Worst-case shape optimization for the Dirichlet energy

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SU UN TIPO DI CONVERGENZA VARIAZIONALE

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Joint work

Worst-case shape optimization for the Dirichlet energy

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Optimization problems:

$$\min\Big\{F(u)\ :\ u\in X\Big\}.$$

Problems in the calculus of variations are of this type, u is the state variable varying in a space of functions and F is usually a functional expressed in an integral form. In this formulation we only observe the system without intervening.

Example:

$$\min\Big\{\int_\Omega j(x,u,\nabla u)\,dx\ :\ u\in H_0^1(\Omega)\Big\}.$$

Optimal control problems:

$$\min\left\{F(u,v)\ :\ u\in X,\ v\in Y,\ u\in\mathcal{A}(v)\right\}.$$

The class $\mathcal{A}(v)$ is usually given through a differential equation (elliptic PDE in our case), called state equation, v is the control variable and represents the way we can operate in the system.

Example:

$$\min_{u\in H_0^1(\Omega),\ v\in L^2(\Omega)}\left\{\int_\Omega |u-u_0|^2+\alpha v^2\,dx\ :\ -\Delta u=f+v\right\}.$$

Let us emphasize the presence of given data in the optimal control problem, that we denote by f. Then the problem is written as

$$\min\Big\{F(u,v)\ :\ u\in\mathcal{A}(v,f)\Big\}.$$

For instance, the example above describes the vertical displacement of a membrane fixed on $\partial\Omega$ under the action of the exterior load f and of an extra load v that we may add.

Consider the case when the data f are only known only up to some degree of uncertainty; nevertheless, we still want to find an optimal solution in some sense.

A possibility (we do not deal with) is to assume that f is known with a probability P; in this case the average cost functional can be optimized and we are in the interesting framework of stochastic optimization.

We want on the contrary consider the **worst case** for f; more precisely, we assume that the data can be perturbed as $f + g$ with $\|g\|_{L^p} \leq \delta$ and we optimize the worst case cost

$$F_{wc}(u, v) = \sup_{\|g\|_{L^p} \leq \delta} \Big\{ F(u, v) \ : \ u \in \mathcal{A}(v, f+g) \Big\}.$$

We are interested in the case when the control is the domain Ω; we are then in the framework of shape optimization problems. Other cases of worst case optimization problems are considered in Allaire-Dapogny (M3AS 2014). We want to show the existence of an optimal domain for

$$\min\Big\{\mathcal{F}(\Omega)\ :\ \Omega\subset D,\ |\Omega|\le m\Big\}$$

where D is a prescribed bounded subset of $\mathbf{R}^d$ and $\mathcal{F}$ is a worst-case functional given by

$$\mathcal{F}(\Omega)=\sup\Big\{F(\Omega,f+g)\ :\ \|g\|_{L^p(D)}\le\delta\Big\},$$

being F a given shape functional.

We start by considering as F the energy functional

$$E(\Omega, f) = \inf_{u \in H_0^1(\Omega)} \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - fu\right) dx.$$

The worst-case functional $\mathcal{F}$ is:

$$\begin{aligned}\mathcal{F}(\Omega) &= \sup_{\|g\|_{L^p(D)} \le \delta} E(\Omega, f+g) \\ &= \inf_{u \in H_0^1(\Omega)} \int_D \left(\frac{1}{2}|\nabla u|^2 - fu\right) dx + \delta \|u\|_{L^{p'}(D)}\end{aligned}$$

and the worst-case shape optimization problem becomes

$$\min \Big\{\mathcal{F}(\Omega) \ : \ \Omega \subset D, \ |\Omega| \le m\Big\}.$$

Before stating the existence of an optimal domain for the problem above let us recall a very general existence theorem based on the Dal Maso-Mosco (AMO 1987) characterization of relaxed Dirichlet problems.

Theorem [Buttazzo-Dal Maso (ARMA 1993)] *Let $F(\Omega)$ be such that:*

- *F is γ-lower semicontinuous;*
- *F is decreasing for set inclusion.*

Then the shape optimization problem

$$\min\Big\{F(\Omega)\ :\ |\Omega|\le m\Big\}$$

admits a solution.

The assumptions above are verified in the worst-case shape optimization problem, and so we have that for every δ and m there exists an optimal domain $\Omega_{\delta,m}$ solving

$$\min\Big\{\mathcal{F}_\delta(\Omega)\ :\ \Omega\subset D,\ |\Omega|\le m\Big\}$$

where

$$\mathcal{F}_\delta(\Omega)=\inf_{u\in H_0^1(\Omega)}\int_D\left(\frac{1}{2}|\nabla u|^2-fu\right)dx+\delta\|u\|_{L^{p'}(D)}.$$

Radial case

We consider the case of a right-hand side f of radial type; more precisely, we assume $f = f(|x|)$ with $f(r)$ decreasing.

Theorem If D is large enough (to contain a ball of measure m) the optimal domain $\Omega_{\delta,m}$ is a ball of measure m (centered at the origin).

Uncertainty only in the state equation

We consider the case of a shape optimal control problem

$$\max_{|\Omega|\le m} \int_\Omega h(x) u_\Omega \, dx$$

where $h \geq 0$ and u_Ω is the solution of

$$-\Delta u = f \quad \text{in } \Omega, \qquad u \in H_0^1(\Omega).$$

We assume that h is perfectly known, while f is uncertain.

Example: best shape for the average temperature under partially known heat sources.

Writing $f+g$ instead of f and taking the worst-case situation, denoting by R the resolvent operator of $-\Delta$, we have the worst case functional

$$\begin{aligned}\mathcal{F}_\delta(\Omega) &= \sup_{\|g\|_p\le\delta} -\int_D hR(f+g)\,dx \\ &= \sup_{\|g\|_p\le\delta} -\int_D \Big(fR(h)+gR(h)\Big)\,dx \\ &= \int_D -f(x)w_\Omega\,dx + \delta\|w_\Omega\|_{L^{p'}(D)}\end{aligned}$$

where

$$-\Delta w_\Omega = h \quad \text{in } \Omega, \qquad w_\Omega \in H_0^1(\Omega).$$

Notice that $\mathcal{F}_\delta$ is **still** γ-lower semicontinuous but it is **not** monotone decreasing. Then the Buttazzo-Dal Maso theorem for the existence of an optimal shape cannot be used. Nevertheless, the following result holds.

Theorem Assume:

- $h \geq 0$ and $h \in L^d(D)$;
- $f \in L^p(D)$ with $p \geq 2d/(d+2)$;
- $f \geq c > 0$ on D.

Then, there exists $\bar{\delta} > 0$ such that for every $0 < \delta \leq \bar{\delta}$, there exists a solution Ω_δ to the worst-case shape optimal control problem.

A numerical example

$$D = [0,1] \times [0,1], \quad p = 2, \quad \delta = 0.25$$
$$f = \begin{cases} 1 \text{ on } [0, \frac{1}{2}] \times [0,1] \\ 2 \text{ on } [\frac{1}{2}, 1] \times [0,1] \end{cases}$$

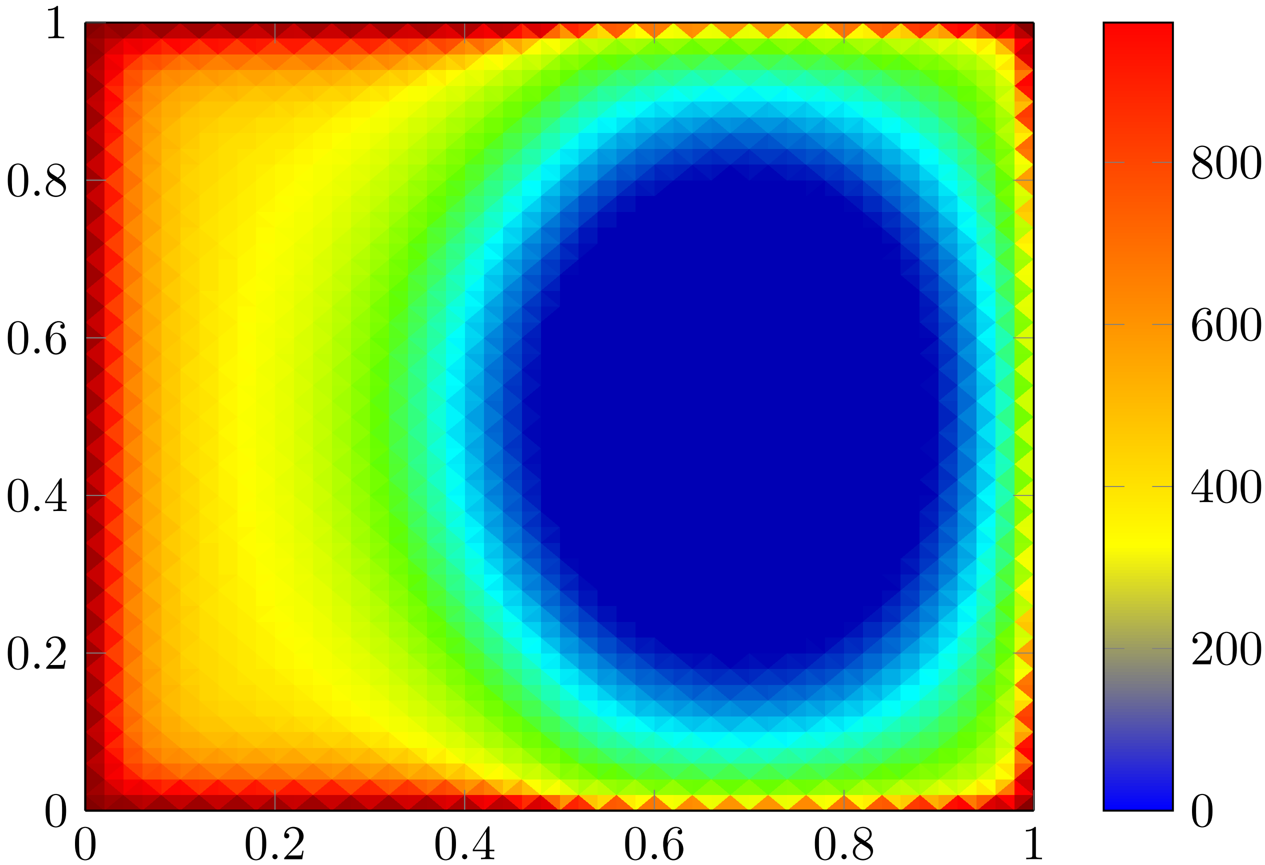
It is numerically convenient to simulate a domain Ω by a potential $V(x)$ taking the value 0 in Ω and $+\infty$ outside. The measure $|\Omega|$ is then simulated through the quantity

$$\int_D e^{-\alpha V(x)}\, dx \quad \text{with } \alpha \text{ small.}$$

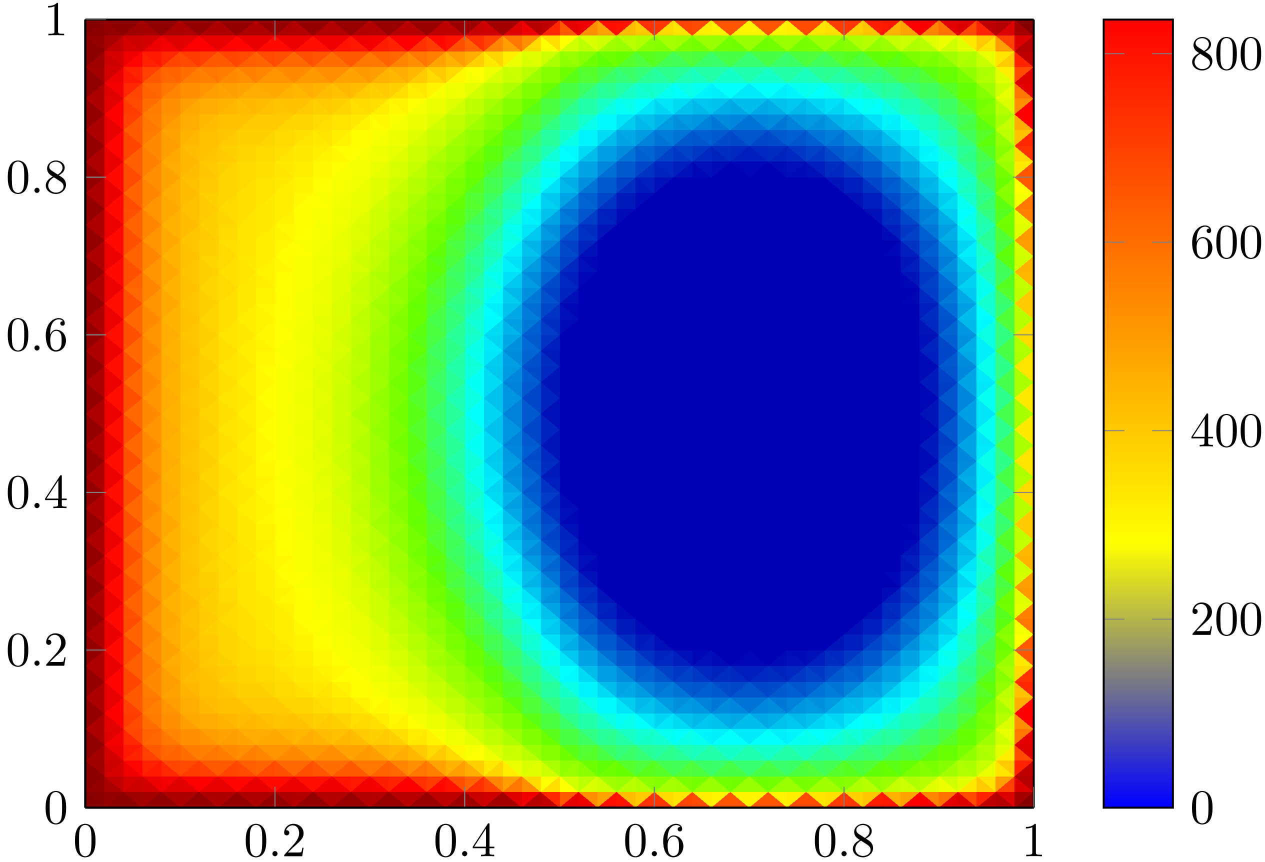
More precisely this approximation has to be stated in terms on Γ-convergence, proved in [BGRV, JEP 2014].

The simulation has been made by J.C. Bellido using:

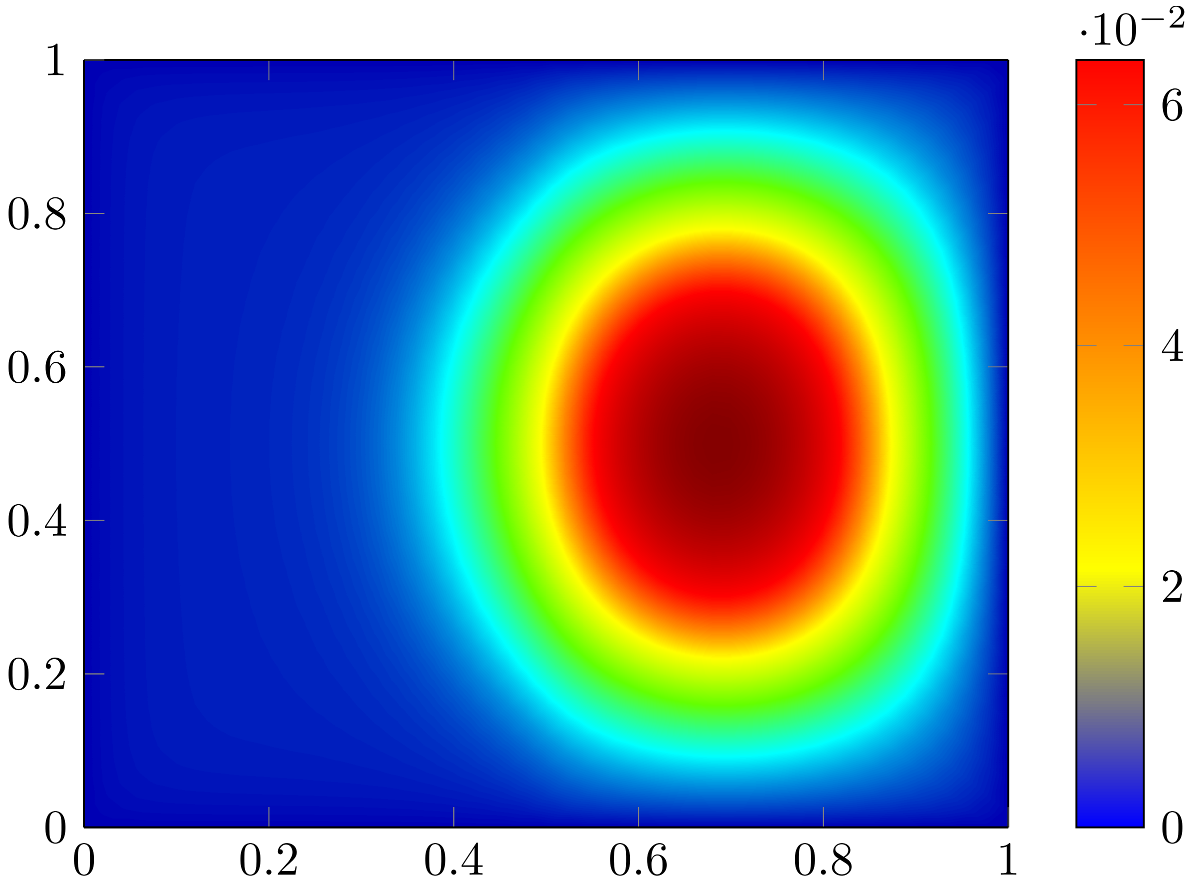
- `FreeFEM++`
- the *Method of Moving Asymptotes* (a kind of gradient method widely used for Topology and Structural Optimization problems)
- a mesh of 50×50 elements.



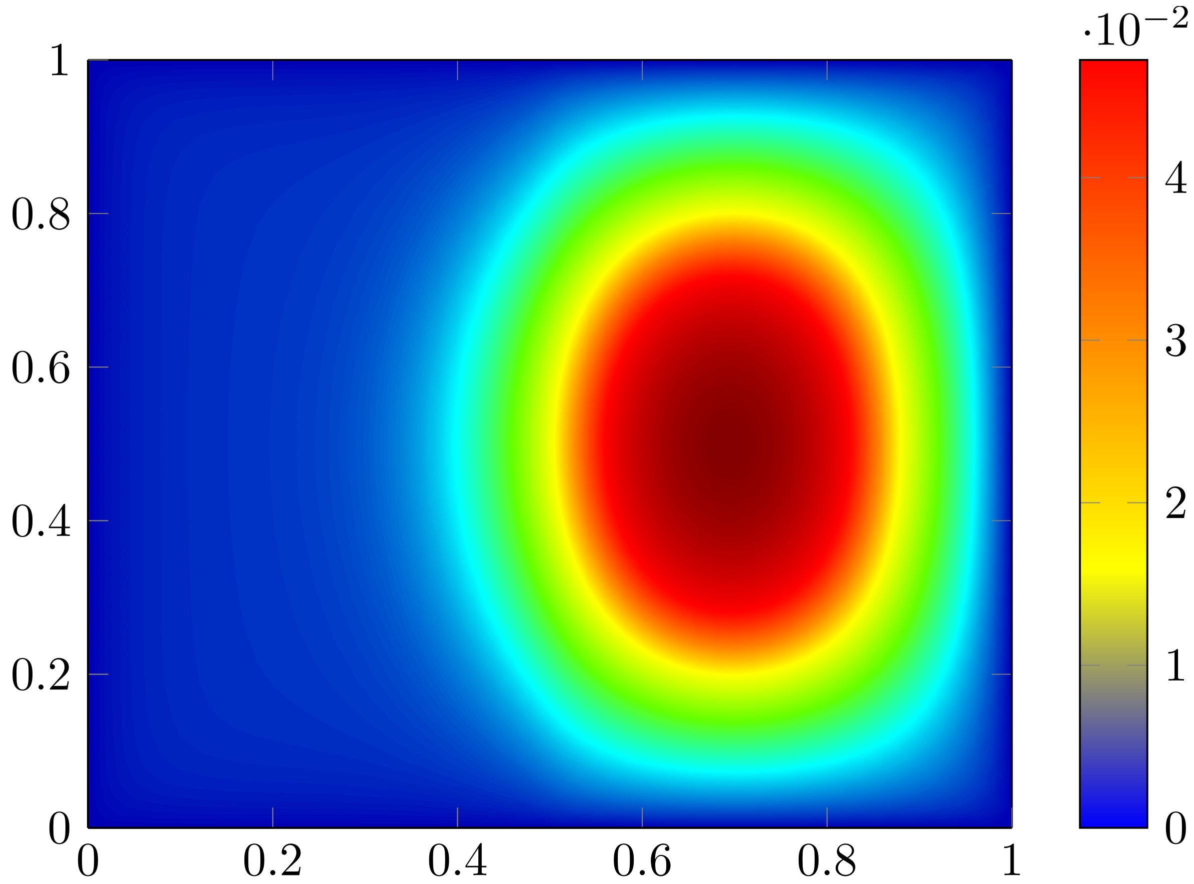
Optimal potential for the unperturbed case



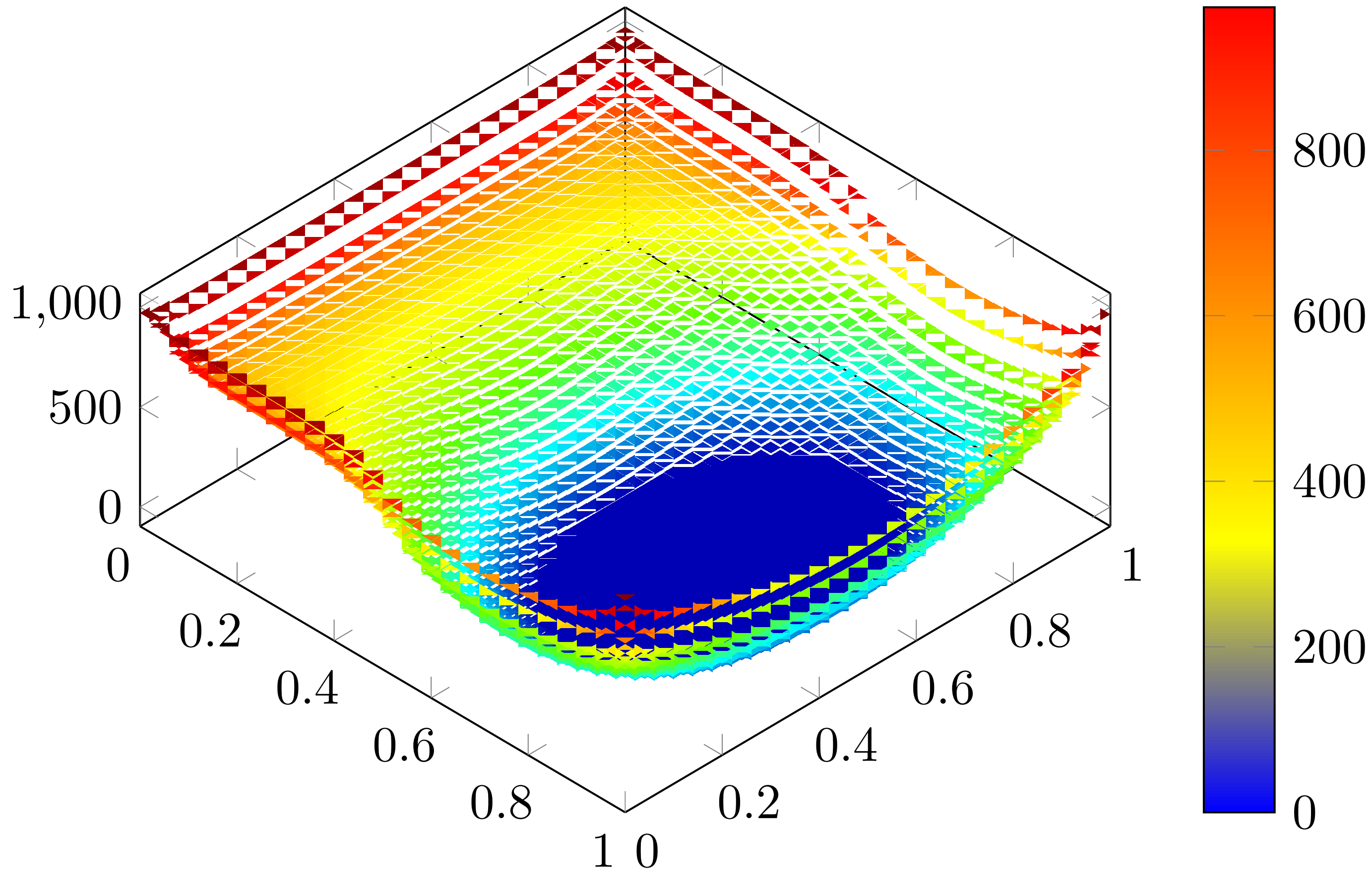
Results for the perturbed case with $\delta = 0.25$



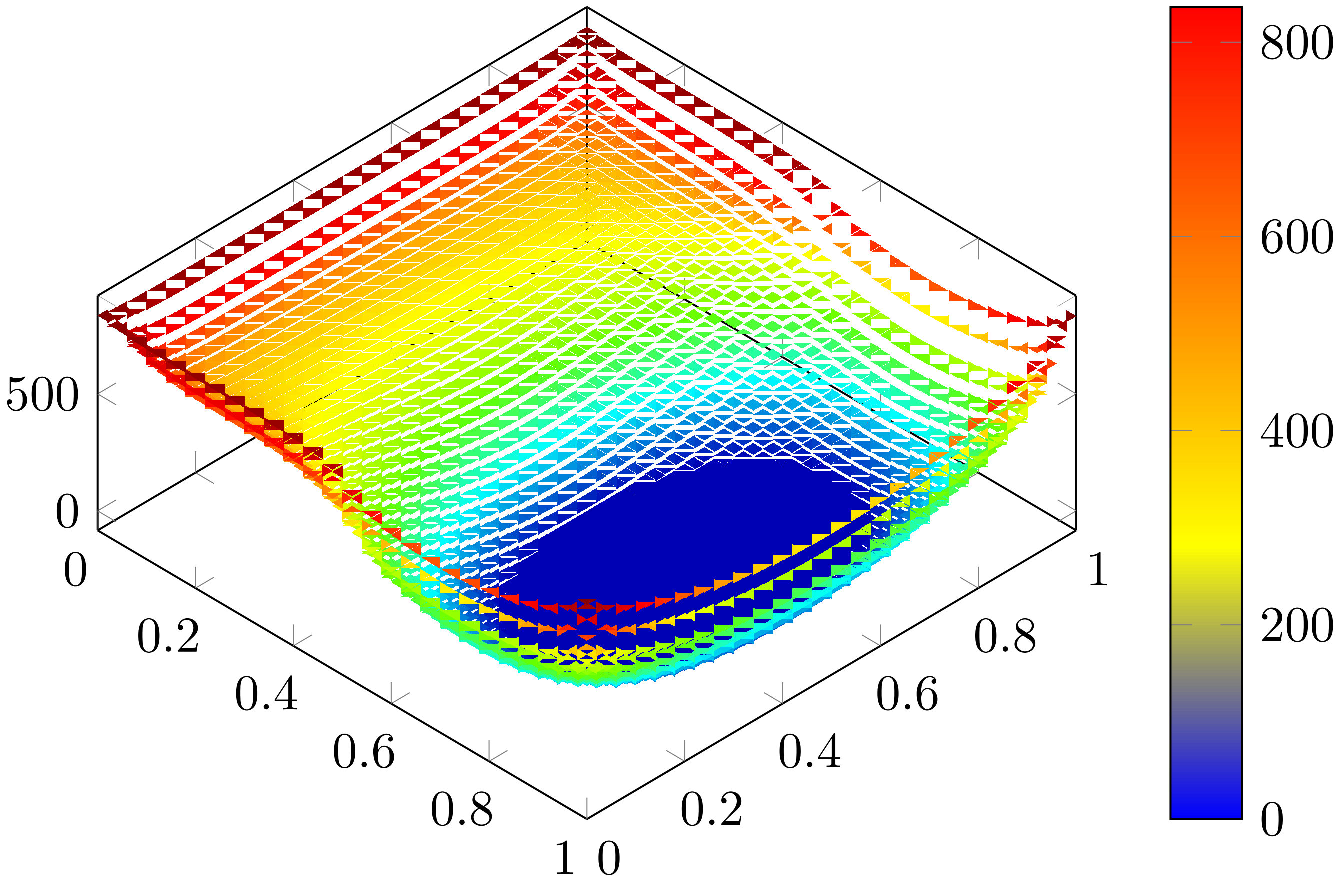
Optimal state for the unperturbed case



Optimal state for the perturbed case with $\delta = 0.25$



Optimal potential (3D view) for the unperturbed case



Optimal potential (3D view) for the case with $\delta = 0.25$

In progress: It would be very interesting to make an asymptotic analysis (often called Γ development) of the sets Ω_δ for δ small.

The expected result is that Ω_δ is (asymptotically) equal to Ω with a boundary layer Σ_δ of local thickness $\delta h(\sigma)$

$$\Sigma_\delta = \left\{ x = t\nu(\sigma),\ \sigma \in \partial\Omega,\ -\delta h^-(\sigma) < t < \delta h^+(\sigma) \right\}$$

for a suitable function h to be characterized, with $\int_{\partial\Omega} h \, d\sigma = 0$.