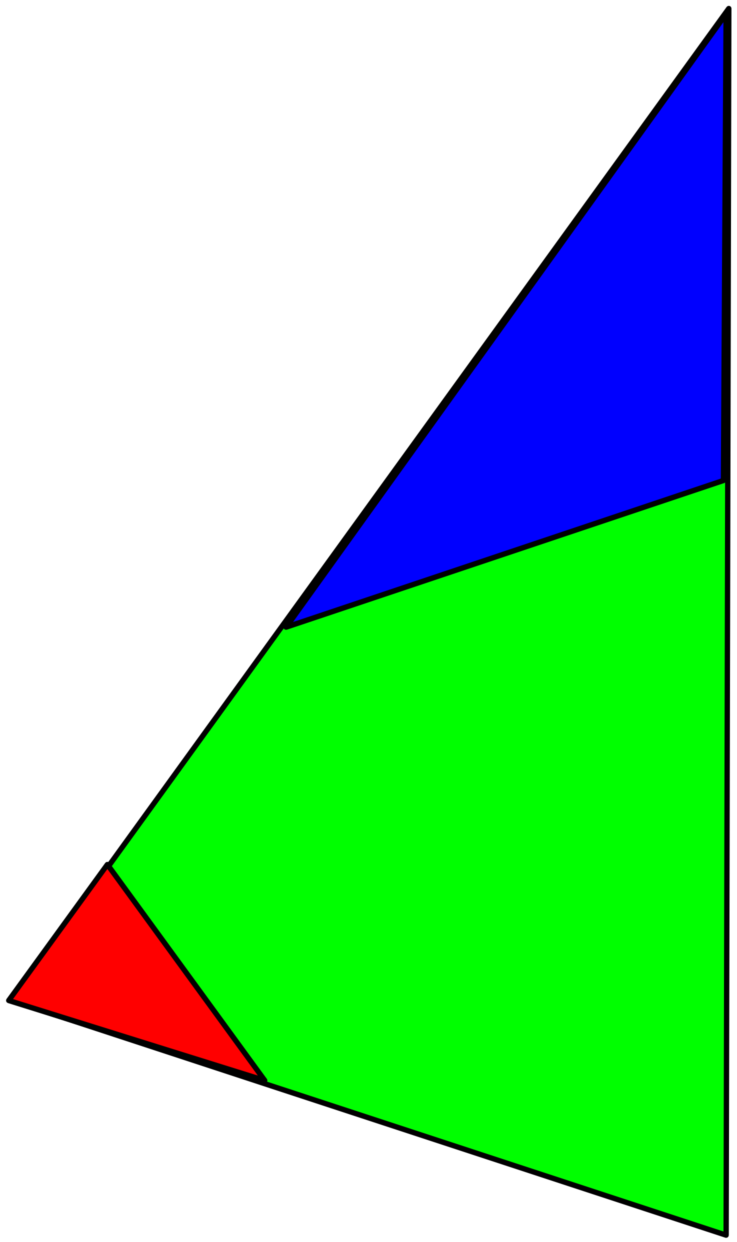
The map is the composition of two involutions: $F = GH$



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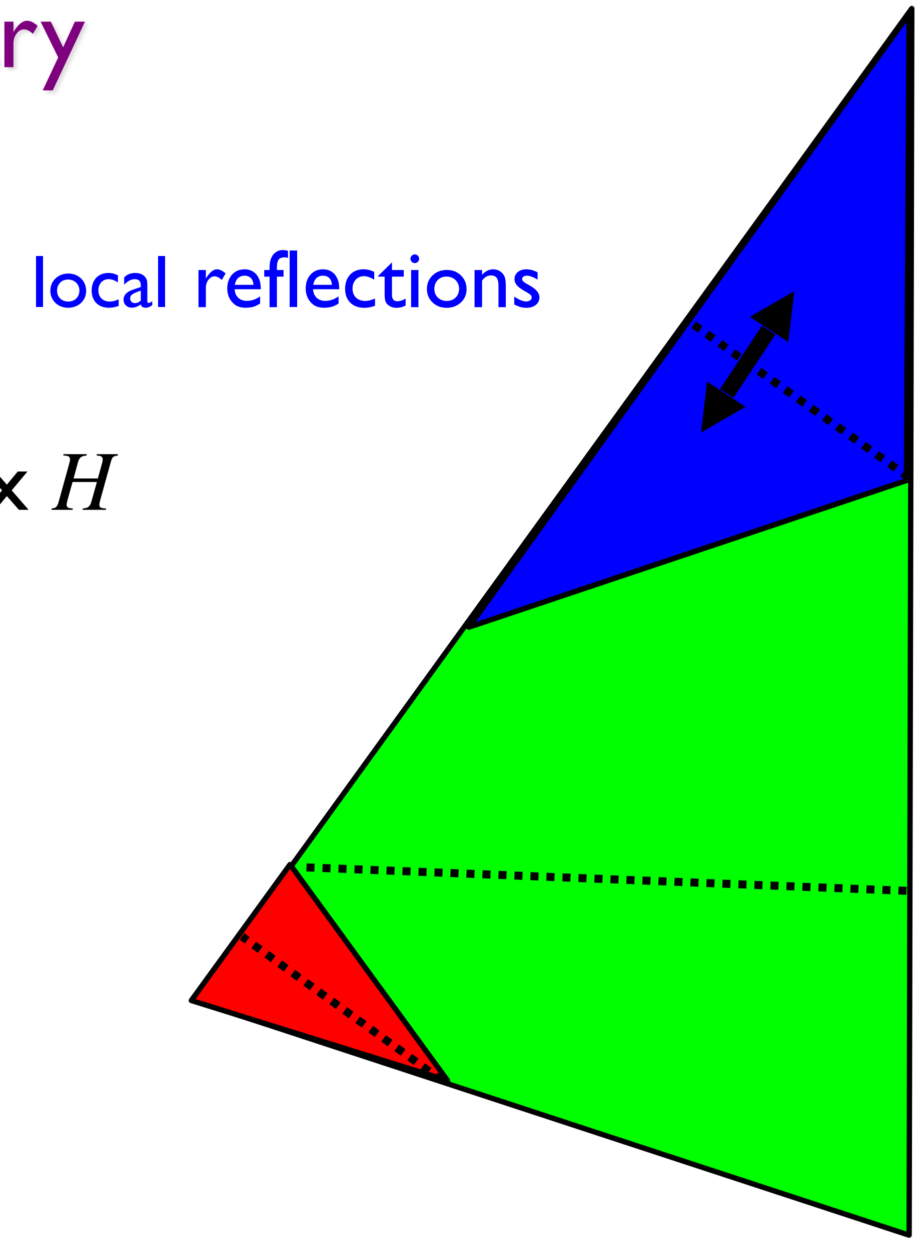

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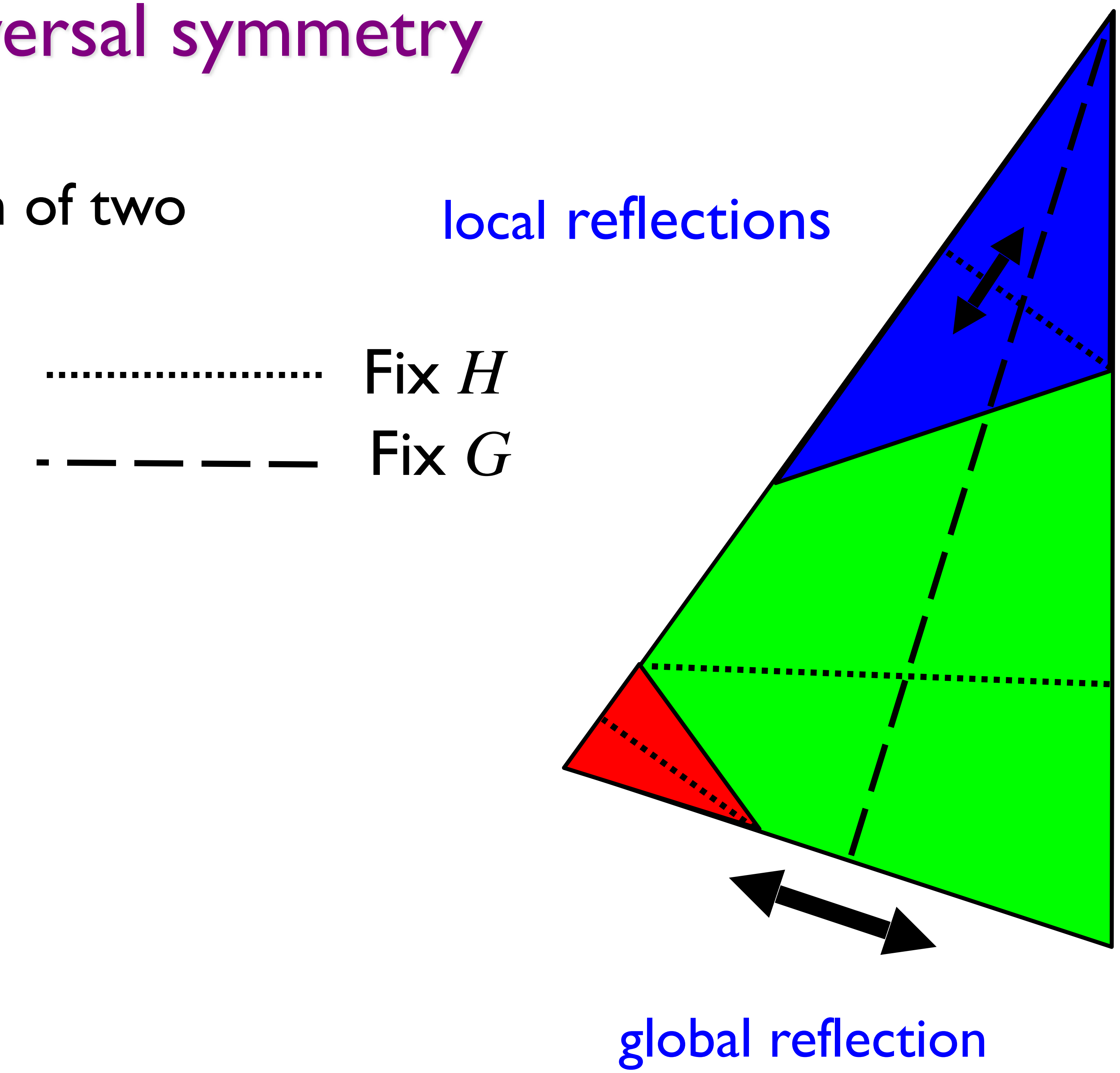

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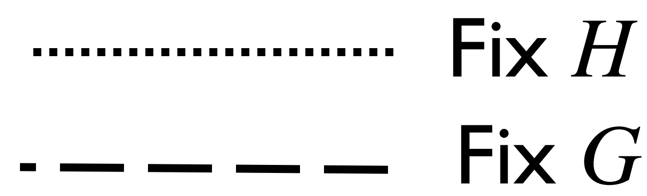
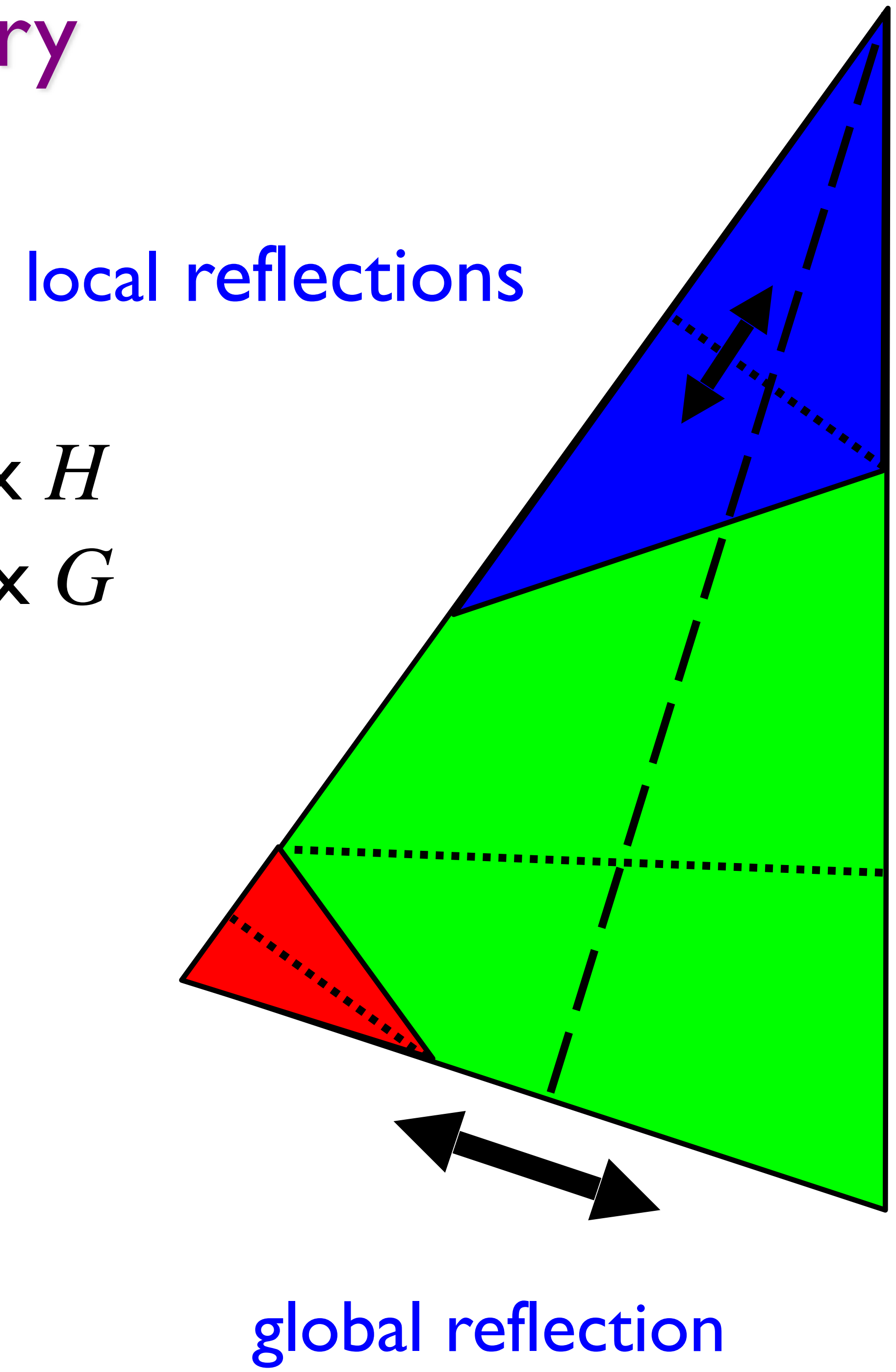
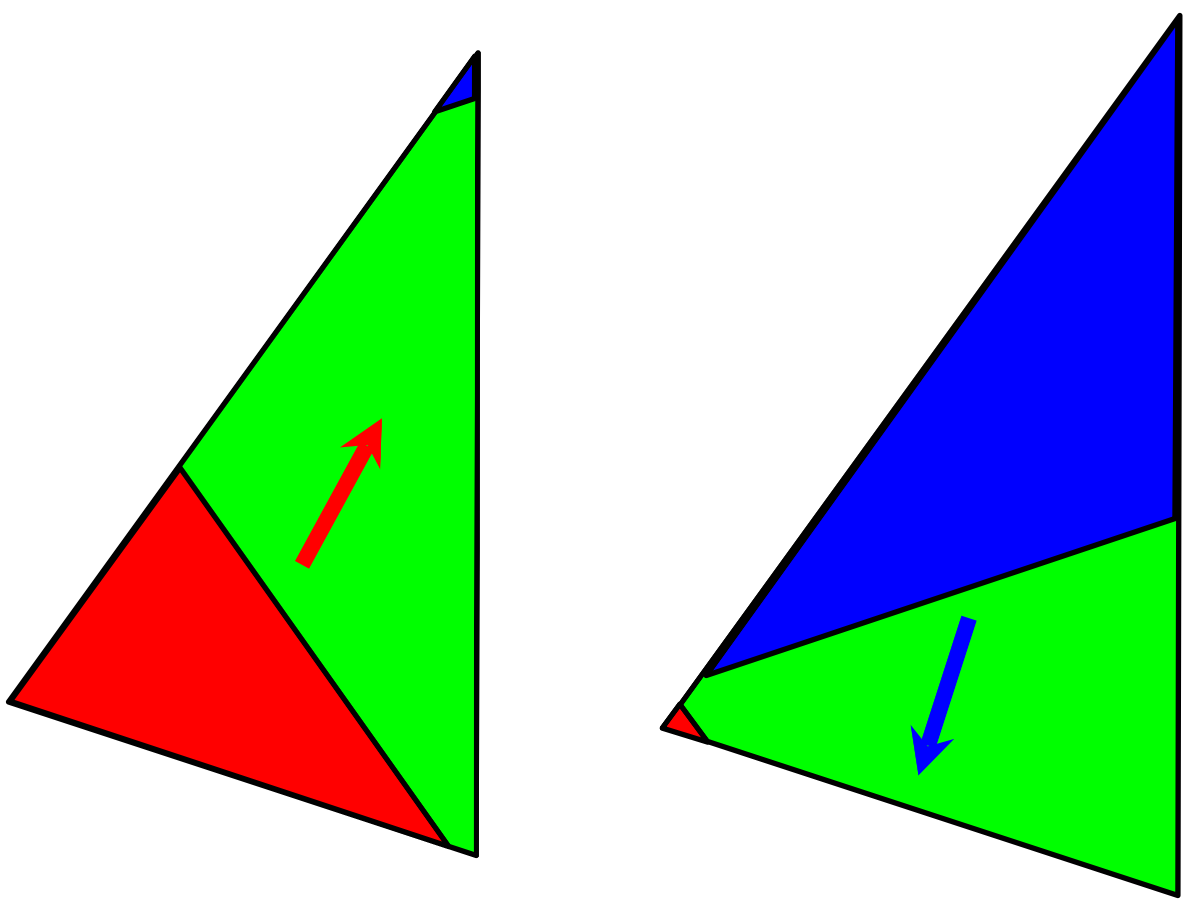




Symmetry results in rigidity under parameter change



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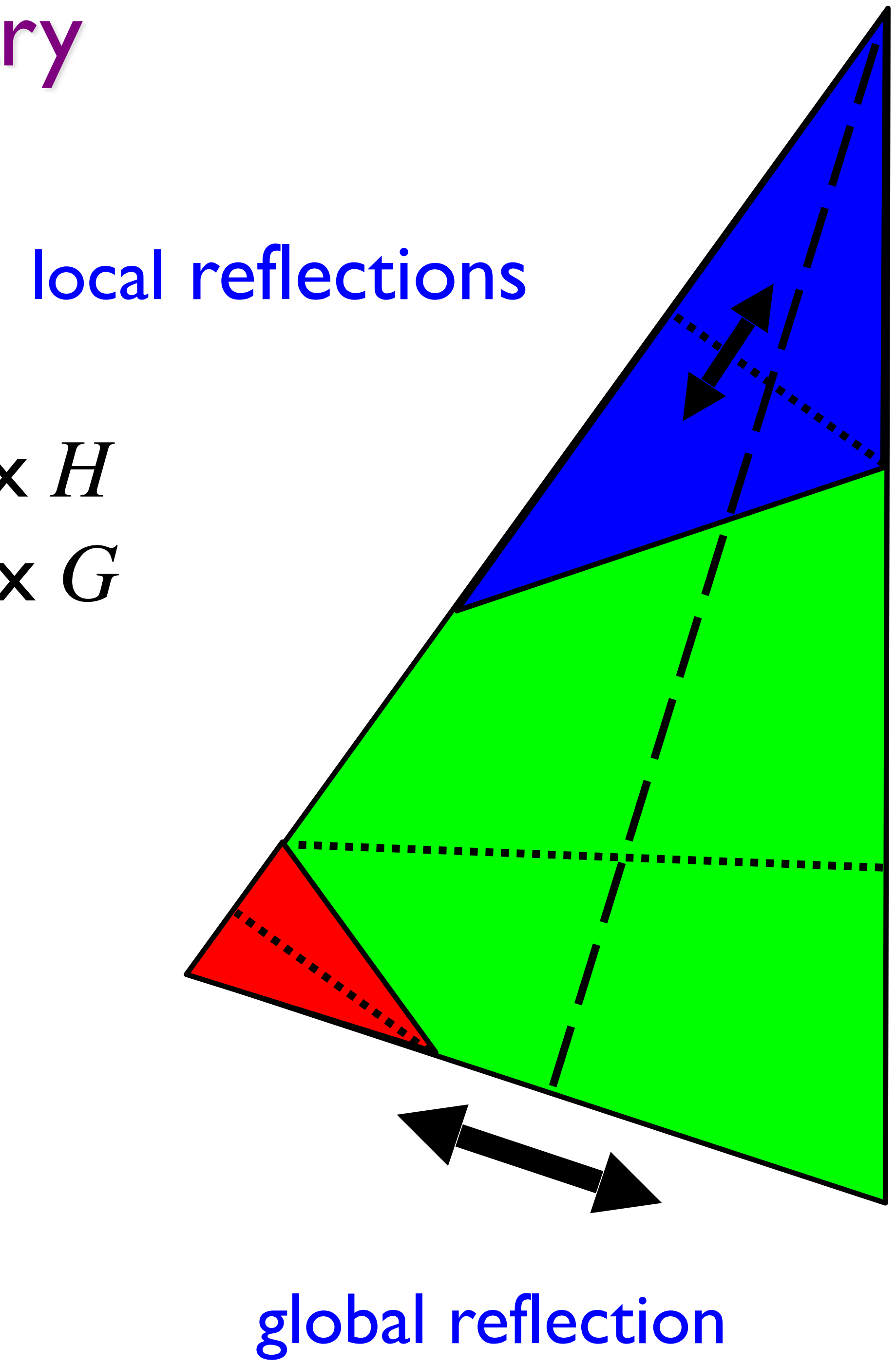

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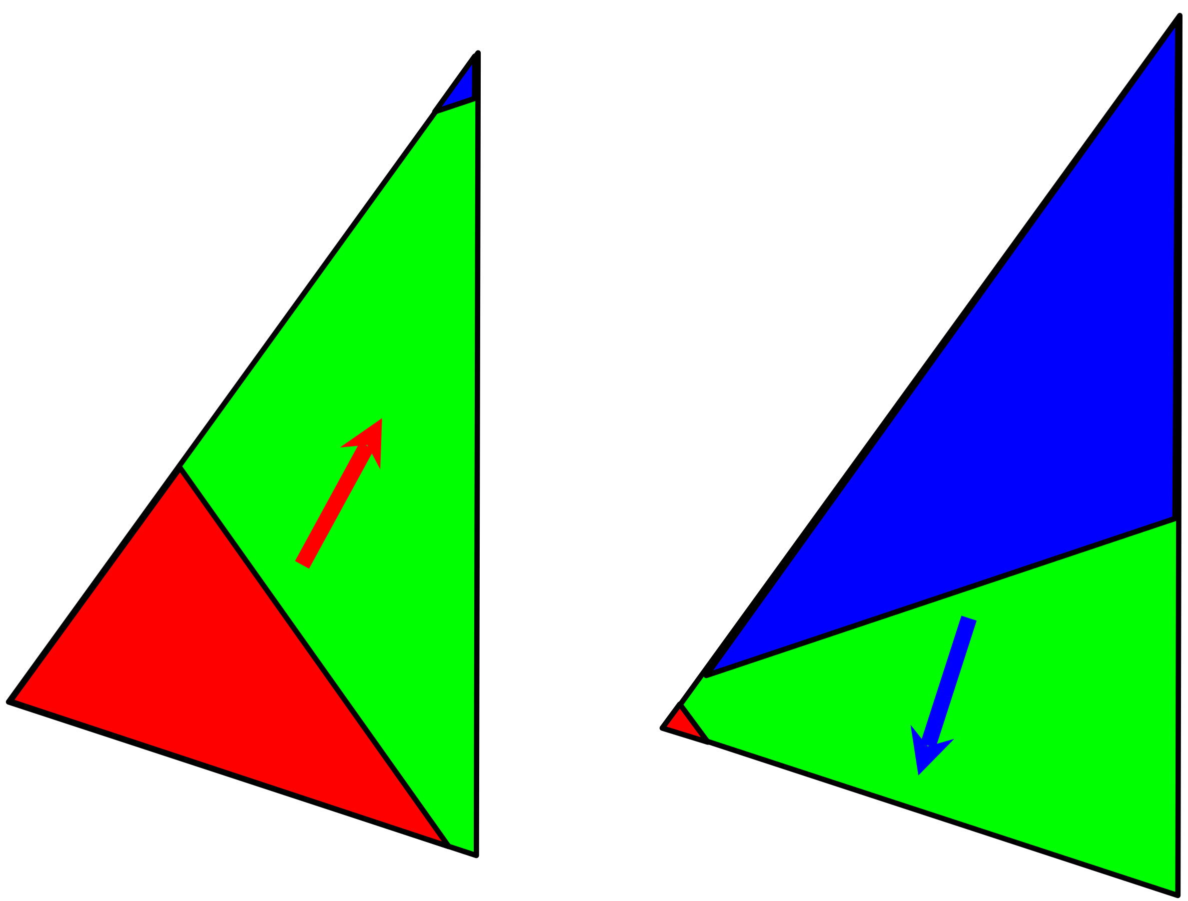
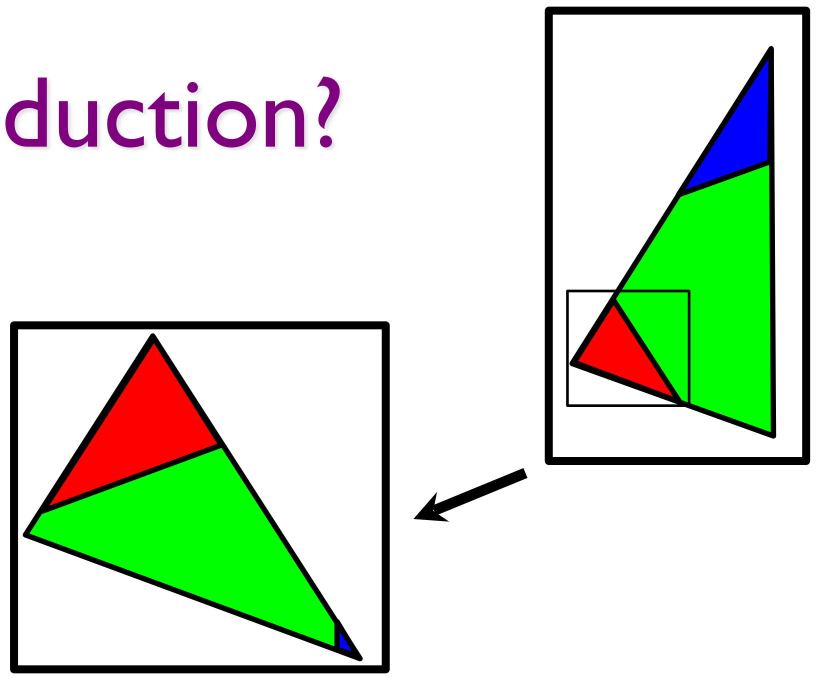


Bifurcations at end-points of parameter interval: one atom disappears.

How does the parameter change under induction?



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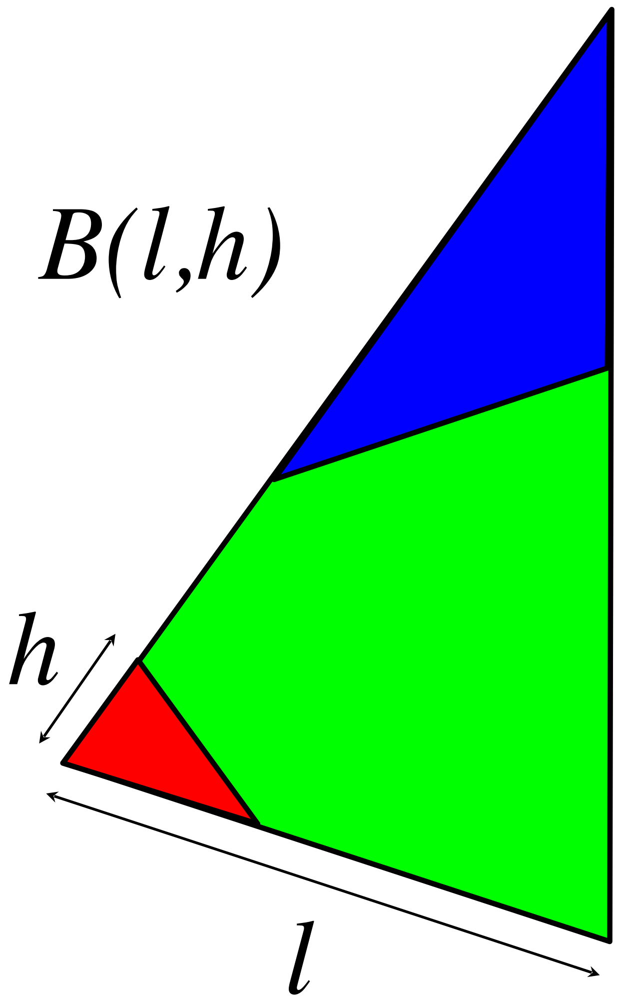

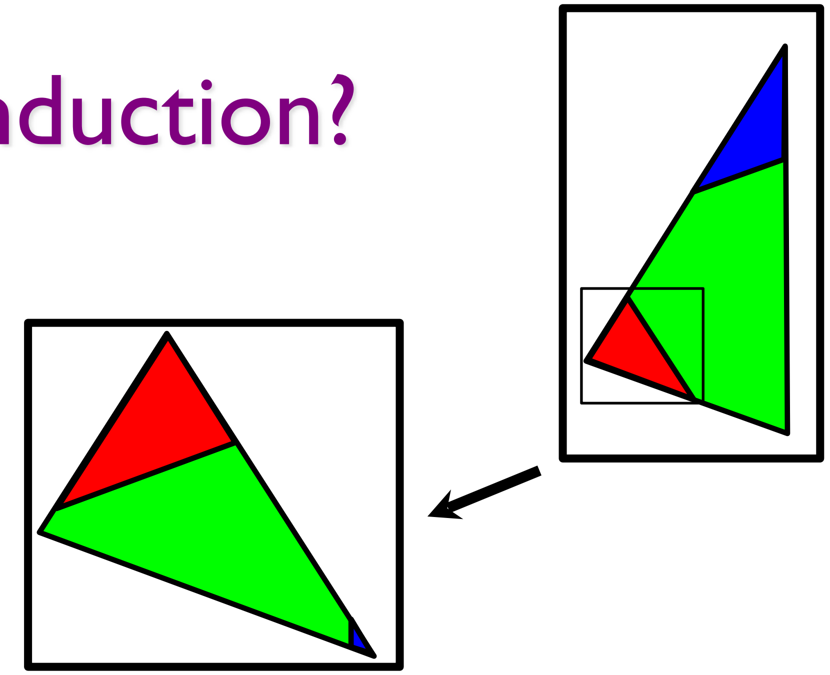

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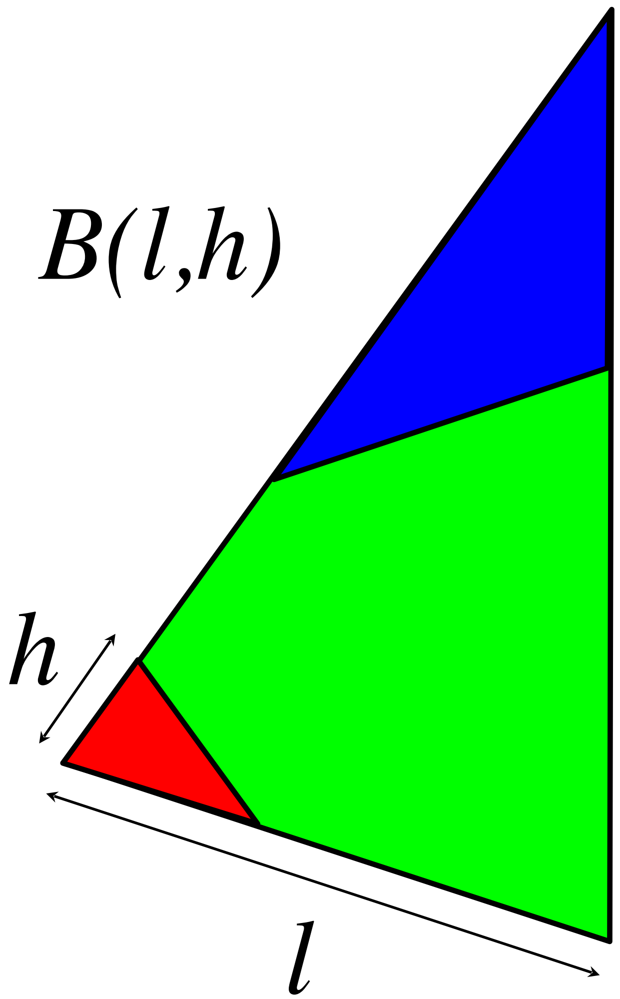

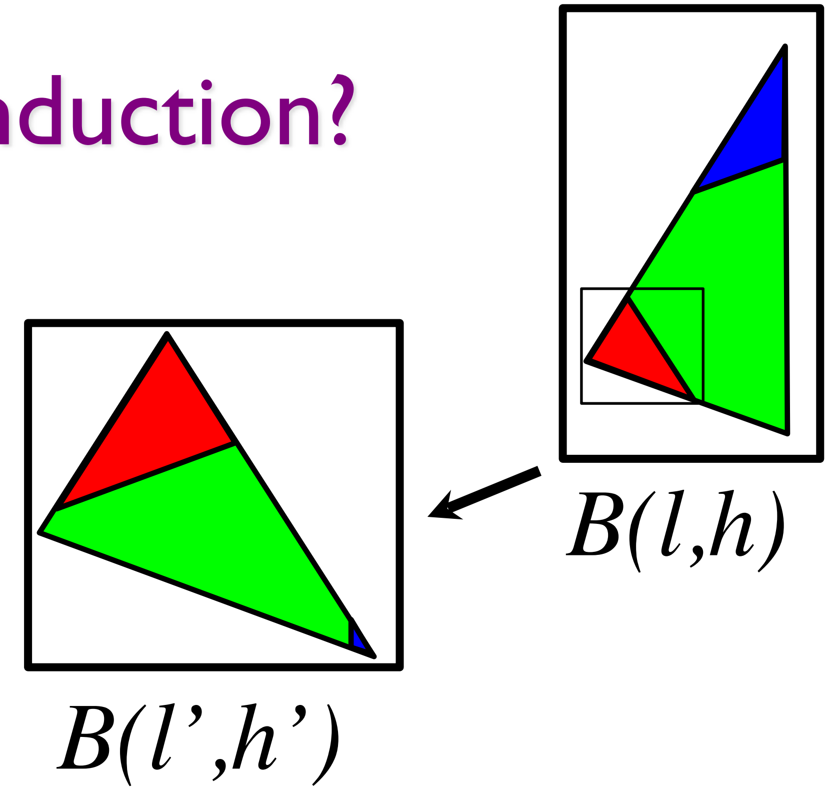

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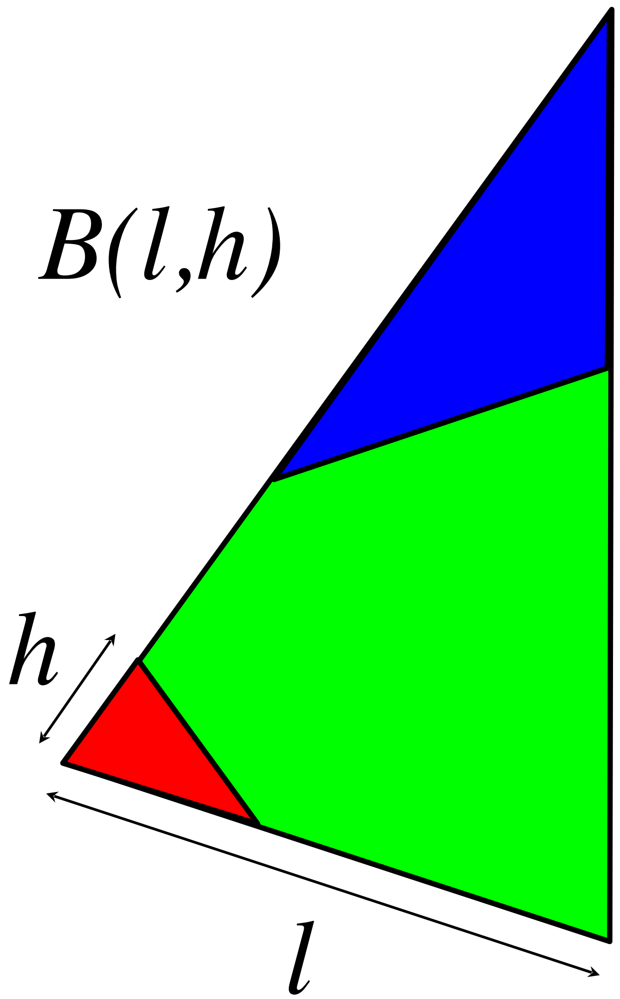

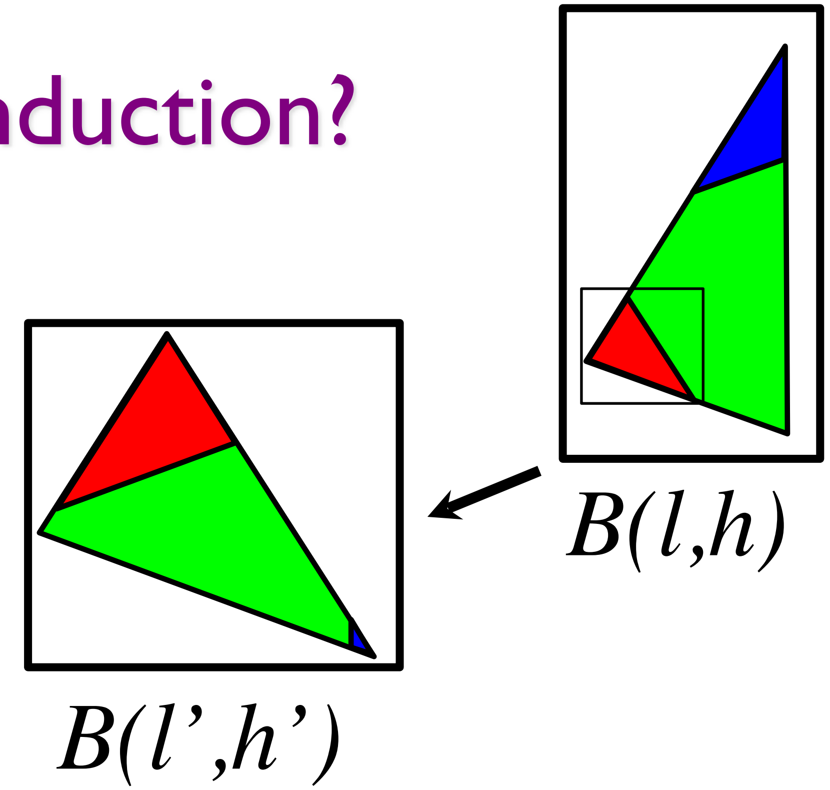

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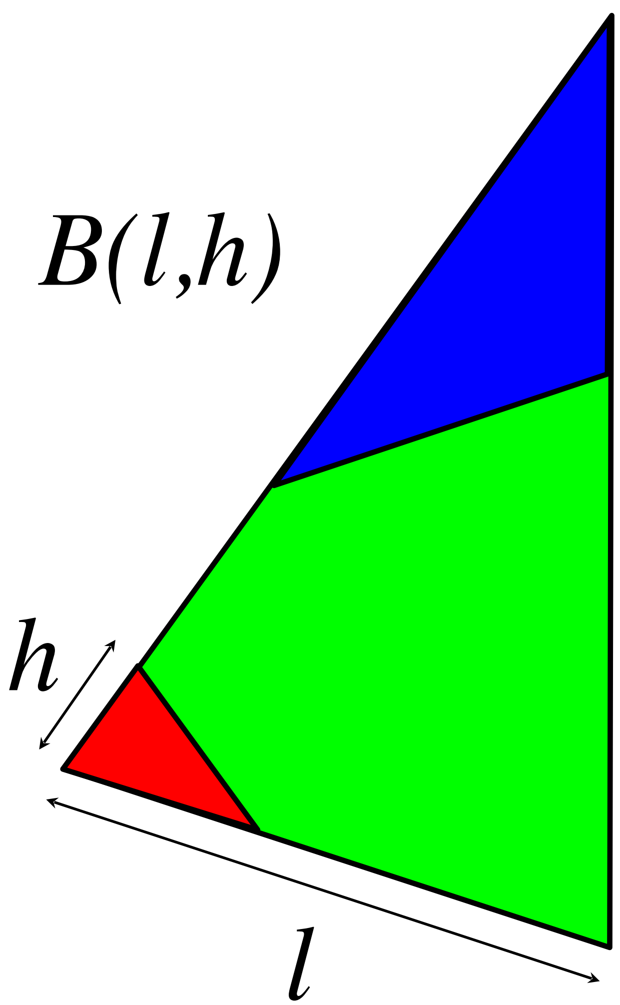

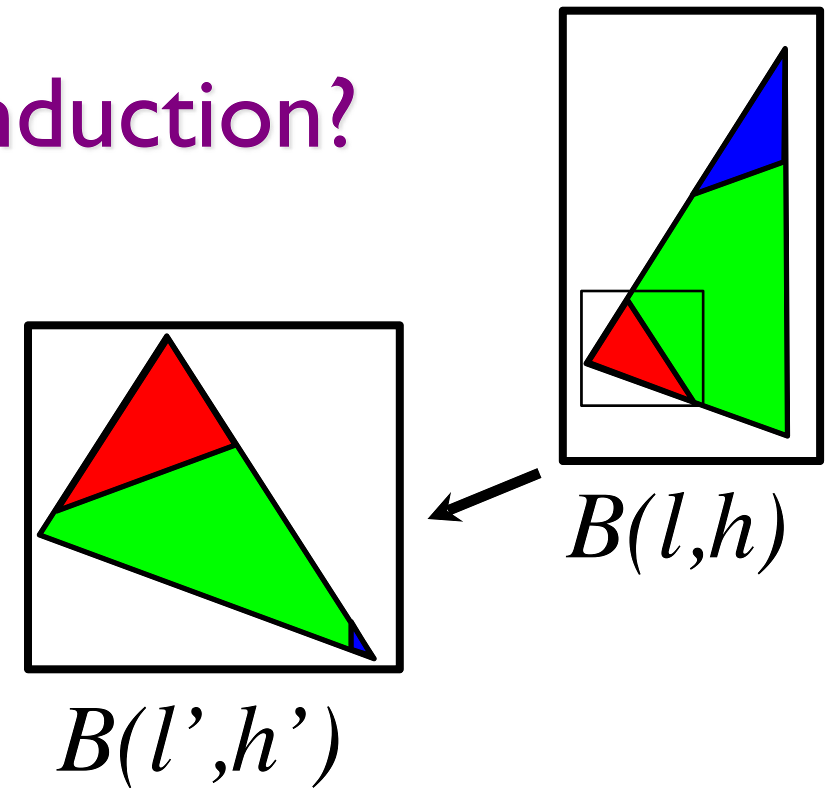

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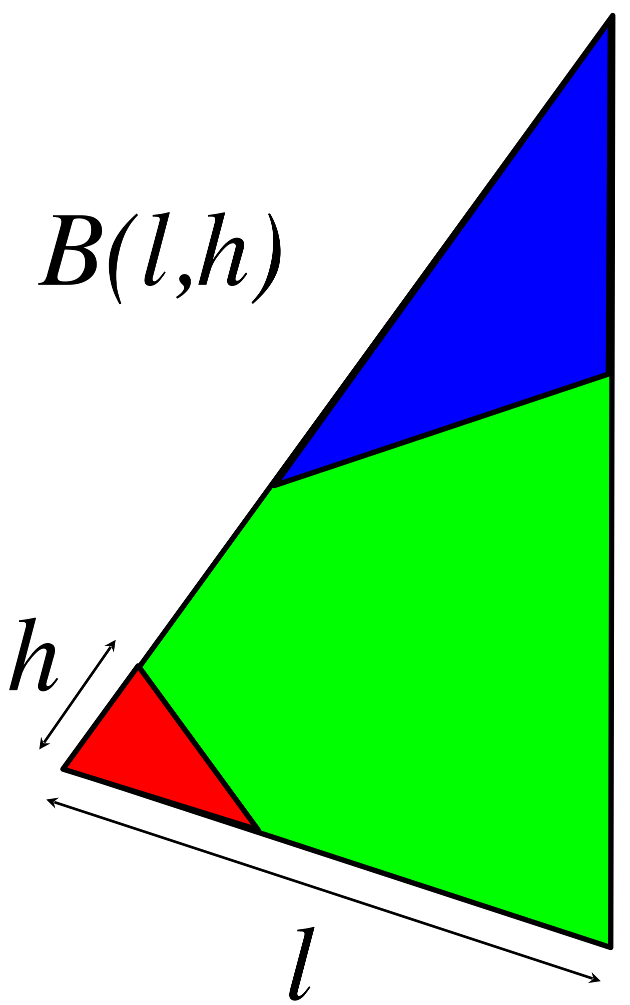

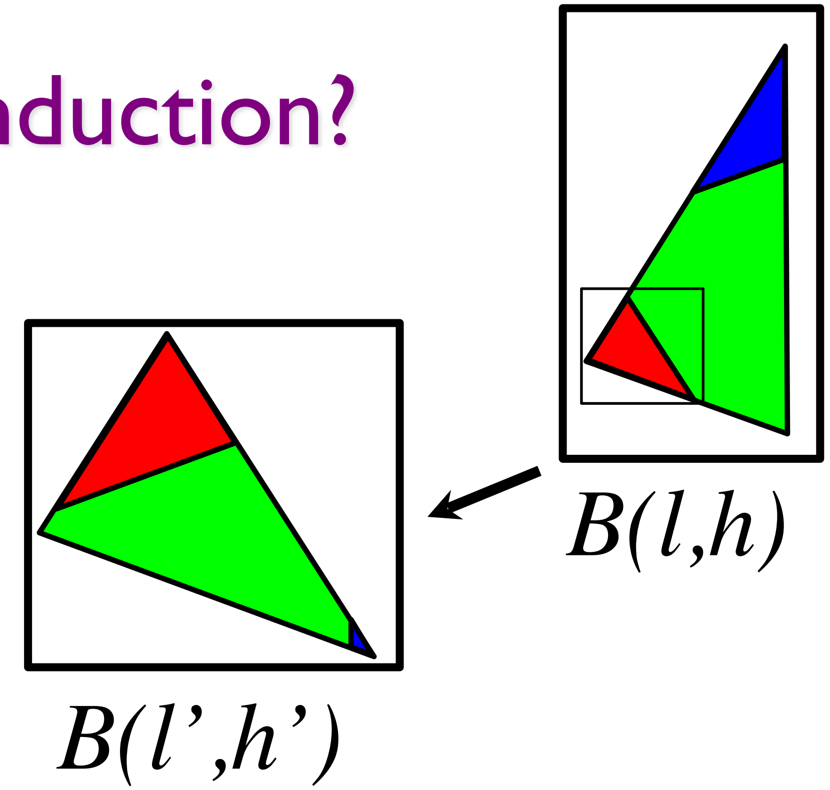

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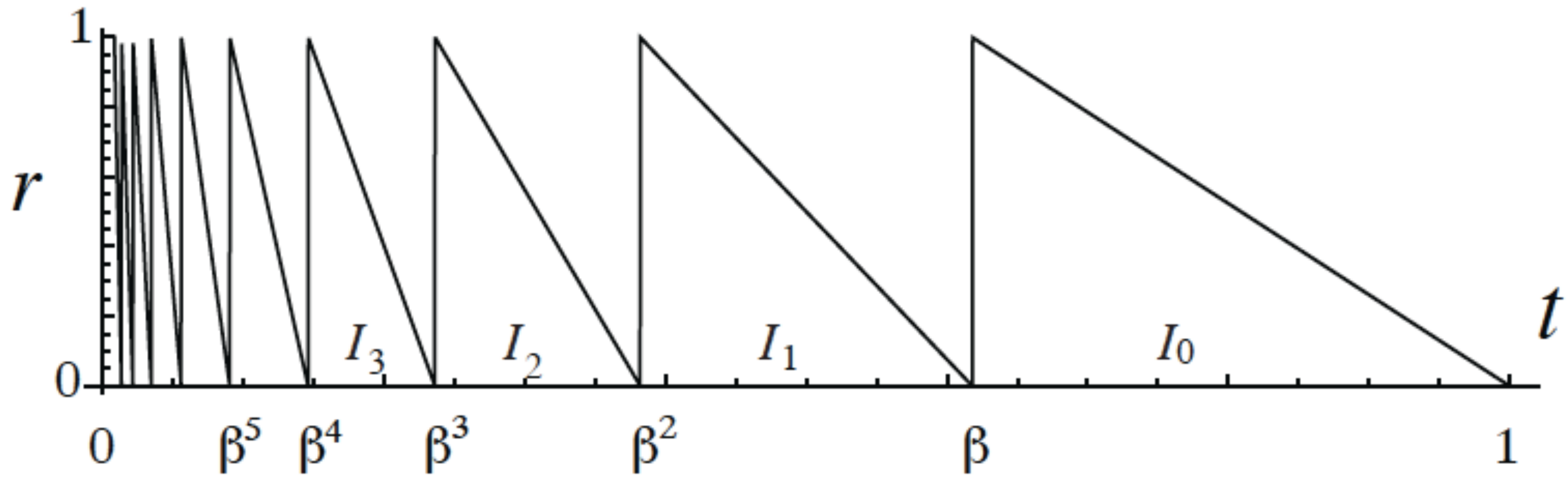
parameter renormalization function

$$t' = r(t)$$

The parameter renormalization function



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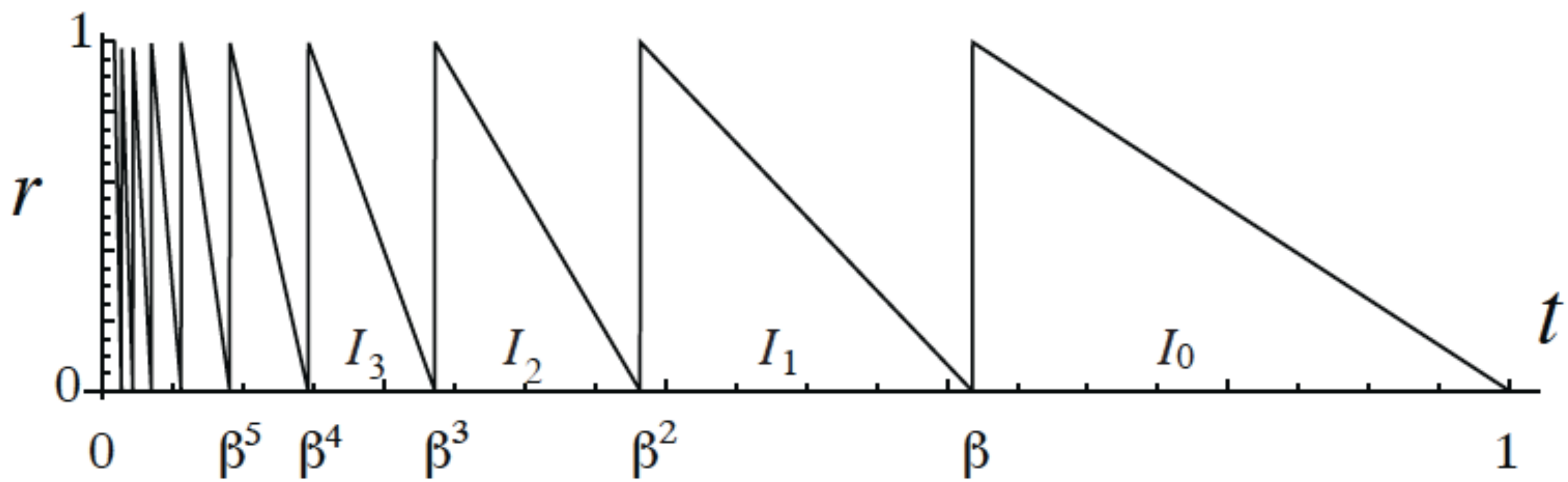

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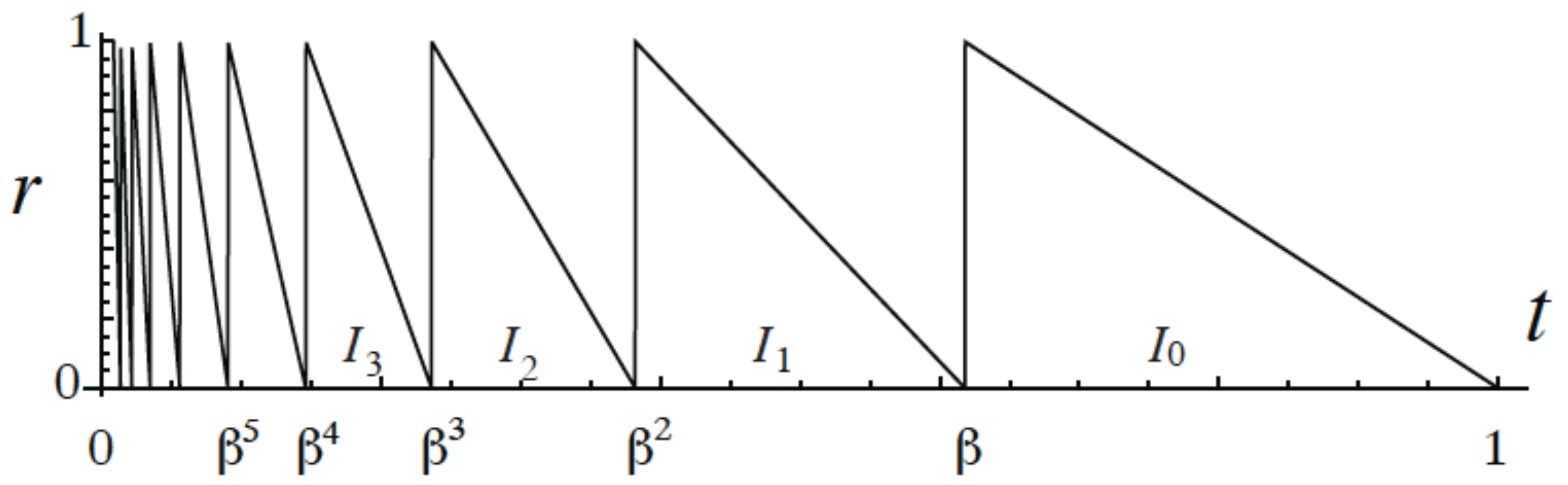

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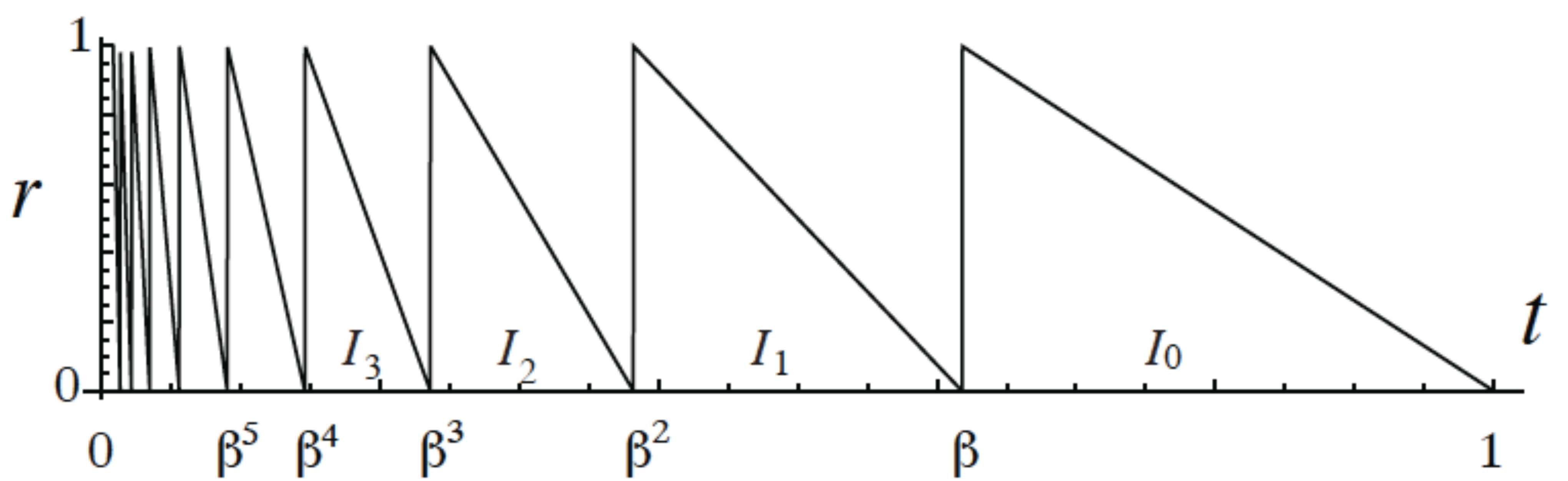

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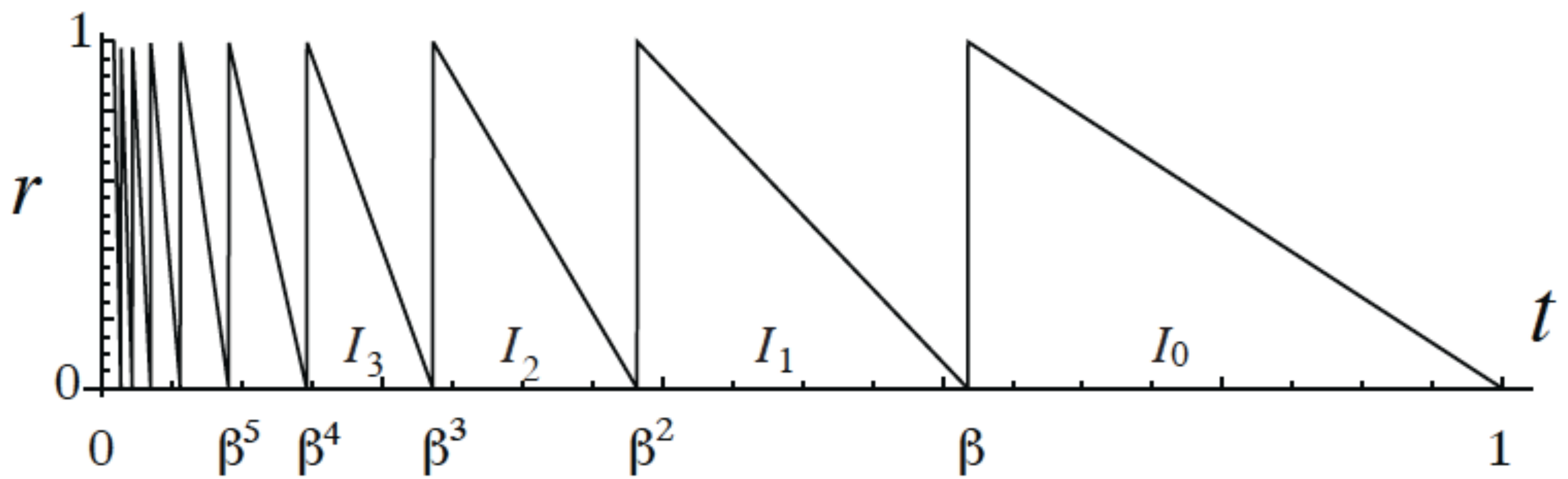

- A piecewise affine map of the unit interval into itself.
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- The eventually periodic points coincide with $\mathbb{Q}(\sqrt{5})$

Generalised Luroth series

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There is a natural symbolic dynamics, recording the sub-intervals visited by an orbit:

$$(i_0, i_1, i_2, \ldots) \quad i_k \in \mathbb{N} \cup \{\infty\}$$

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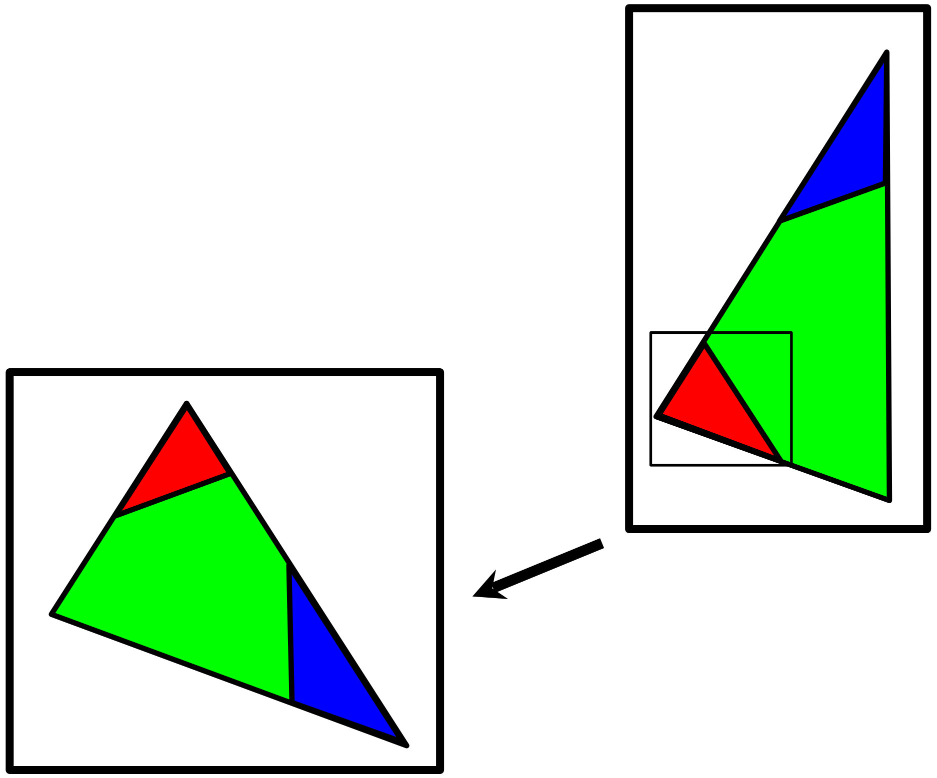
A contraction argument shows that the field condition is sufficient for periodicity.

Self-similarity



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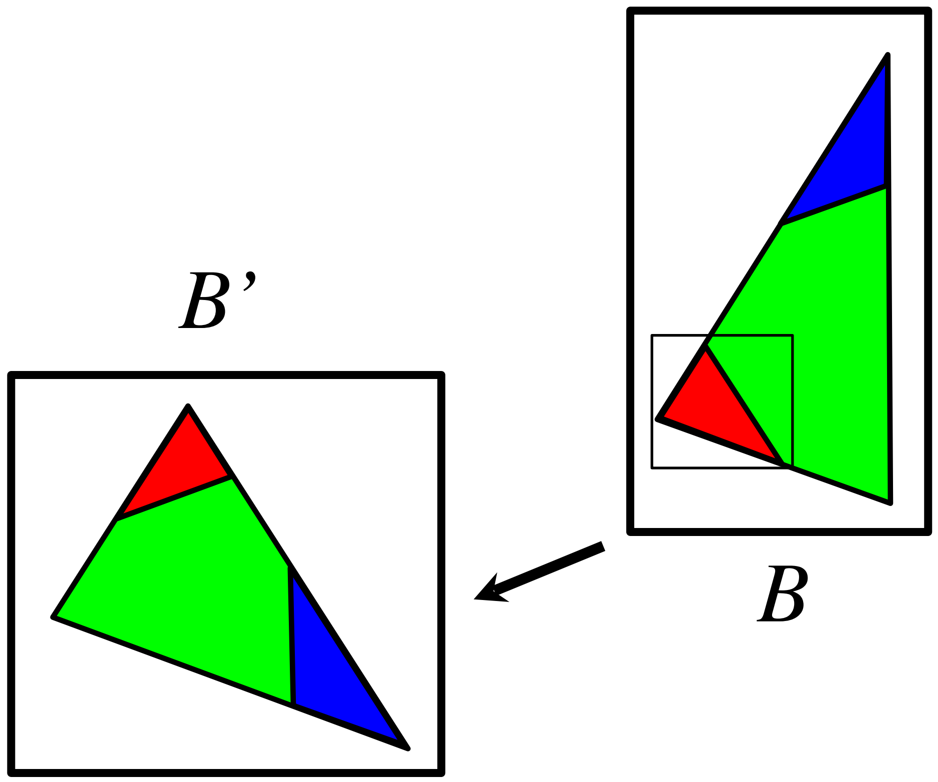
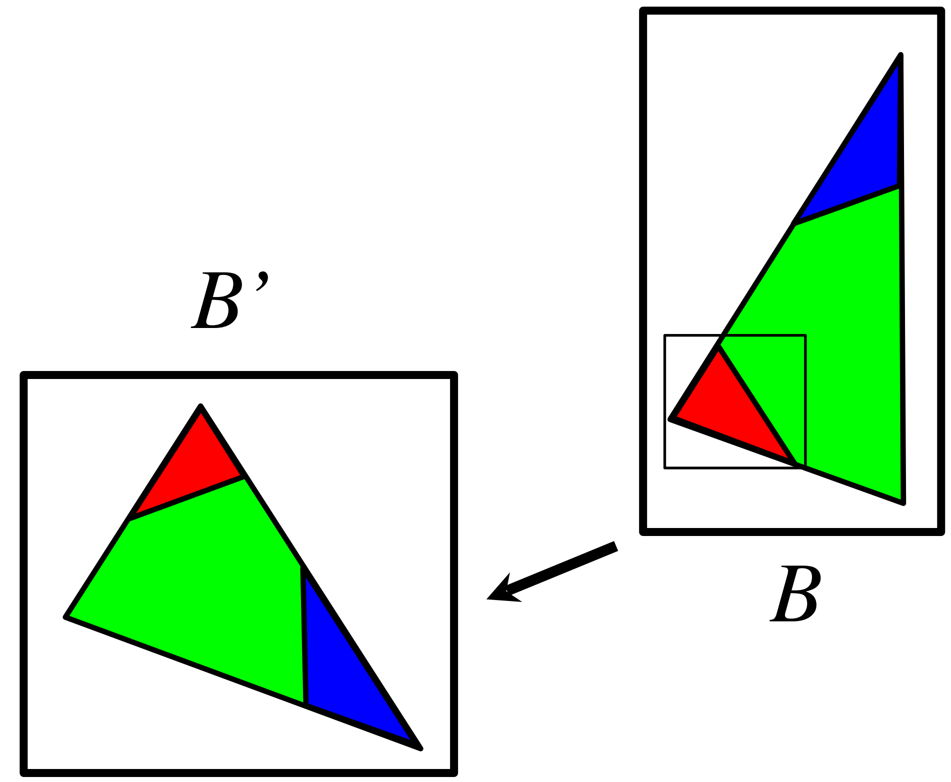


Self-similarity

$B \sim B'$



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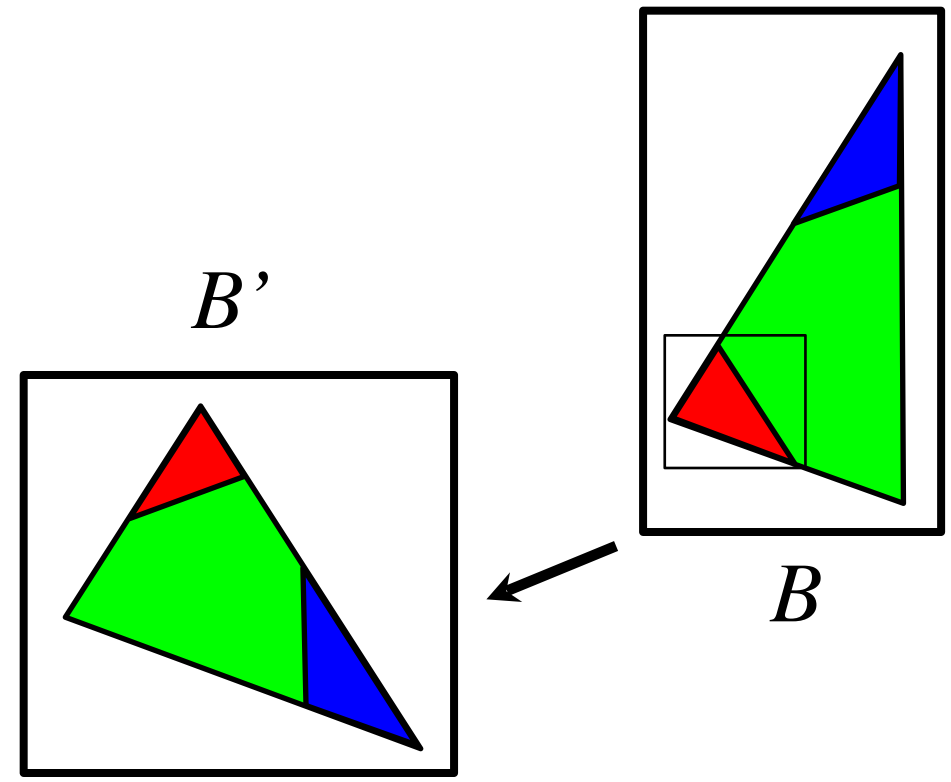

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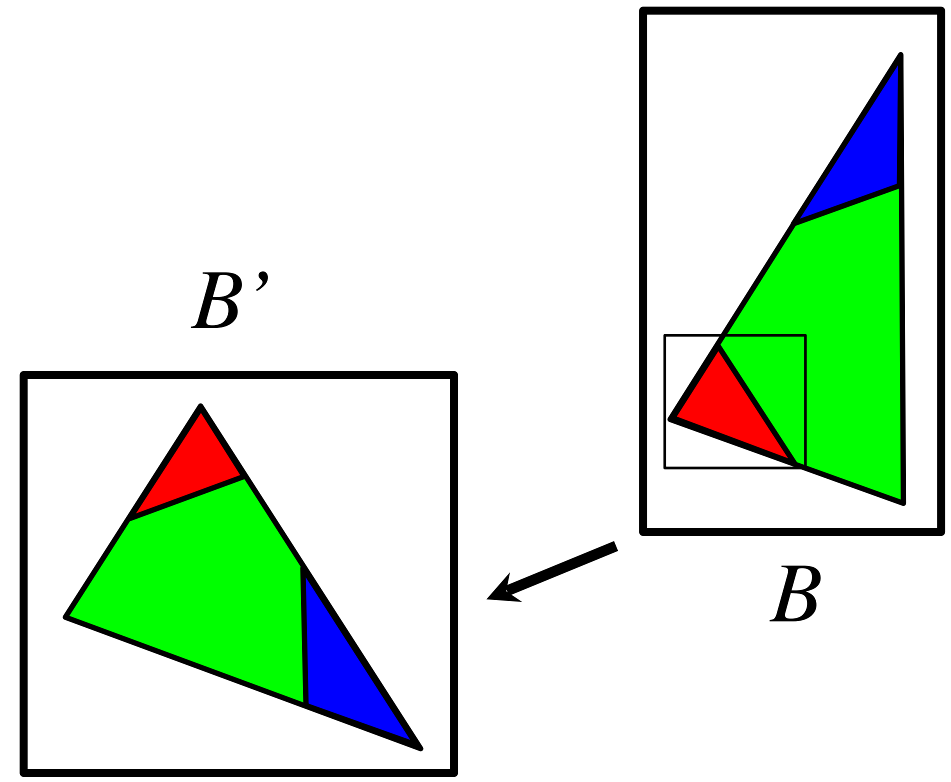

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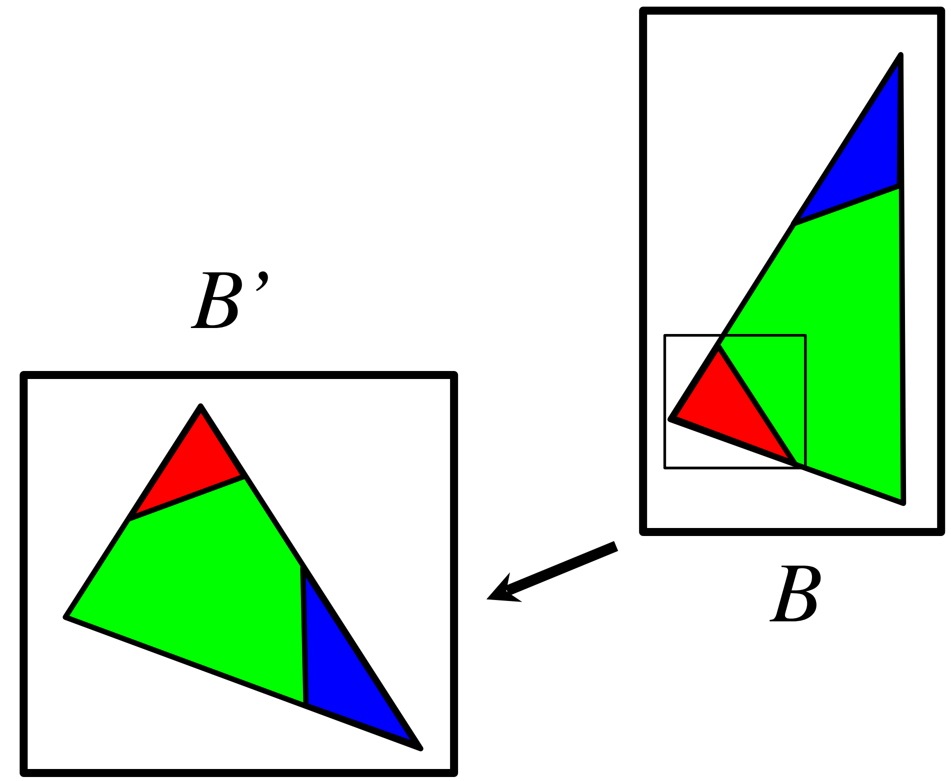
- We write *B*~*B'* to denote congruence with respect to the following transformations:
  - symmetry group of the pentagon

Self-similarity



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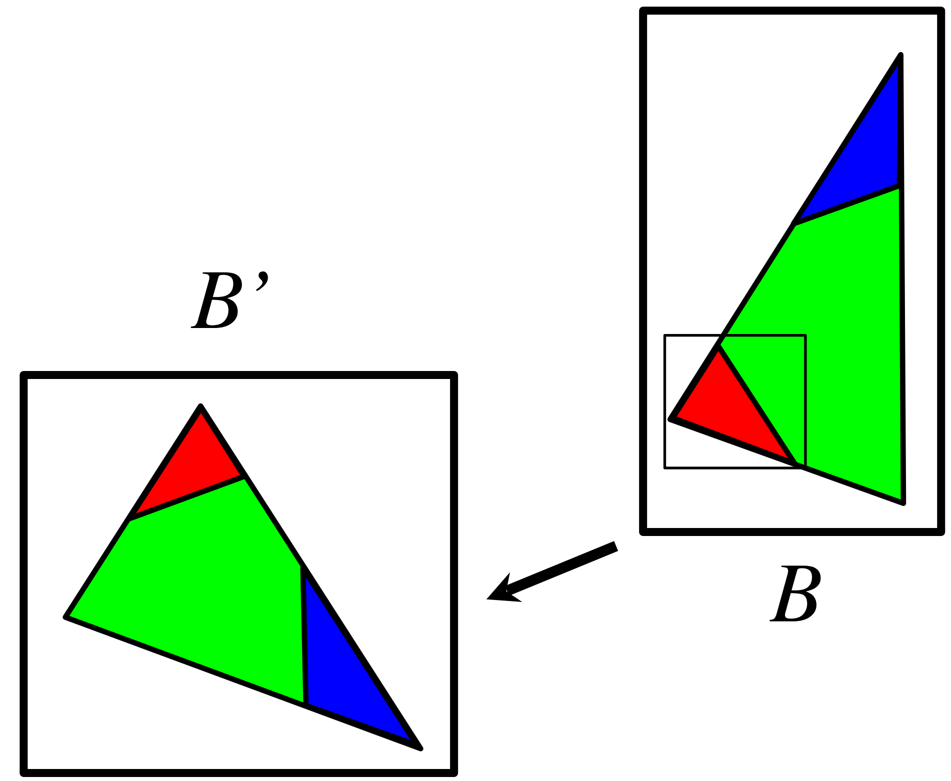

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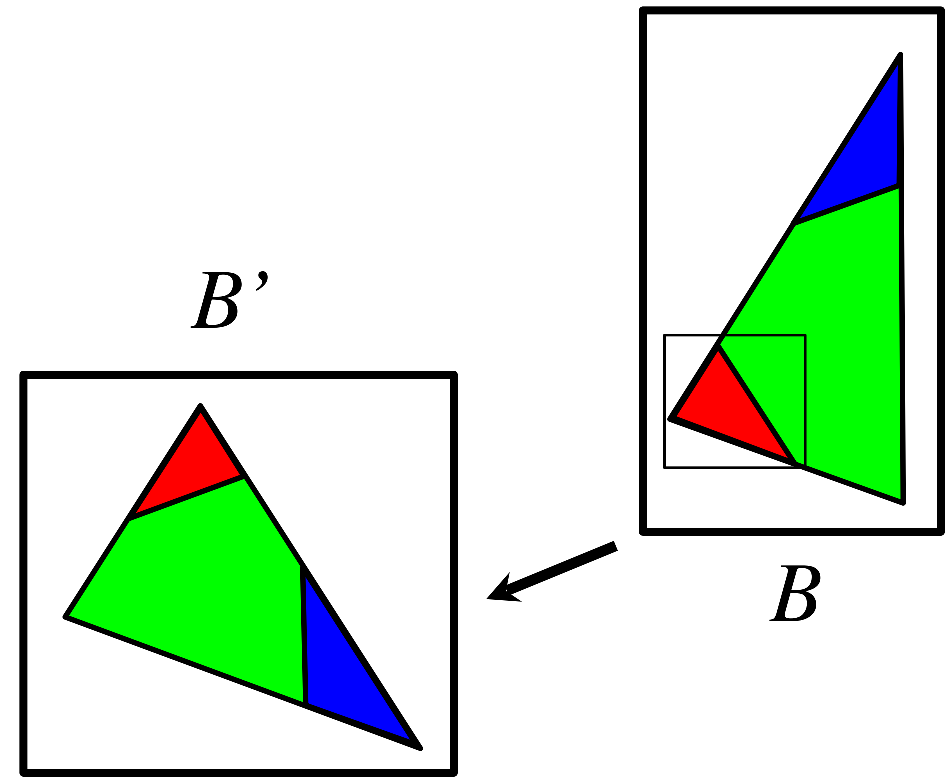

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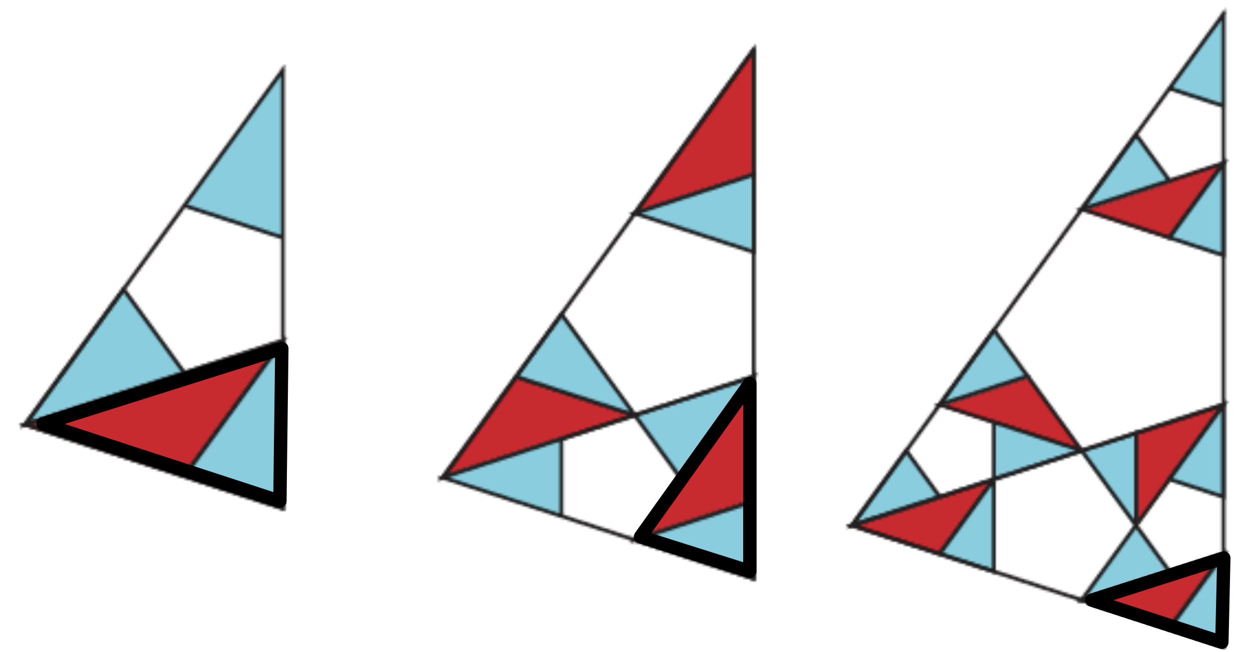
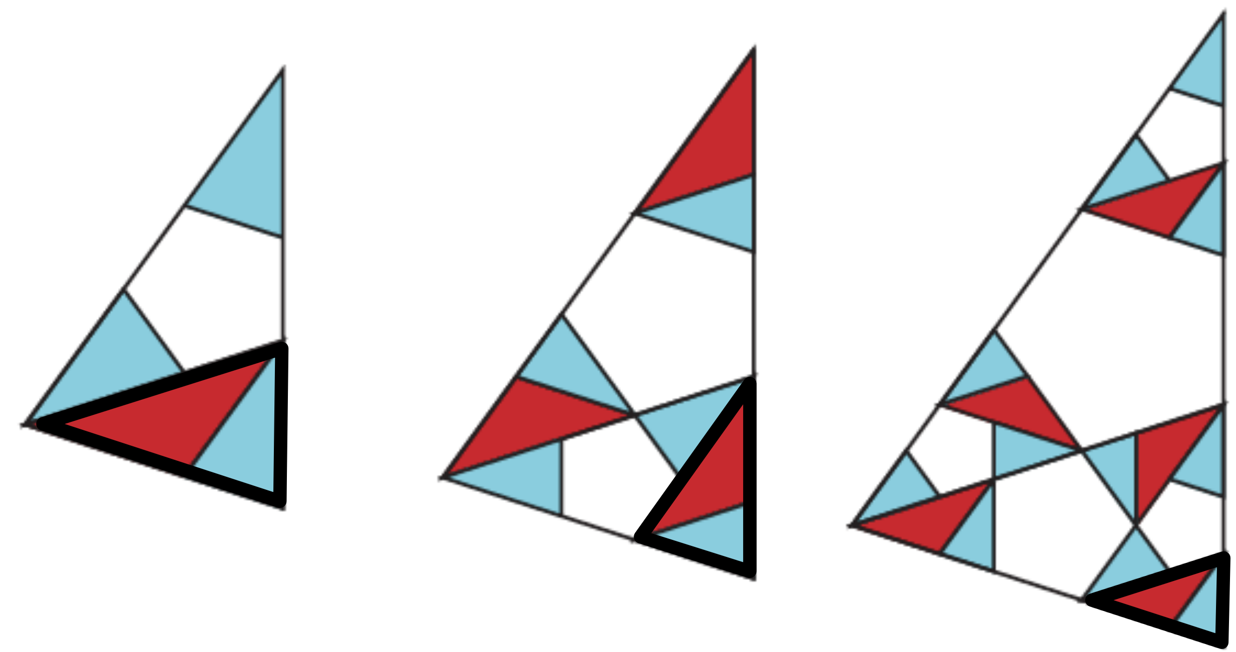
  - scaling by units.
- This equivalence extends naturally to the PWIs on B and B' (written $\mathbf{B}$, $\mathbf{B'}$), by matching the corresponding atoms and their images: $\mathbf{B} \sim \mathbf{B'}$.

Tiling and renormalizability



Tiling and renormalizability

Refined coverings of the exceptional set, via recursive tiling.



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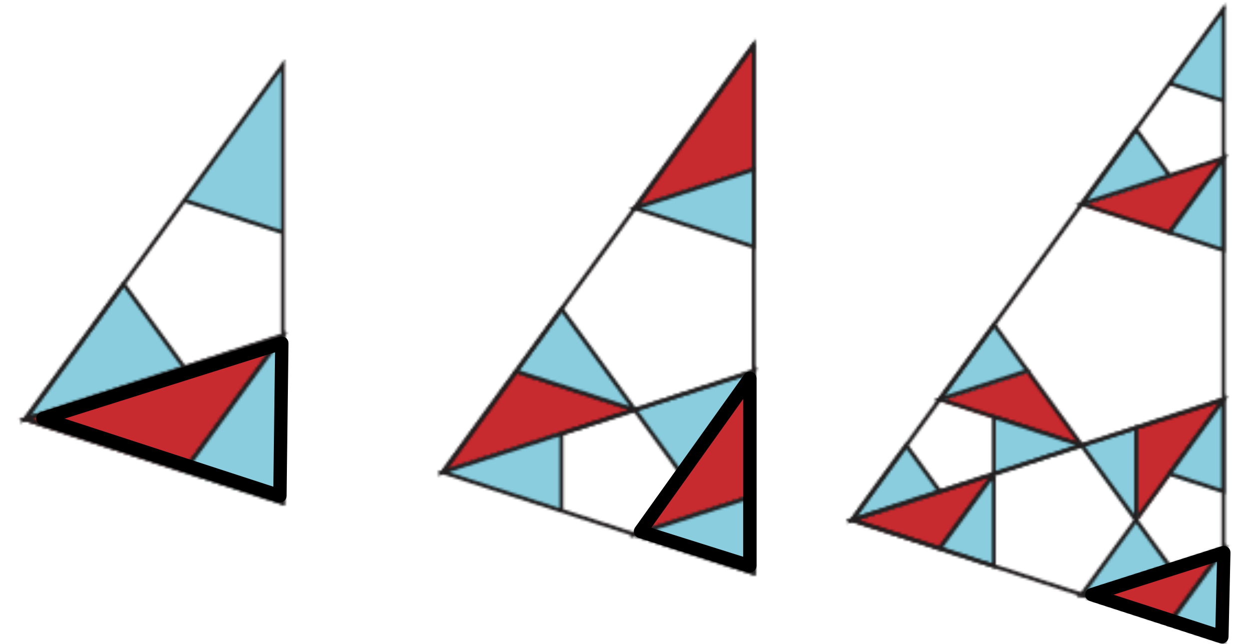

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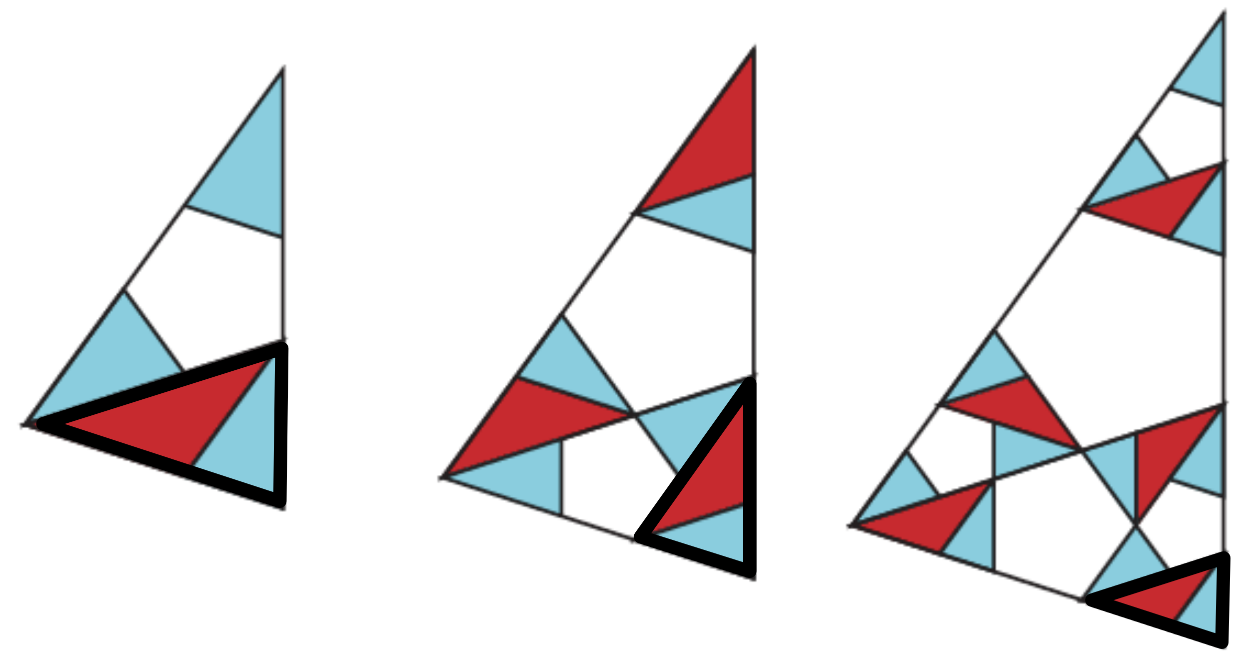

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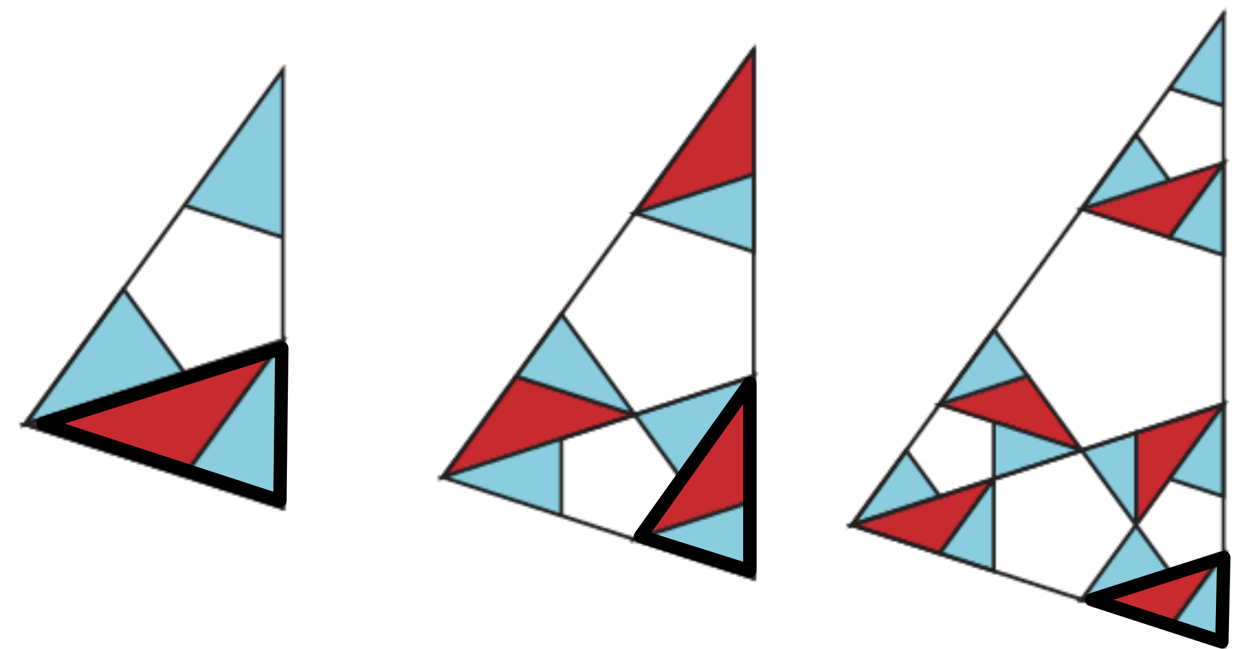

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Theorem. The one-parameter pentagonal model is renormalizable if and only if the parameter belongs to the field $\mathbb{Q}(\sqrt{5})$.

The octagonal model (field $\mathbb{Q}(\sqrt{2})$)

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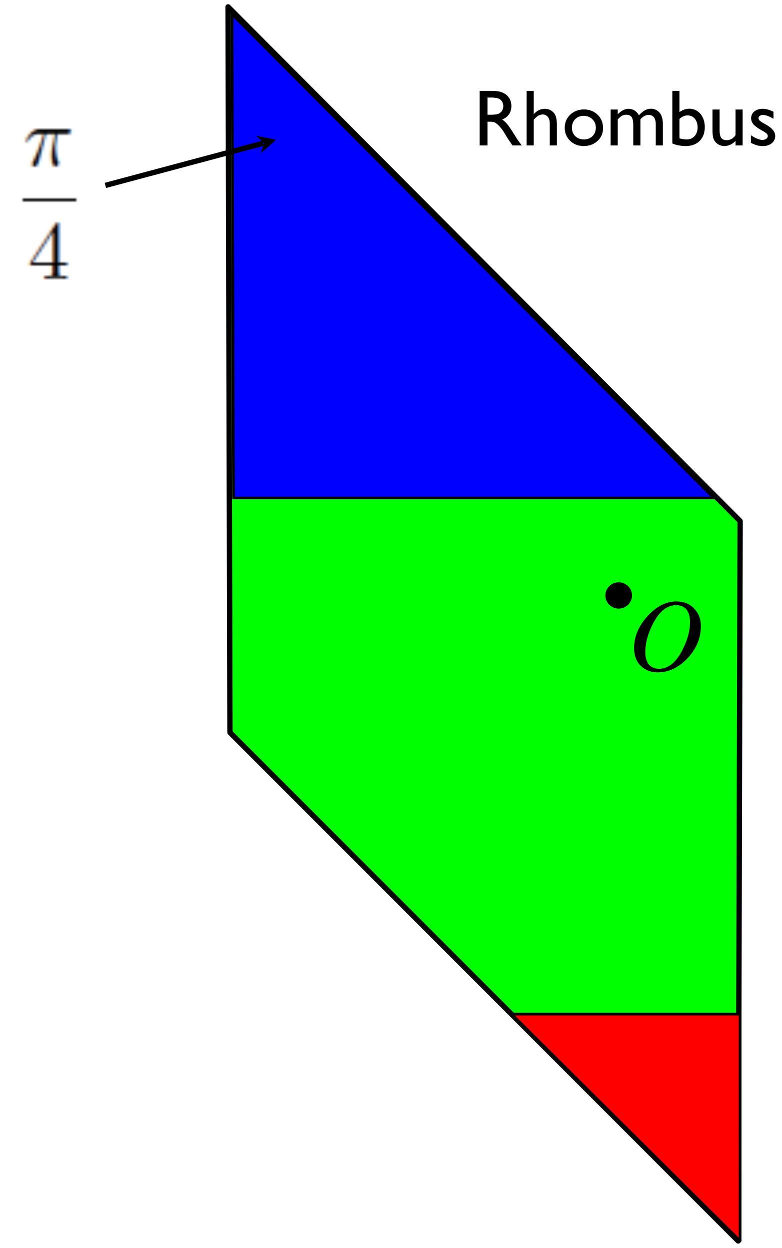

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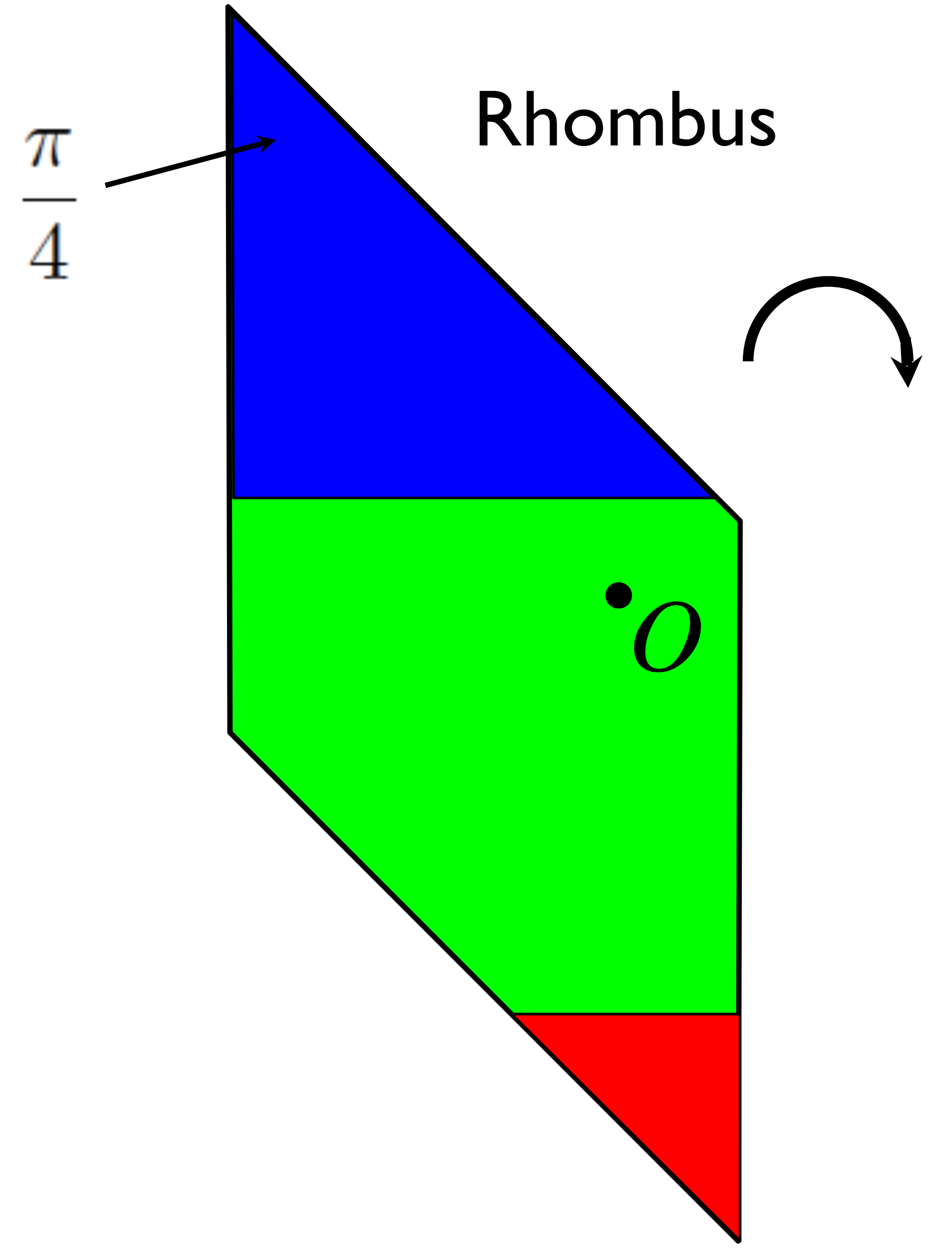

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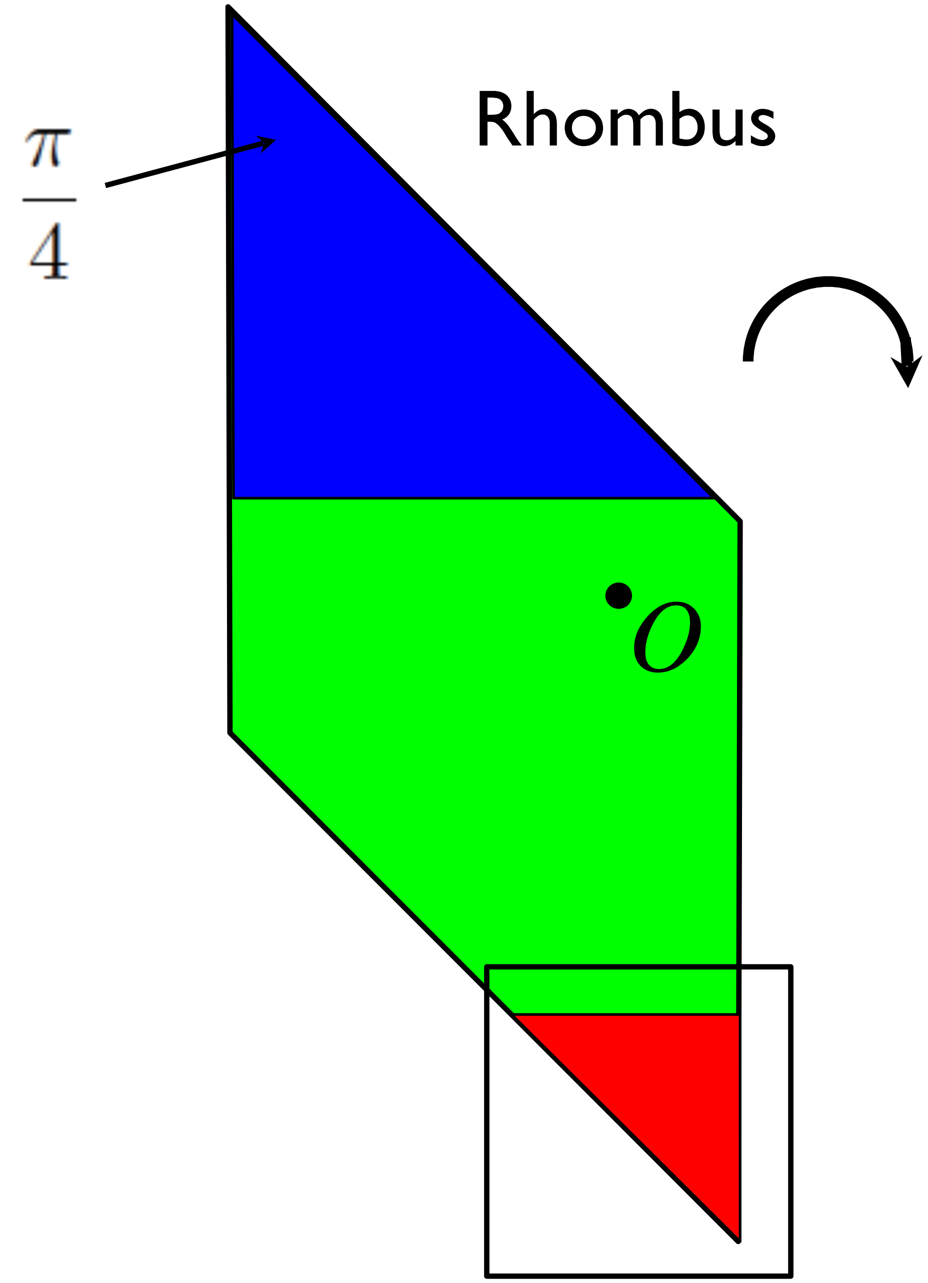
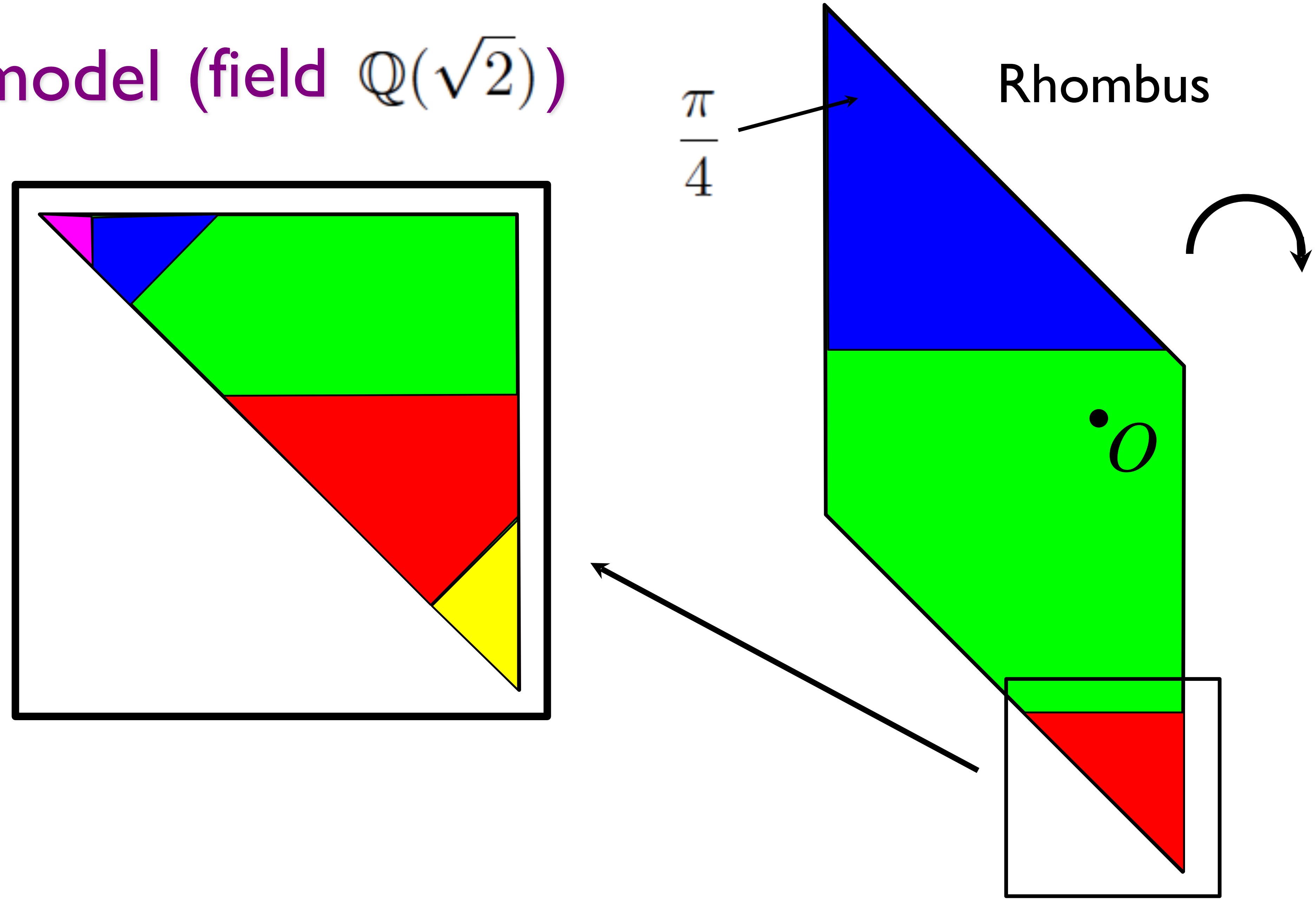


The octagonal model (field $\mathbb{Q}(\sqrt{2})$)

5-atom base triangle



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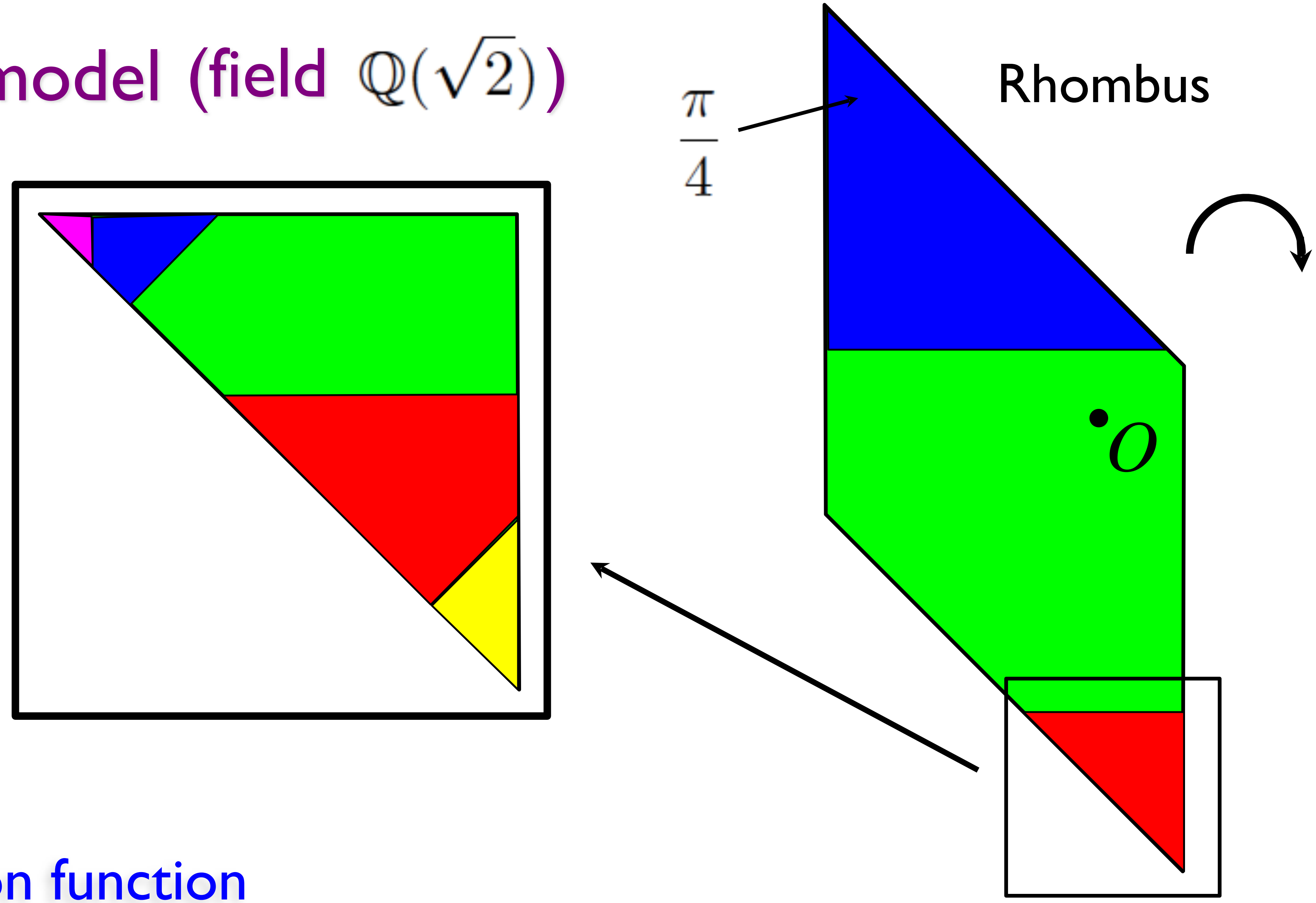
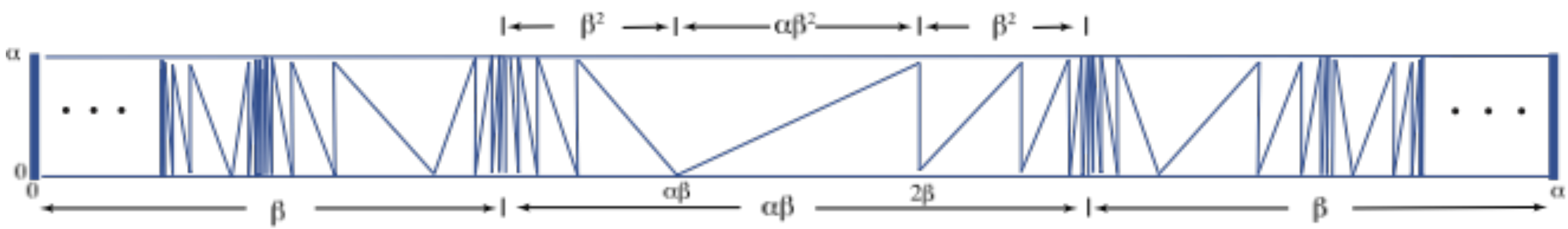


The renormalization function



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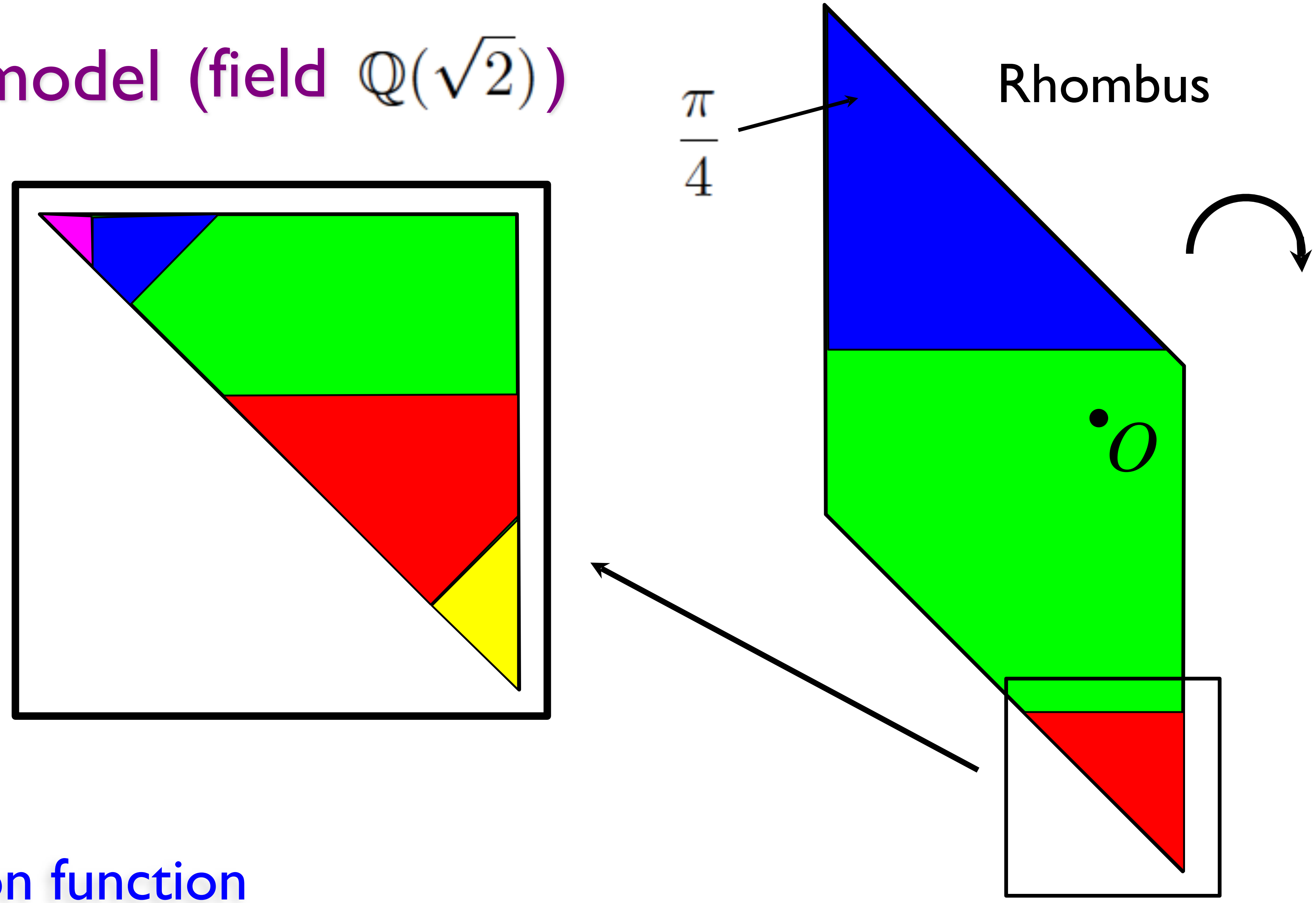

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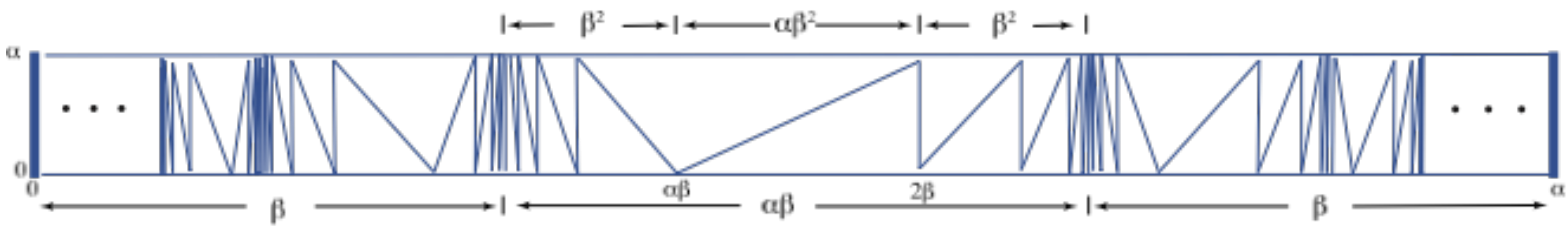

$$r(s) = f^2(s)$$

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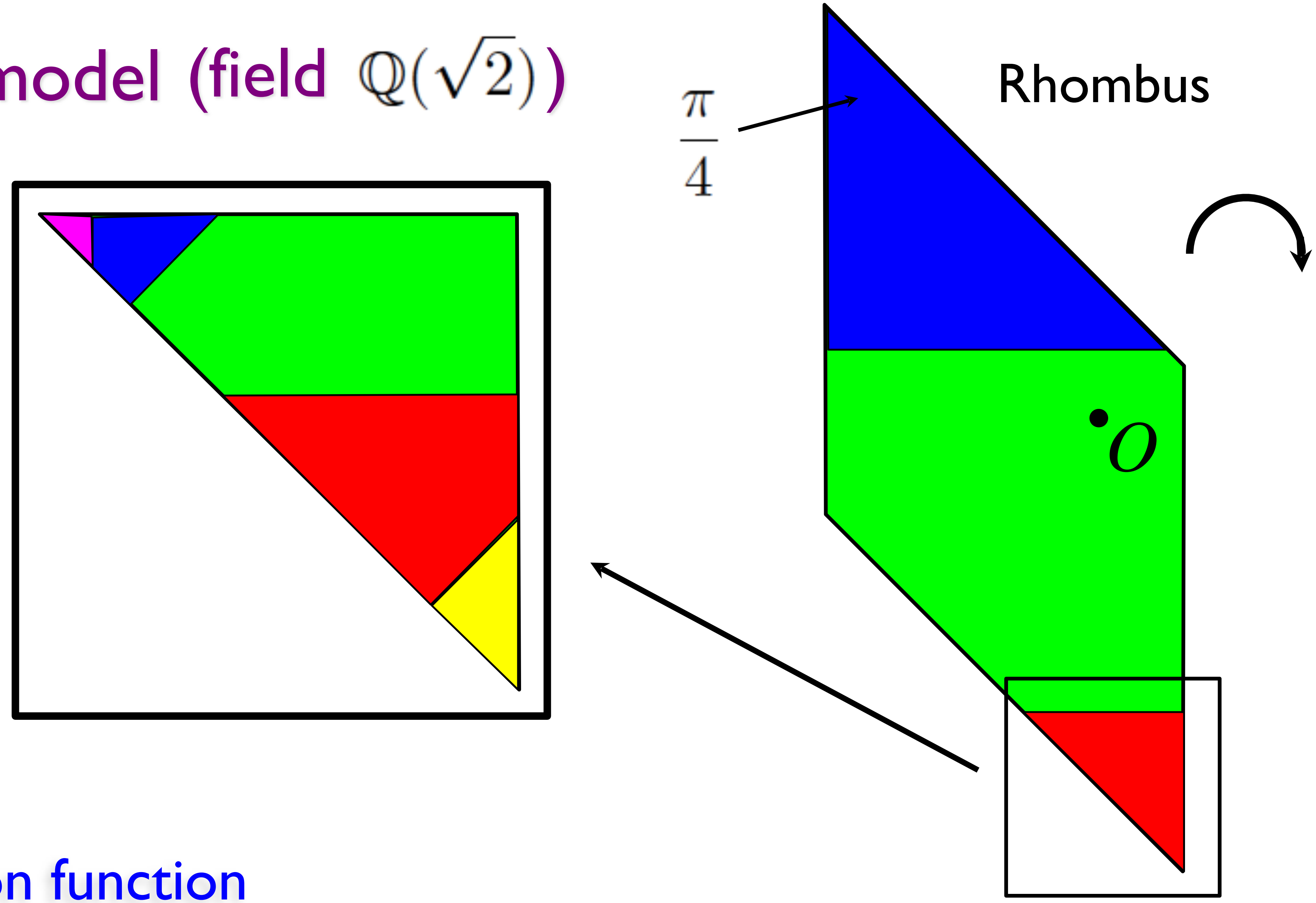

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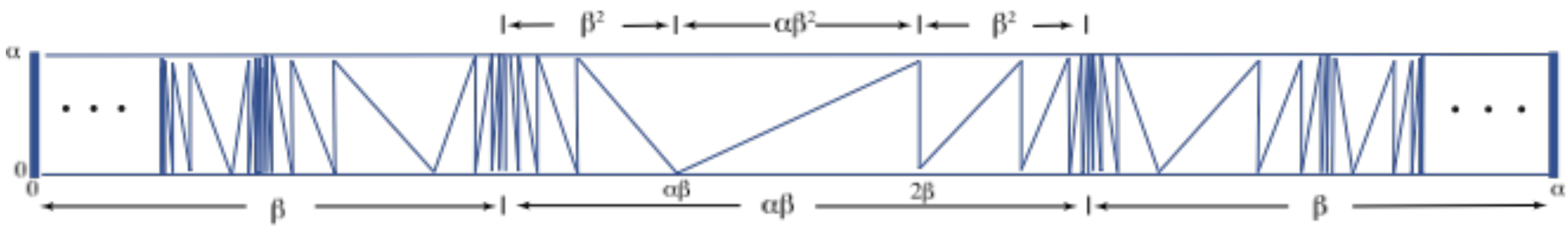

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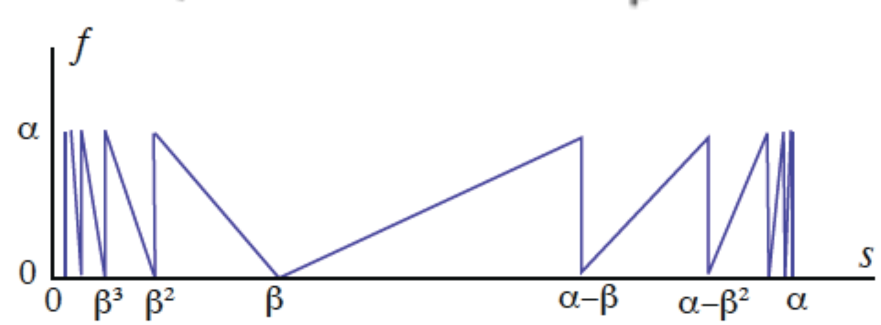

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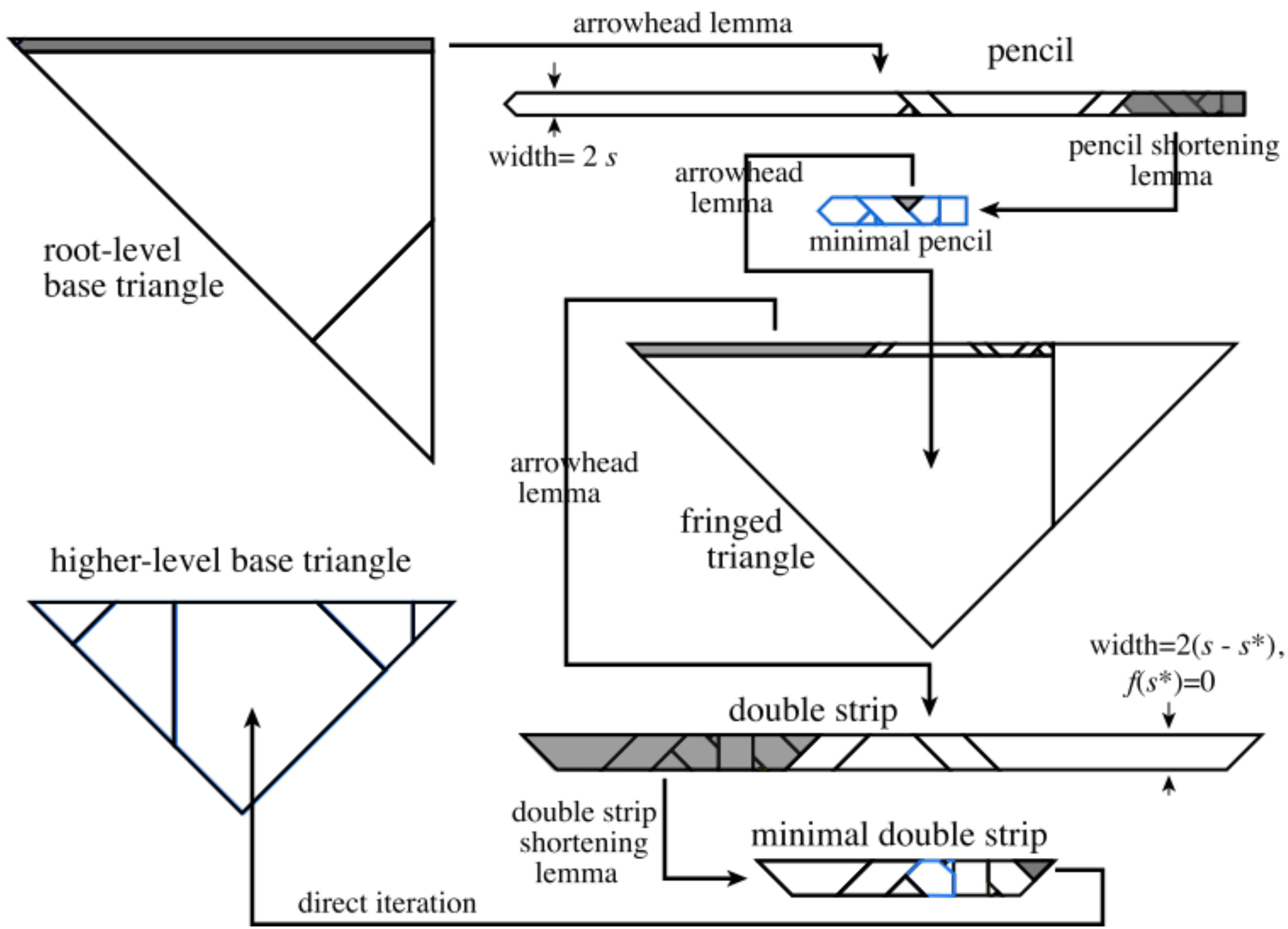

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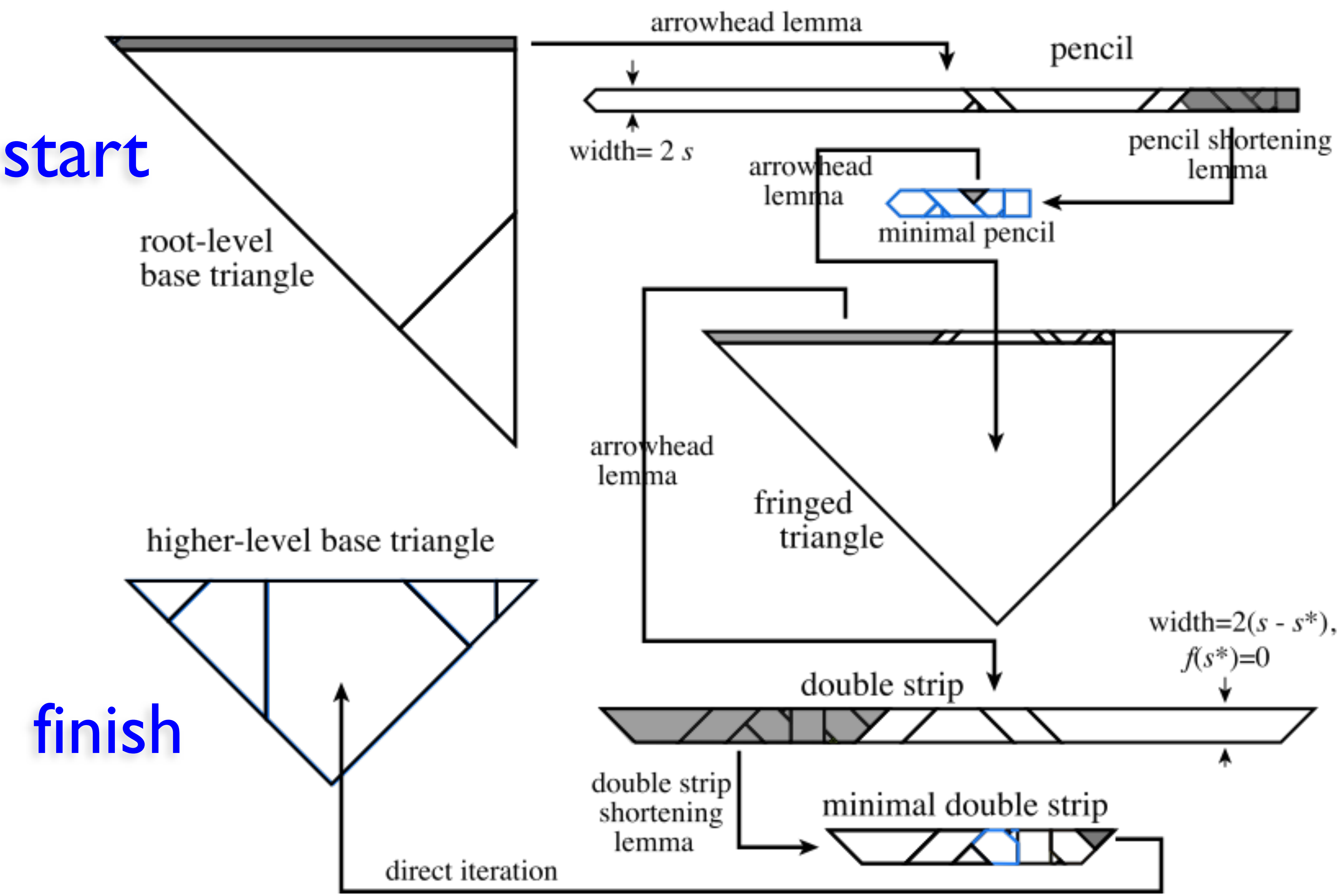

More than one parameter: re-combination of atoms

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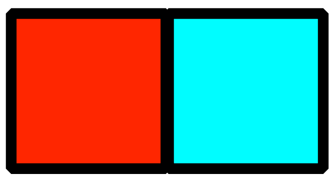

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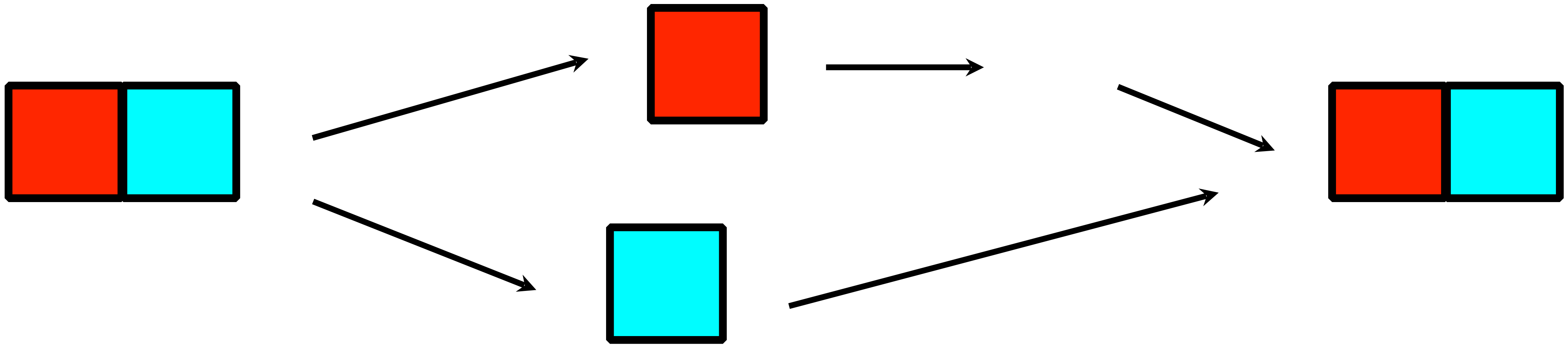

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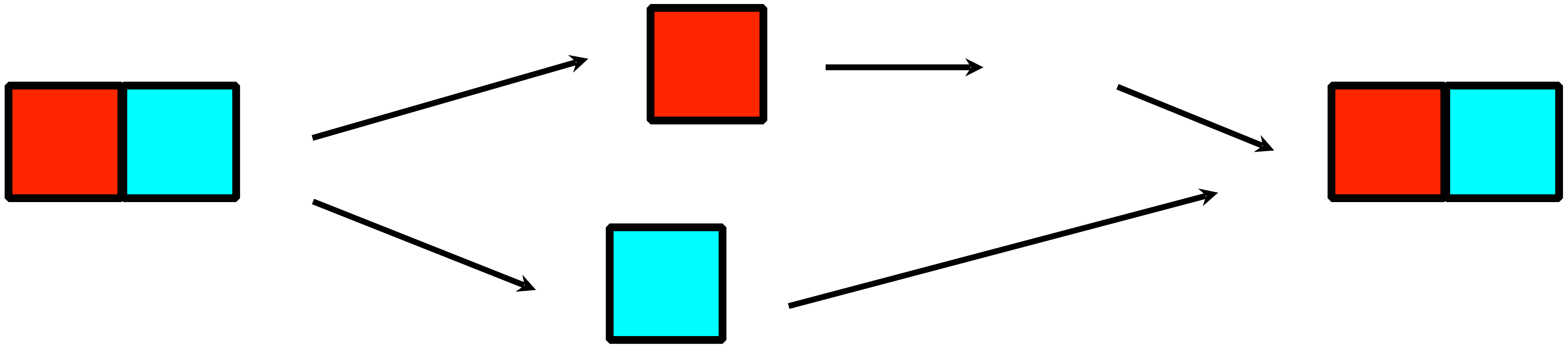

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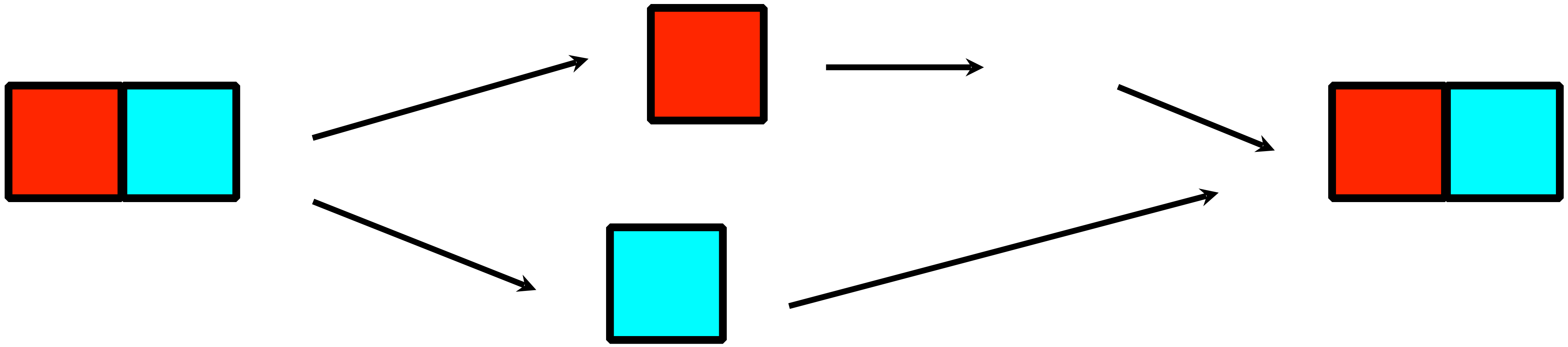

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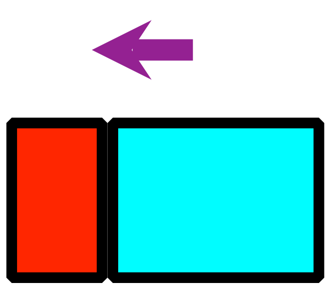

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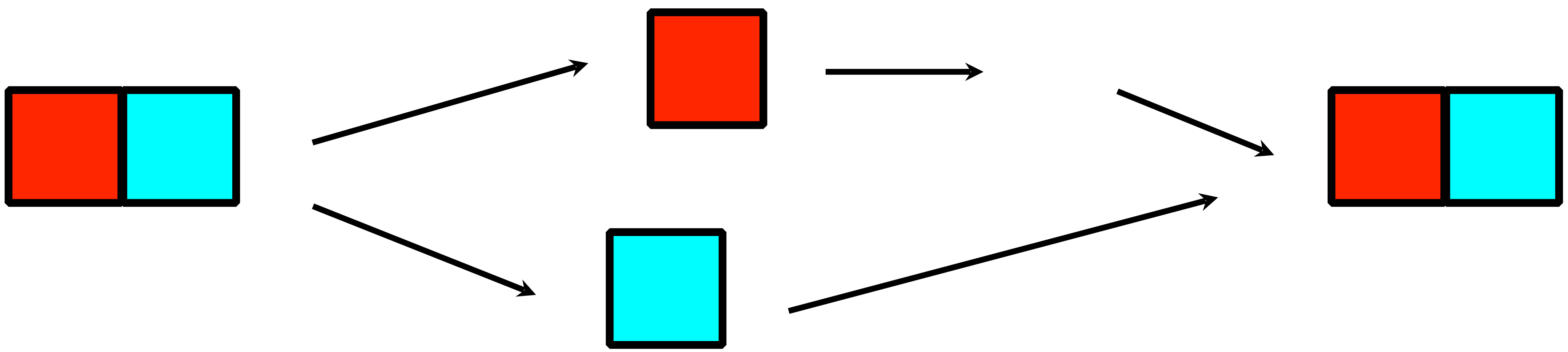

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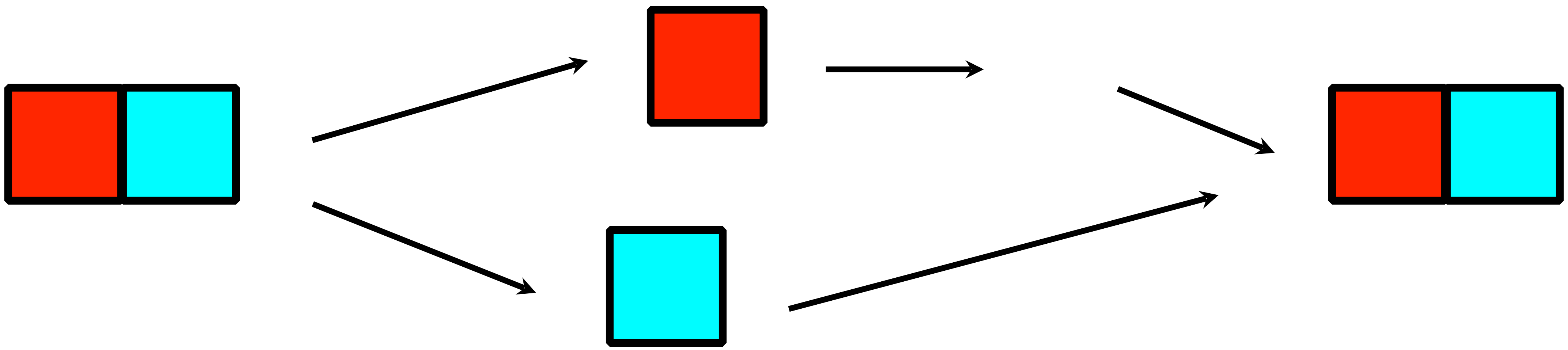

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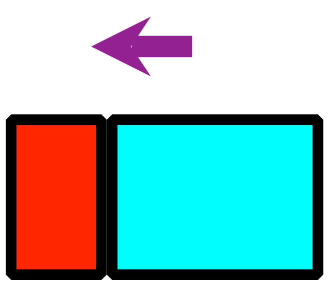

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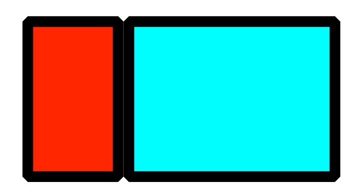

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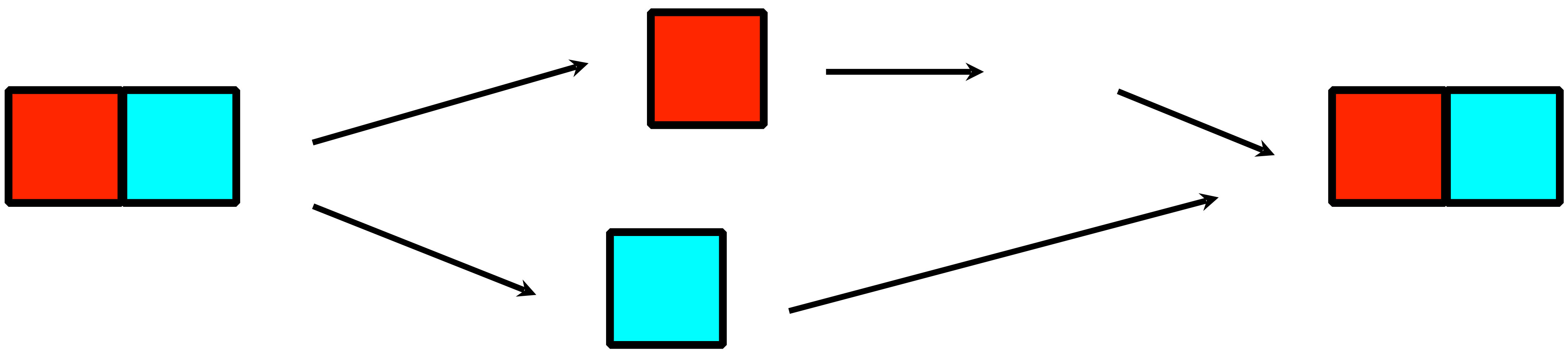

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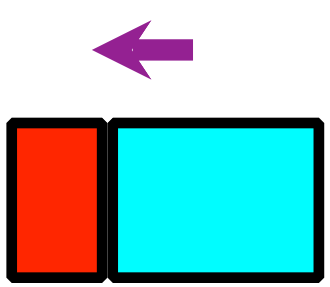

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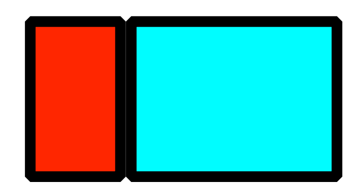
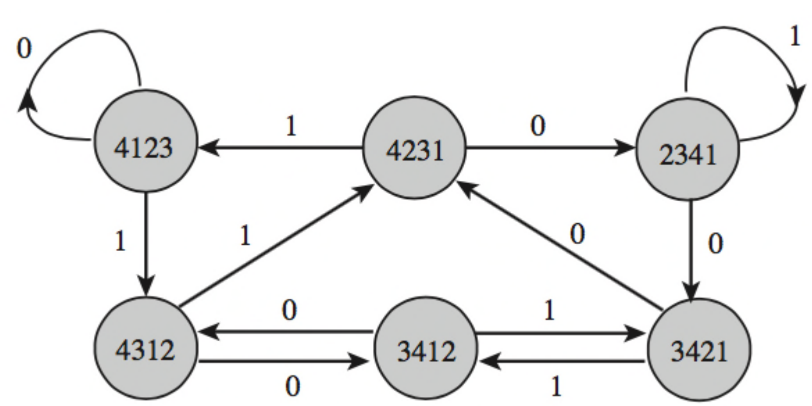


IETs: Rauzy class



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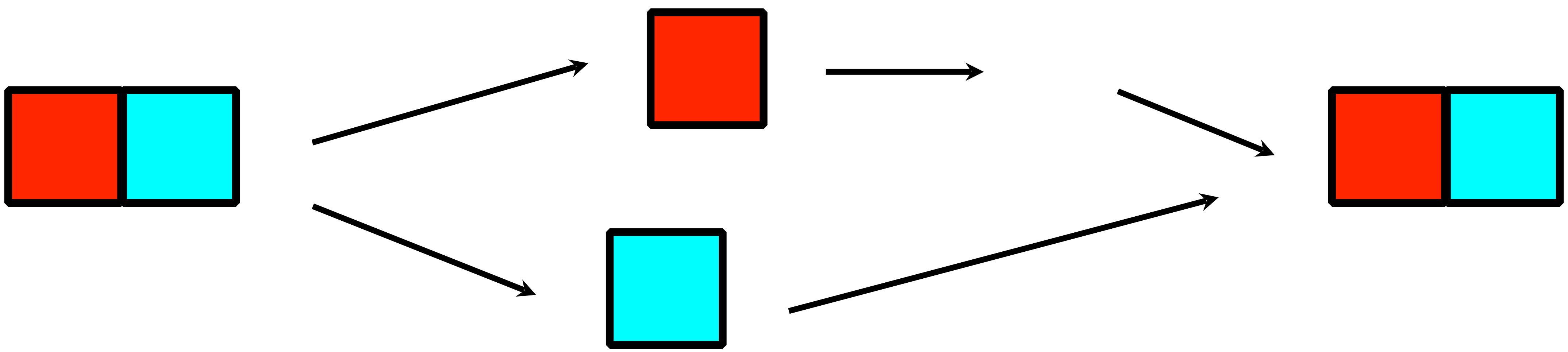

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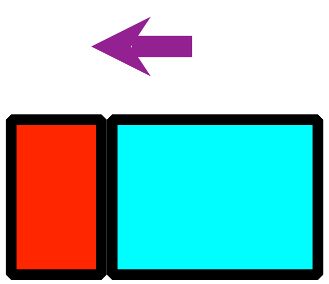

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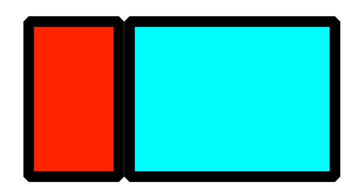

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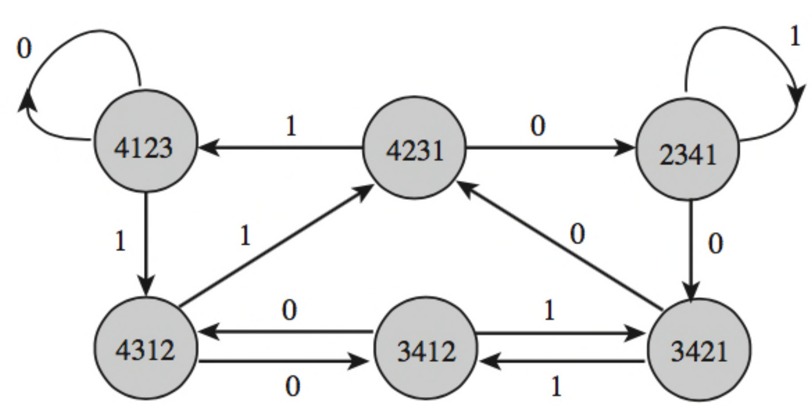

All permutations have a pair
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Two-parameter PETs: degenerate renormalisation (field $\mathrm{K} = \mathbb{Q}(\sqrt{5})$)

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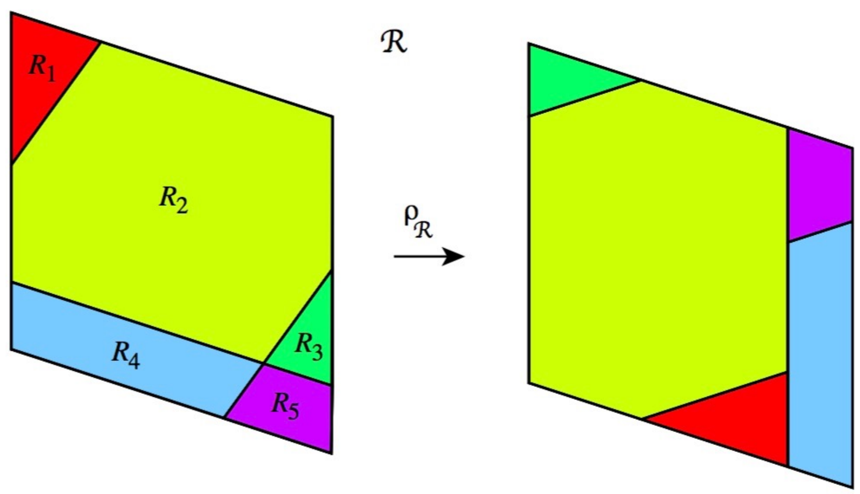

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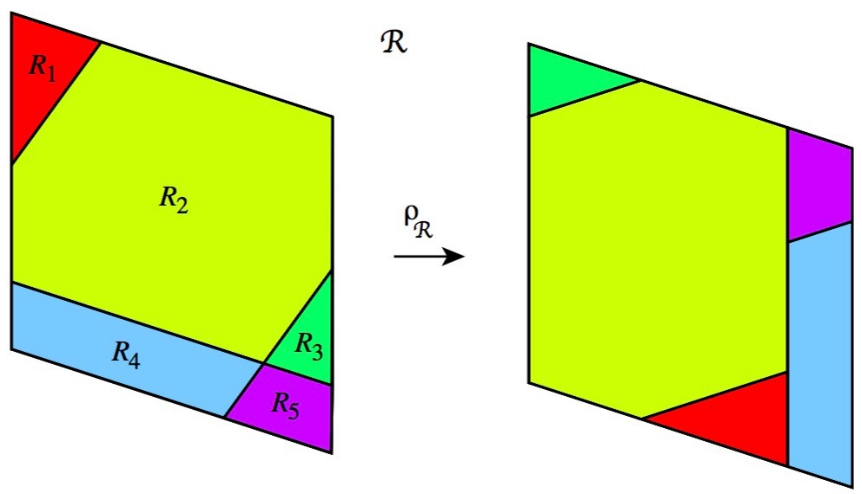

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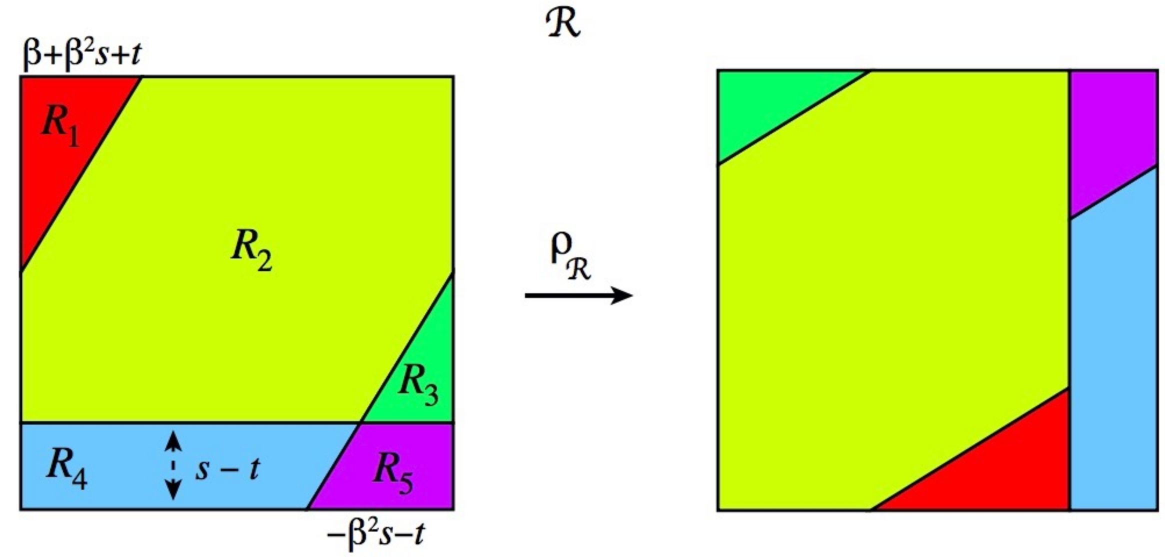

two parameters: s and t

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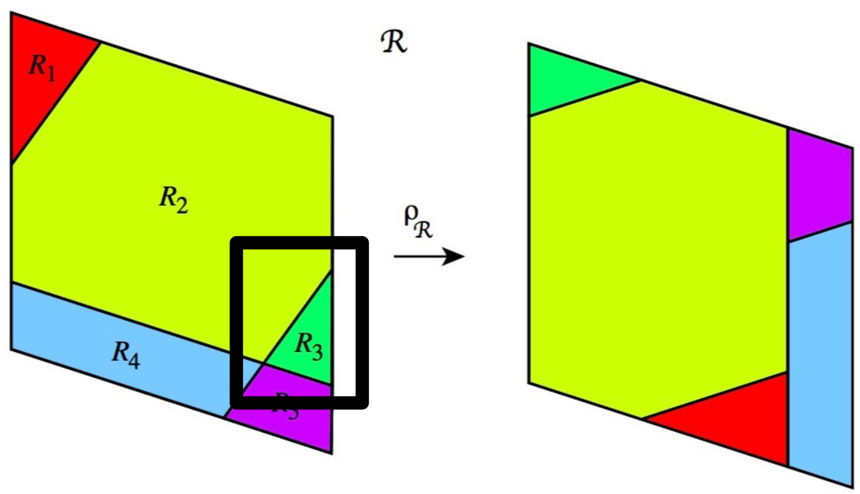

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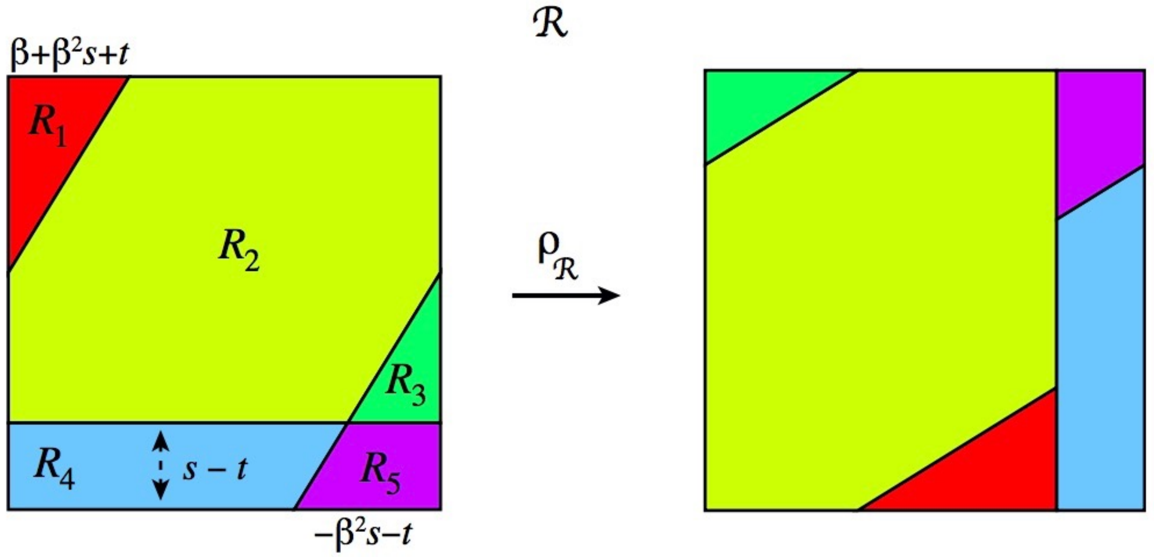

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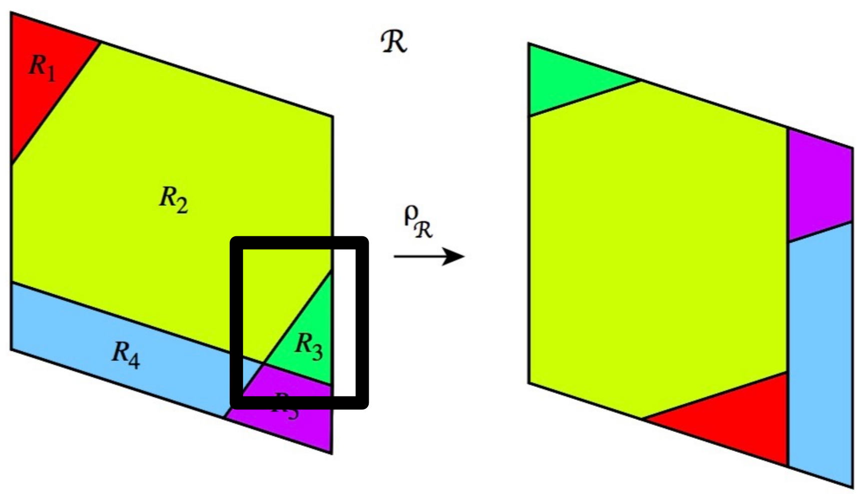

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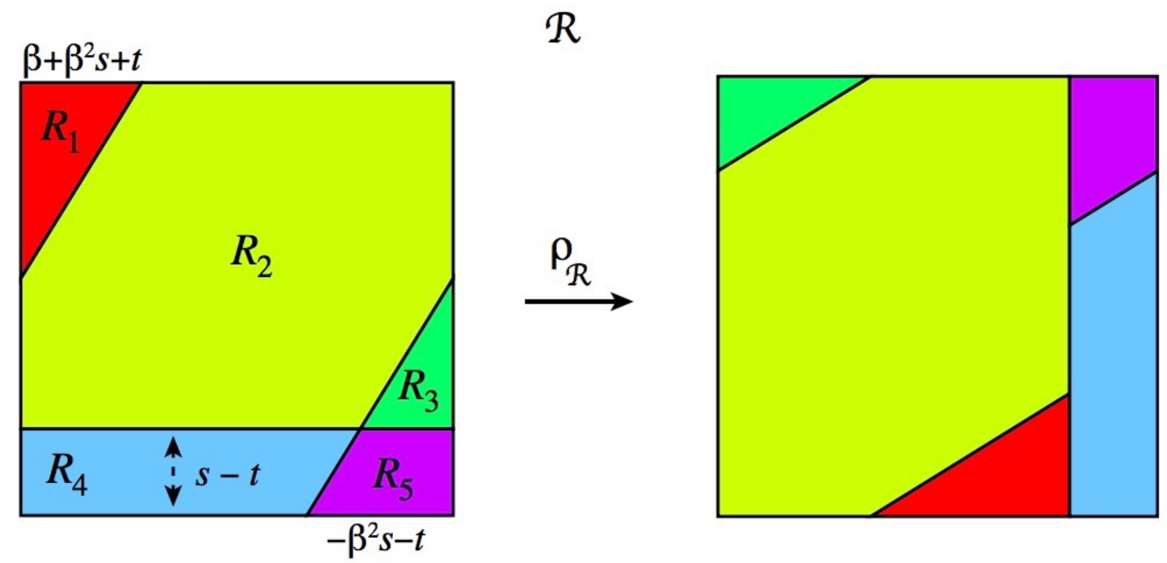

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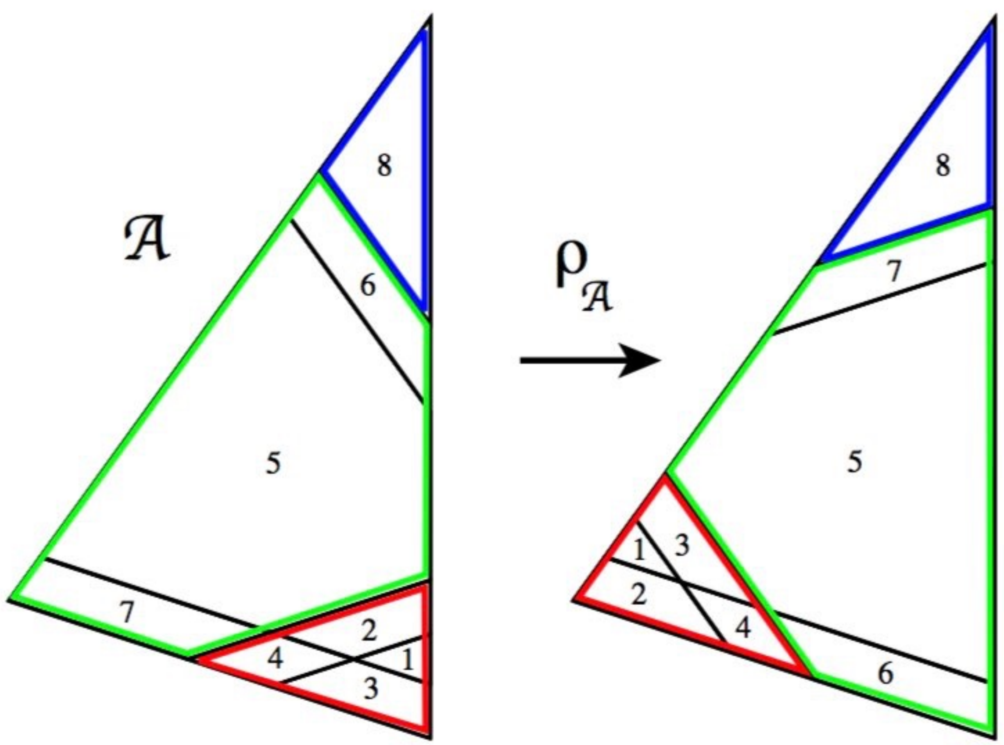

re-combination of atoms

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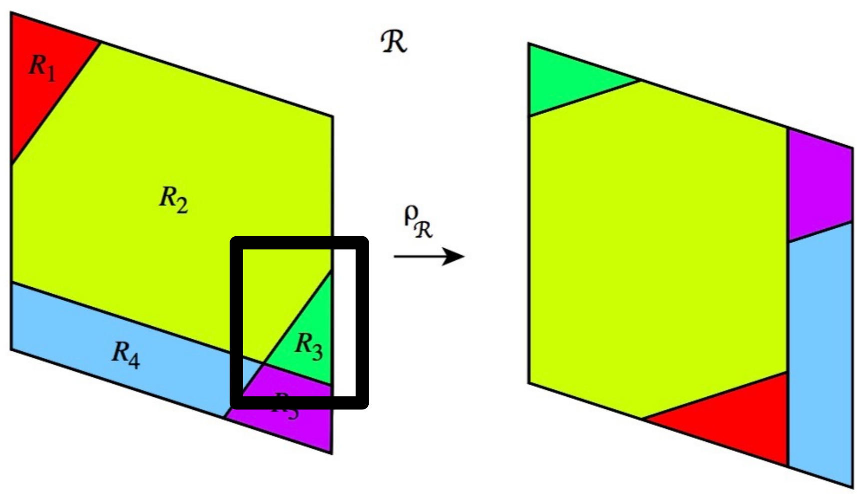

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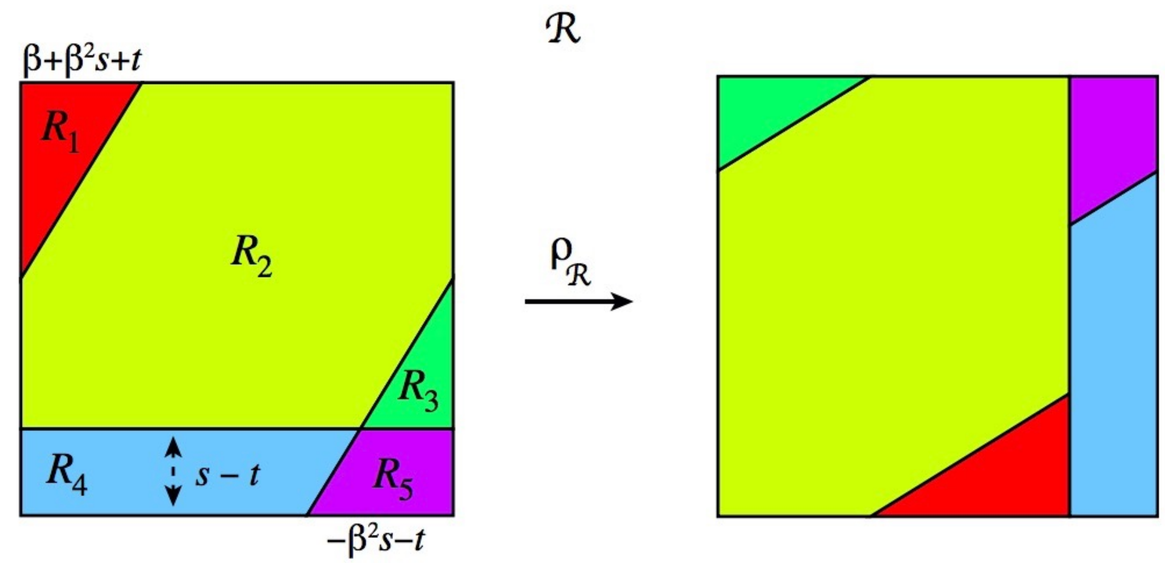

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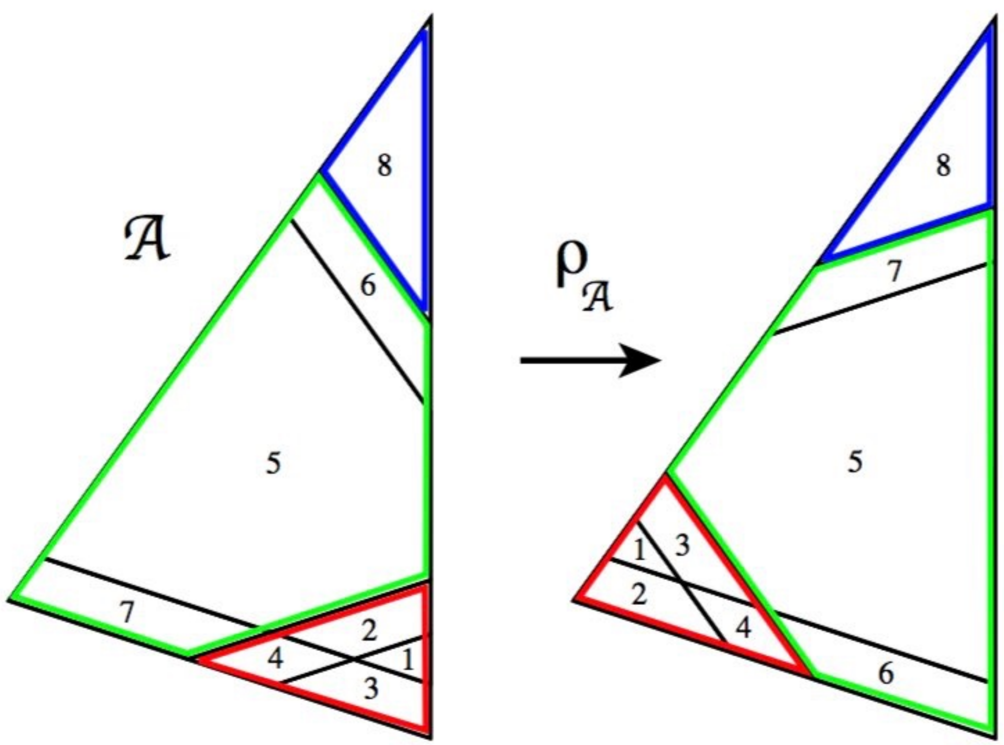

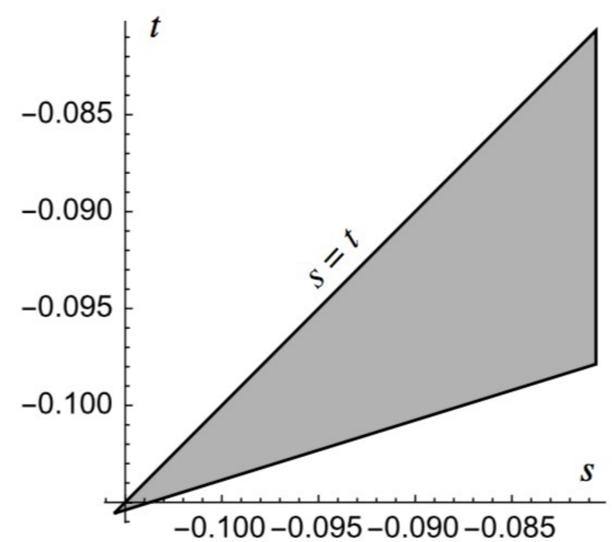

re-combination of atoms

bifurcation-free parametric domain

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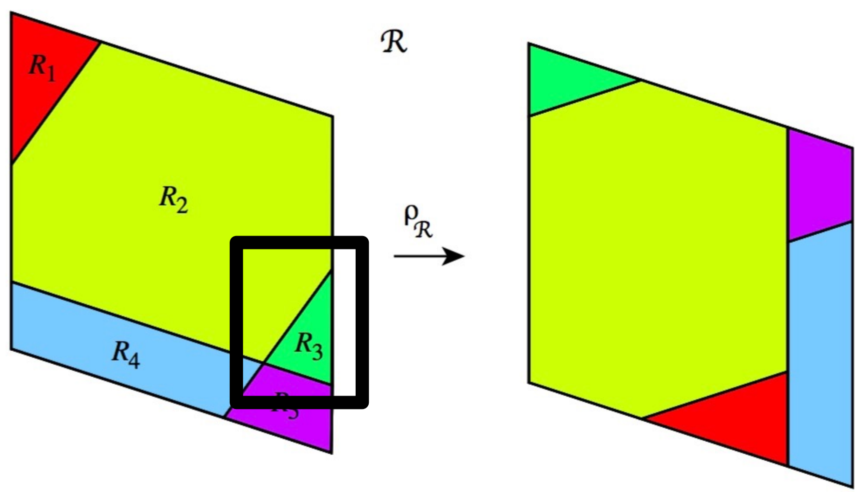

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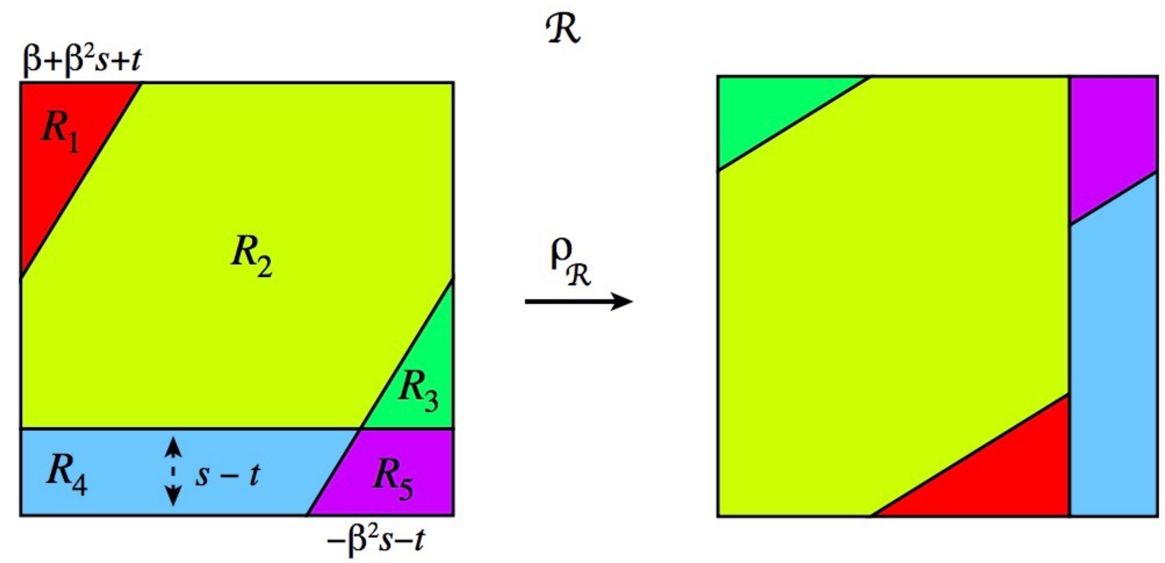

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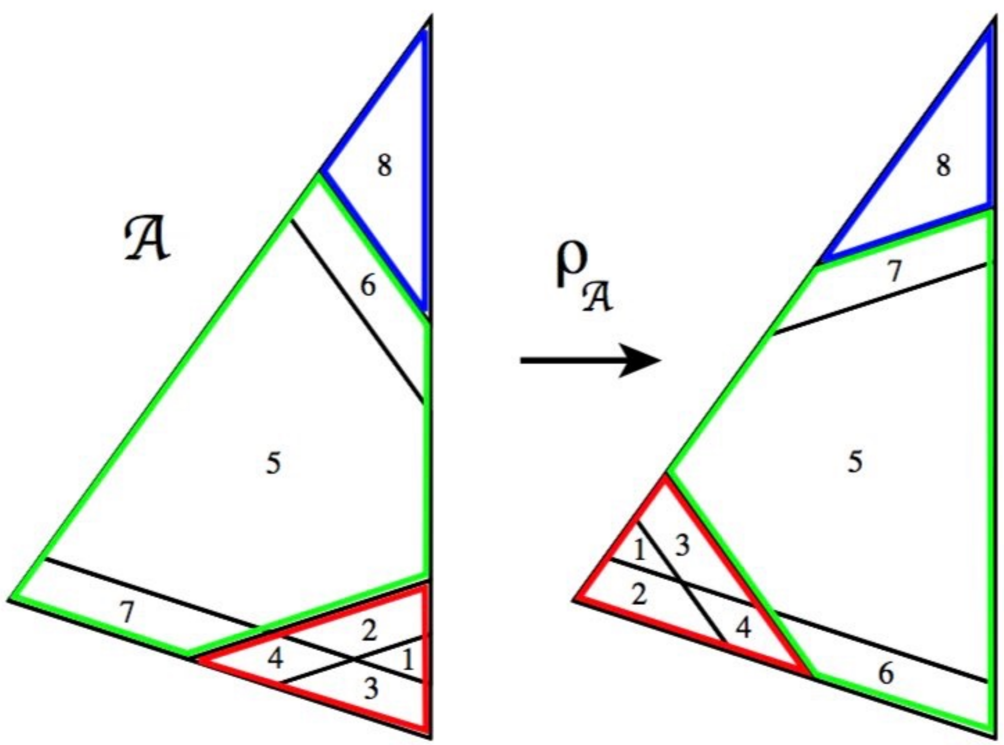
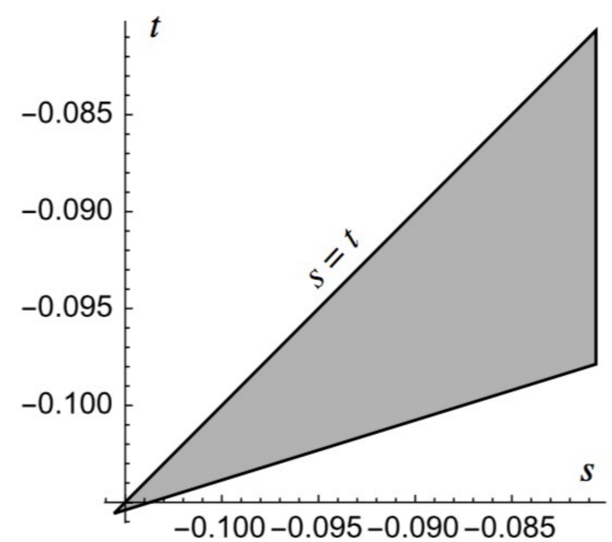


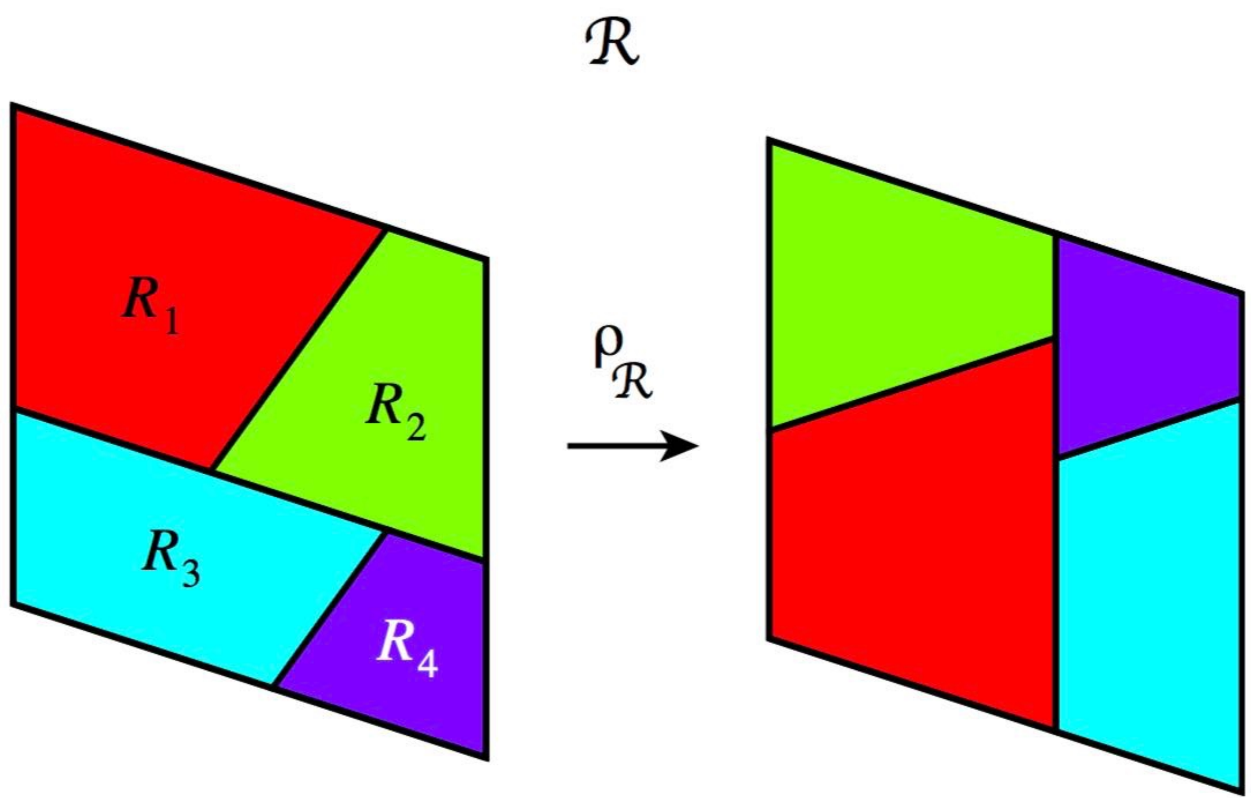


For renormalizability, s must be constrained to K, while t is unconstrained.

re-combination of atoms

bifurcation-free parametric domain

A weakly non-degenerate renormalisation, with two parameters



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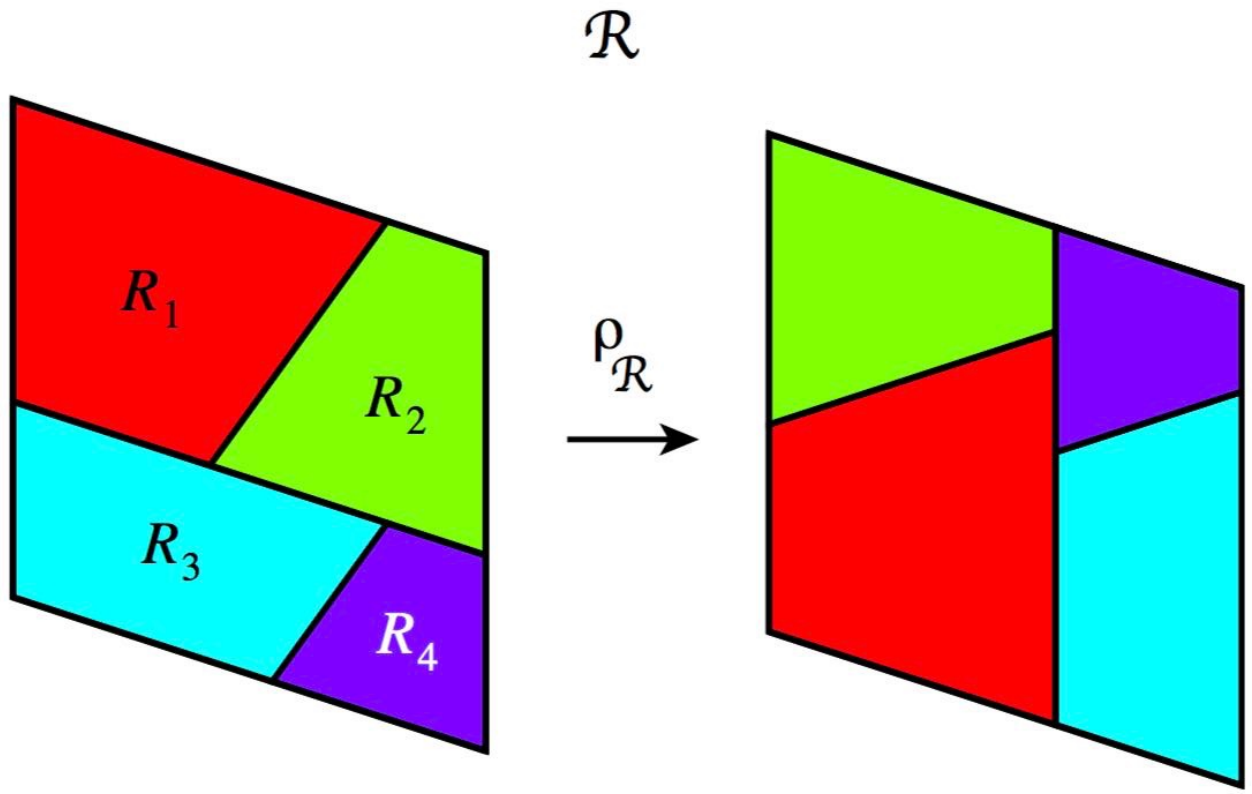

- The renormalization domain splits into three components.

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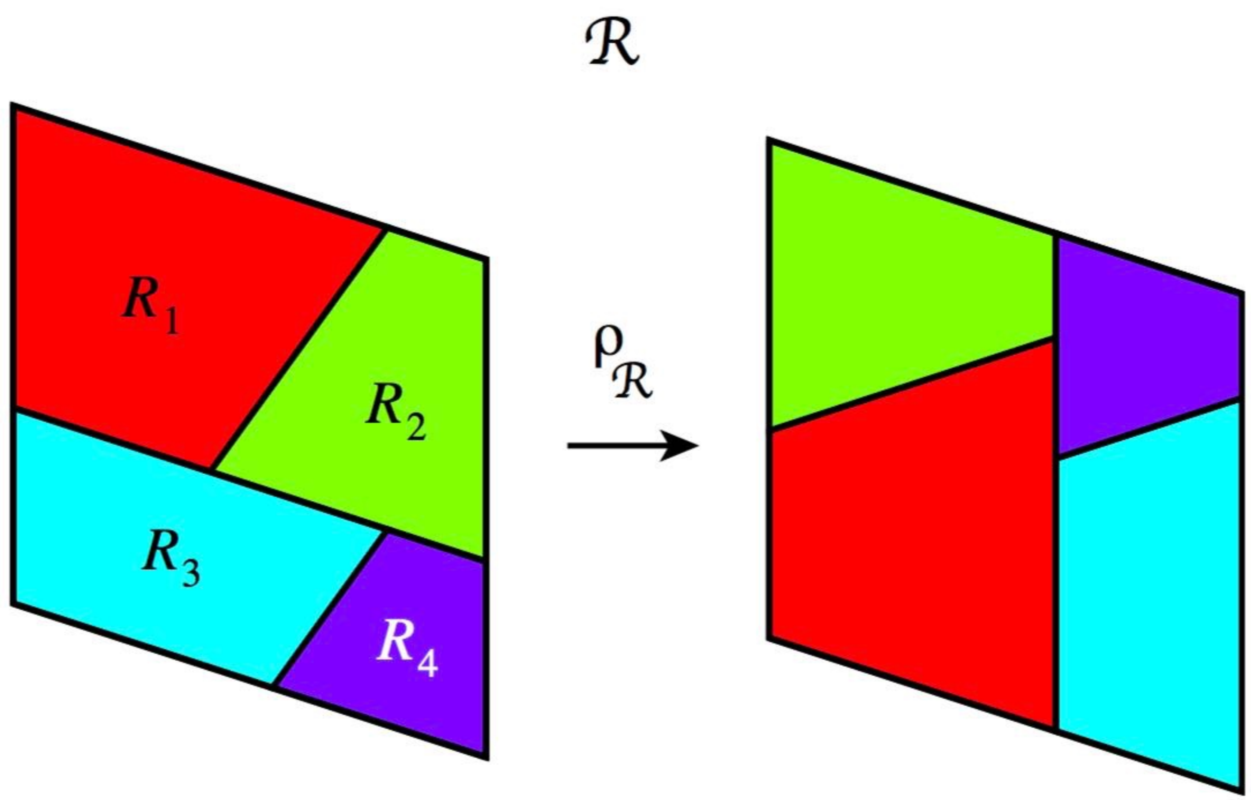

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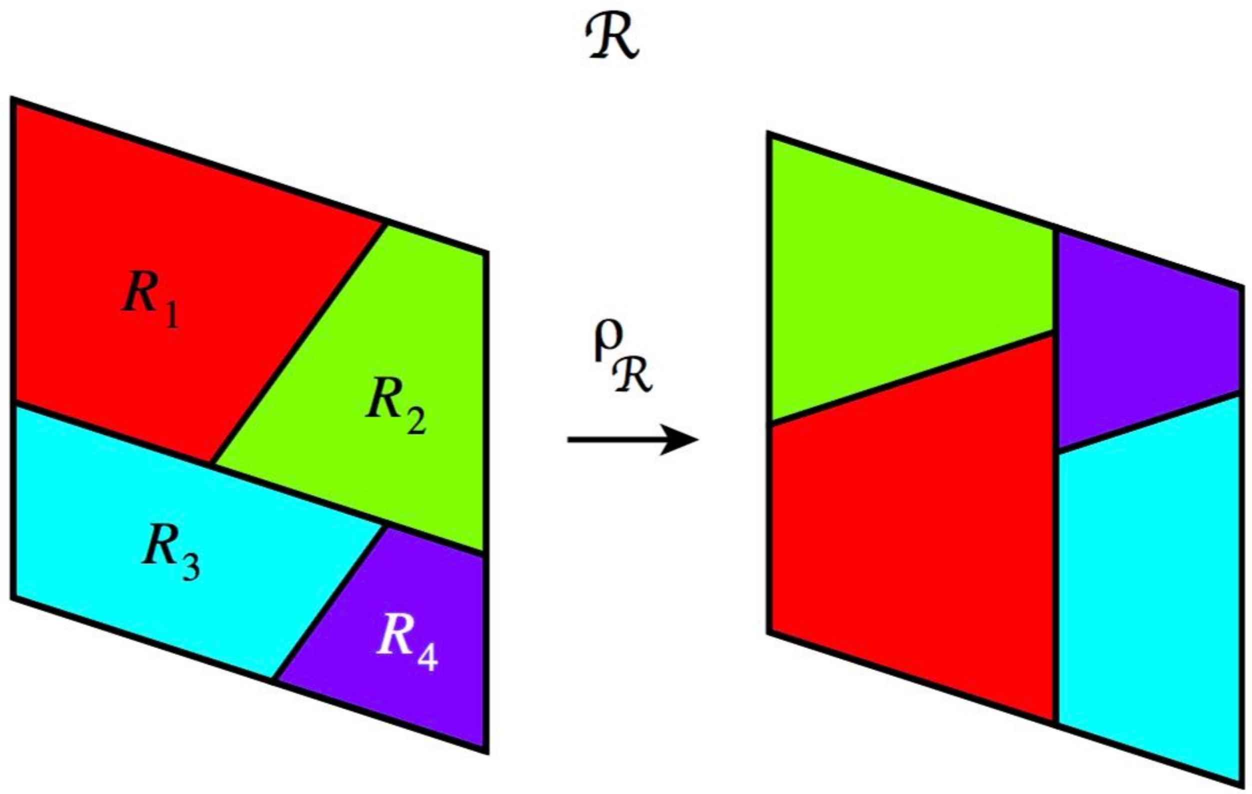

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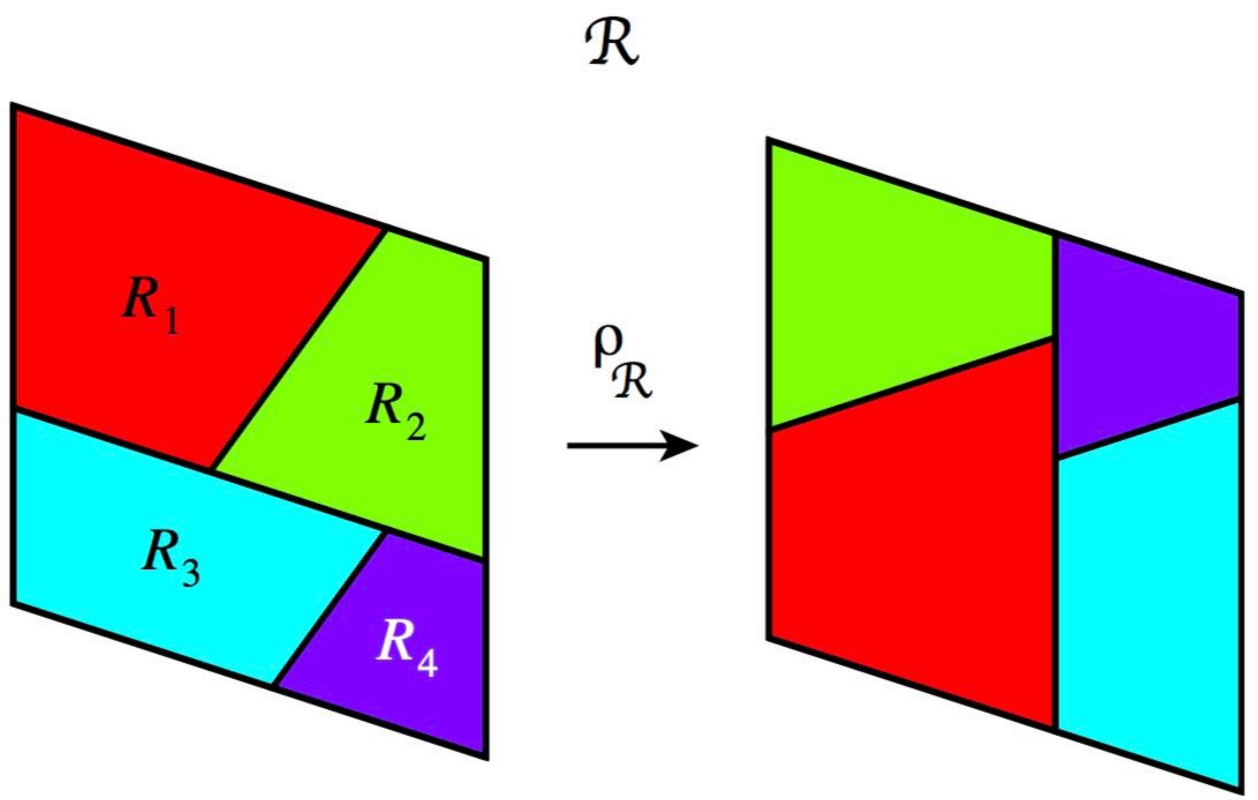

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- However, the free parameters are transversal in parameter space.
- The system is renormalisable iff both parameters belong to K.

Thank you for your attention