Min-Max methods for surfaces with boundary

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A classical fact (cf. mountain pass, Morse theory etc.):

Theorem

A function with two local minima has always a third critical point (provided some reasonable assumptions are fulfilled).

Problem

Let Γ be an $n-1$-dim. closed surface in $\mathbb{R}^{n+1}$ for which there exist two distinct strictly stable embedded minimal hypersurfaces Σ_1 and Σ_2 with

$$\partial \Sigma_i = \Gamma\,.$$

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Minimality: zero first variation for any normal perturbation which fixes the boundary.

$$\delta\Sigma(\chi) = \left.\frac{d}{dt}\right|_{t=0} \underbrace{\mathrm{Vol}^n(\Phi_t(\Sigma))}_{=:V(t)}$$

$$\frac{d\Phi_t}{dt} = \chi(\Phi_t)$$

$$\chi = f\nu \qquad \chi = 0 \quad \text{on} \quad \partial\Sigma\,.$$

Stability: nonnegative second variation, $\delta^2\Sigma(\chi) = V''(0) \geq 0$.

Strict stability: $V''(0) > 0$ when $f \neq 0$.

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Main theorems

Yes for $n = 2$ (without embeddedness but control on topology):

- Douglas-Rado approach, Tomi-Tromba, Tromba, etc.;
- Harmonic map flow, Struwe.

$n = 2$ special: existence of conformal parametrization, criticality of the Dirichlet energy.
No result in higher dimension.

Theorem (De Lellis - Ramic, 2016)

YES for $n \leq 6$ if

(C) *Γ lies on the boundary of a bounded, uniformly convex open set.*

(N) *Σ_1 and Σ_2 do not intersect in the interior.*

NB: smoothness up to the boundary. Hence no multiplicity.

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Note: strict stability can be replaced by local minimality.

In fact, an important step of the proof is:

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YES for $n \geq 7$ IF

(C) *Same convexity assumption as before;*

(N) *Same nonintersection assumption;*

(S1) *Σ_1 and Σ_2 are smooth;*

(S2) *Σ_3 is allowed to have an $(n-7)$-dimensional singular set.*

Note:

- (S2) natural because of the celebrated example of Bombieri, De Giorgi and Giusti.
- (S1) not natural because of the same example!

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First obstruction: the proof of the White-type local minimality result needs smoothness.

Assuming local minimality the unnatural assumption (S1) can be removed (not done in the paper for a technical reason: much longer proof, see below).

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Final remarks:

- all results proved in a general ambient Riemannian manifold.
- similar methods work to produce free boundary minimal surfaces (actually easier!), generalizing 2-d works of Grüter-Jost.

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Min-Max construction

Basic (old) idea: min-max method.

Consider all continuous 1-parameter families of surfaces S_t joining Σ_1 and Σ_2.

Local minimality of Σ_1-Σ_2:

$$\max \mathrm{Vol}^n(S_t) \geq \underbrace{\Delta}_{\text{fixed!}} + \max\{\mathrm{Vol}^n(\Sigma_1), \mathrm{Vol}^n(\Sigma_2)\}\,.$$

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More precisely:

- Denote by M_0 the minmax value.
- Let $\{S^j_t\}$ be a minimizing sequence of paths.
- If $\mathrm{Vol}^n(S^{t_j}_j) \to M_0$, then $\{S^{t_j}_j\}$ is a minmax sequence.

Problem

Show the existence of a min-max sequence which converges to a minimal surface Σ.

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A third minimal surface Σ_3

Observe:

- Regularity (and Boundary regularity!) of $\Sigma \Rightarrow \Sigma$ taken with multiplicity 1.
- $\Rightarrow \mathrm{Vol}^n(\Sigma) = M_0$.
- $M_0 > \max\{\mathrm{Vol}^n(\Sigma_1), \mathrm{Vol}^n(\Sigma_2)\} \Rightarrow \Sigma$ is a third minimal surface distinct from Σ_1 and Σ_2.

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Closed minimal submanifolds

The idea goes back at least to Birkhoff in the case of geodesics to produce simple closed geodesics in surfaces diffeomorphic to $\mathbb{S}^2$.

A classical problem: generalize Birkhoff's construction to produce minimal closed (hyper)-surfaces in Riemannian manifolds.

Abstract geometric measure theory: producing a stationary varifold is rather simple (Almgren's pull-tight lemma). The version with fixed boundaries is a minor modification.

REAL ISSUE : regularity!

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Pitts' breakthrough

Pitts' Monograph (1981): Full regularity for $n \leq 5$.

Three fundamental ingredients:

- Rather loose concept of continuity in the parameter, to allow many deformations.
- Local minimality/local stability of the min-max surface.
- Schoen-Simon-Yau curvature estimates for stable hypersurfaces.

Last ingredient reason for $n \leq 5$. Schoen-Simon compactness theorem $\Rightarrow$ extension of Pitts' theory to any n.

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Pitts' and alternative families

Ingredients 2 and 3 rather efficiently treated (see Schoen-Simon, De Lellis-Colding).

Pitts' families: "discretized" one-parameter families of currents, very hard to work with. Big source of technical problems and hard GMT.

De Lellis-Tasnady 2009: alternative proposal

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A generalized family $\{S_t\}$ varies smoothly in the real parameter t except for finitely many t where the smoothness fails at finitely many points of S_t.

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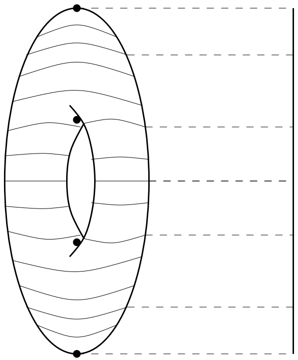

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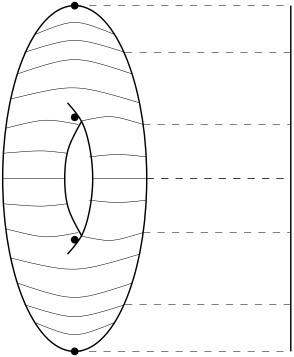

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Theorem (De Lellis - Tasnady 2009)

The alternative families give a quicker and less technical approach to Pitts' existence of closed minimal hypersurfaces in any (closed) Riemannian manifold.

Unfortunately: not clear if as powerful as Pitts' theory (main weakness: Marques-Neves proof of Willmore needs Pitts' theory).

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Technical conditions (N) (nonintersection of Σ_1 and Σ_2) and (S1) (smoothness of Σ_1, Σ_2)
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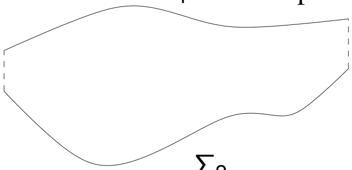
Σ_1 above, Σ_2 below.

Move Σ_1 slightly upward.



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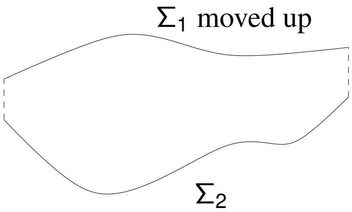

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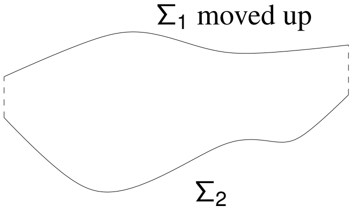

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The real issue

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Almgren's combinatorial lemma

Fix Σ_3, produced by the min-max algorithm (not yet known to be regular).

Pitts: based on Almgren (beautiful combination of analysis and combinatorics).

Lemma

At most points (i.e. except for finitely many) it is impossible to deform continuously Σ_3 decreasing its area.

$\Rightarrow$ At most points Σ_3 is stable

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Pitts' key idea:

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Pitts' strategy is meant for the interior regularity.
"Checklist" at the boundary:

- Almgren-Pitts combinatorial lemma: OK.
- Existence of "local replacements": OK (few technical adjustments).
- Regularity at the boundary: even better! No singularity at the boundary even for $n \geq 7$.
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The min-max surface is contained in the convex hull $\mathrm{ch}\,(\Gamma)$ of its boundary Γ.

$\Gamma \subset \partial\Omega$ and Ω bounded uniformly convex.

$\Rightarrow \mathrm{ch}\,(\Gamma)$ meets $\partial\Omega$ transversally.

For any point $p \in \partial\Omega \cap \mathrm{ch}(\Gamma)$, $\mathrm{ch}(\Gamma)$ is contained in a suitable “wedge” centered at p.

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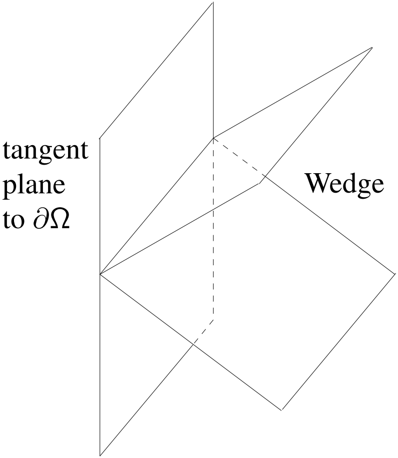

Blow-up strategy

"Classical approach": if the needed compactness statement fails (compactness $\iff$ Regularity) there is a global non planar stable minimal surface Σ in the wedge such that

$$\partial\Sigma = \ell \qquad \text{the tip of the wedge.}$$

For $n = 2$ (Tasnady, PhD thesis 2009):

- reflection of Σ gives a complete minimal surface in $\mathbb{R}^3$;
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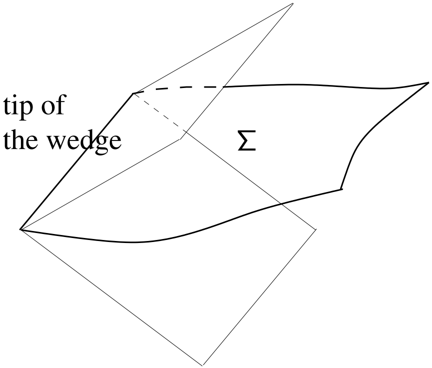

Blow-up – Blow-down strategy

Another classical strategy (Fleming, cf. his talk!).

Monotonicity formula: tangent cone Σ_∞ at infinity (NB: in the sense of varifolds).

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Take at a tangent cone $\Sigma_{\infty,p}$ of Σ_∞ at any point p at the tip.

If $\Sigma_{\infty,p}$ is planar (with multiplicity 1) at most points THEN Σ_∞ is planar with multiplicity 1 (the reason is nontrivial...).

Almgren's stratification theorem: at most points p $\Sigma_{\infty,p}$ is the union of finitely many halfplanes:

$$\Sigma_{\infty,p} = \sum_{i=1}^{m} \pi_i$$

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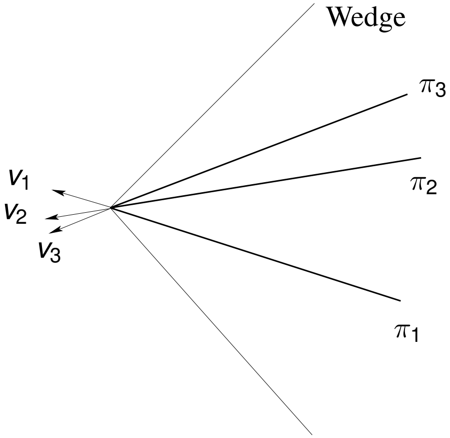

White's trick

Allowing deformations which are nonzero on the tip ℓ

$$\delta\Sigma_{\infty,p} = \sum_{i=1}^{m} v_i \mathcal{H}^{n-1} \llcorner \ell\,.$$

Recall: $\Sigma_{\infty,p}$ is the limit of some classical surfaces S_j with classical boundaries.

- Consequence 1 (elementary intersection theory): m is odd.
- Consequence 2: lower semicontinuity of the first variation

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