A Wiener-Type Condition for Boundary Continuity of Quasi-Minima of Variational Integrals

E. DiBenedetto and U. Gianazza

Pisa, 9-19-2016; A Tribute to E. DeGiorgi

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$$J(u) = \int_E f(x, u, Du)dx$$

if there exists $Q \geq 1$ such that

$$J(u) \leq QJ\big(u + \varphi\big),$$

for all $\varphi \in W^{1,p}_{\text{loc}}(\mathbb{R}^N)$ with $\operatorname{supp}\varphi \subset \bar{E}$. Can also define sub-super Q-minima.

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Significance for Functionals

The significance of a Wiener condition for Q-minima, is that the structure of ∂E near a boundary point $y \in \partial E$, for u to be continuous up to y, hinges on minimizing a functional, rather than solving an elliptic p.d.e.

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$$c_N\big(E^c \cap \bar{B}_\rho\big) = \inf_{\substack{\psi \in W_o^{1,N}(B_2) \cap C_o(B_2)) \\ E^c \cap \bar{B}_\rho \subset [\psi \geq 1]}} \int_{B_2} |D\psi|^N dx.$$

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Theorem

Let u be a Q-minimum for the functional $J(u)$, for $1 < p \leq N$ taking a continuous datum $u = g$ on ∂E. There exists $\epsilon \in (0, 1)$, and $\gamma > 1$, depending only on N, p, Q and the ellipticity ratio C_1/C_o, of $J(\cdot)$, such that at "any point" $0 \in \partial E$, and all $\rho \in (0, 1)$

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If $p > N$ the continuity of u, is insured by the Sobolev embedding theorem.

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The desirable $\epsilon = (p-1)$ still elusive.

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$$\int_{B_\rho \cap E} |Du|^p dx \le Q \frac{C_1}{C_o} \int_{B_\rho \cap E} |D(u - u\varphi)|^p dx,$$

for all non-negative $\varphi \in W_o^{1,p}(B_\rho)$. Since u vanishes on $B_\rho \cap \partial E$, the test function $u\varphi$ is admissible.

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Freedom to play with the parameter $h \geq 0$.

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Result is more general and related to the DeGiorgi classes.

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Let $v \in W^{1,p}(B_{2\rho})$ be non-negative and satisfying

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For (*) to hold only the membership of v to the DG-Classes is needed. We will use only the weak version of (*)

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and the assertion of the theorem follows. Assuming the first holds

$$\big(u - (\mu_{2\rho}^+ - \tfrac{1}{4}\omega_{2\rho}) - (1-k)\tfrac{1}{4}\omega_{2\rho}\big)_+$$

is a non-negative Q-subminimum, for J, in $\bar{B}_{2\rho} \cap \bar{E}$, for all $0 < k \le 1$, **vanishing** on $B_{2\rho} \cap \partial E$.

Renormalizing and the Weak Harnack Inequality-I

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Introduce the renormalization

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$$\int_{B_{2\rho}\cap E} |D\big(w - (1-k)\big)_+|^p \varphi^p dx$$
$$\le \gamma_o \int_{B_{2\rho}\cap E} \big(w - (1-k)\big)_+^p |D\varphi|^p dx,$$

for all non-negative $\varphi \in W_o^{1,p}(B_{2\rho})$, and all $0 \le k \le 1$.

Renormalizing and the Weak Harnack Inequality-II

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By the Corollary of Tolkdsorff lemma

$$\int_{B_{2\rho}} |D\big[\big(w-(1-k)\big)_+ + h\big]|^p \varphi^p dx \\ \le \gamma_o \int_{B_{2\rho}} \big[\big(w-(1-k)\big)_+ + h\big]^p |D\varphi|^p dx.$$

Therefore the previous inequality holds for all $k \ge 0$ and no $h \ge 0$.

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Therefore the previous inequality holds for all $k \ge 0$ and no $h \ge 0$.
Set $v = 1 - w$ and rewrite the previous inequality as

$$\int_{B_r(z)} |D(v-k)_-|^p dx \le \frac{\gamma}{r^p} \int_{B_{2r(z)}} (v-k)_-^p dx, \qquad (v = 1 - w)$$

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for all balls $B_{2r}(z) \subset B_{2\rho}$ and all $k \ge 0$, for a constant γ independent of k, z and r.

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for all balls $B_{2r}(z) \subset B_{2\rho}$ and all $k \ge 0$, for a constant γ independent of k, z and r. Thus the weak Harnack inequality holds for v, i.e.,

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\begin{aligned}
\fint_{B_{2\rho}} v^\epsilon dx &\le \gamma^\epsilon \inf_{B_\rho} v^\epsilon = \gamma^\epsilon \inf_{B_\rho} (1-w)^\epsilon \\
&= \gamma^\epsilon \inf_{B_\rho} \Big(1 - \frac{\big(u - (\mu_{2\rho}^+ - \frac{1}{4}\omega_{2\rho})\big)_+}{\frac{1}{4}\omega_{2\rho}} \Big)^\epsilon \qquad (*) \\
&= \gamma^\epsilon \inf_{B_\rho} \Big(\frac{\mu_{2\rho}^+ - u}{\frac{1}{4}\omega_{2\rho}} \Big)^\epsilon = \gamma^\epsilon \Big(\frac{\mu_{2\rho}^+ - \mu_\rho^+}{\frac{1}{4}\omega_{2\rho}} \Big)^\epsilon .
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Whence ϵ has been identified, (*) continues to hold for a smaller ϵ, with the same γ. (Hölder's inequ.)

Lemma

$$\int_{B_{2\rho}} |D[v^{\frac{\epsilon}{p}} \varphi]|^p \, dx \le \gamma(\epsilon) \int_{B_{2\rho}} v^\epsilon |D\varphi|^p dx \qquad (**)$$

and for all non-negative $\varphi \in W^{1,p}_o(B_{2\rho})$.

Concluding the Proof from (*) and (**)

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$$\int_{B_{2\rho}} |D[v^{\frac{\epsilon}{p}}\varphi]|^p dx \le \gamma' \rho^{N-p} \Big(\frac{\mu_{2\rho}^+ - \mu_\rho^+}{\frac{1}{4}\omega_{2\rho}} \Big)^\epsilon .$$

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Thus $v^{\frac{\epsilon}{p}}\varphi$ vanishes outside $B_{2\rho}$ and equals 1 on $E^c \cap B_{2\rho}$. Therefore (for $1 < p < N$)

$$c_p(E^c \cap \bar{B}_\rho) = \inf_{\substack{\psi \in W_o^{1,p}(\mathbb{R}^N) \cap C(\mathbb{R}^N) \\ E^c \cap \bar{B}_\rho \subset [\psi \ge 1]}} \int_{\mathbb{R}^N} |D\psi|^p dx \le \int_{B_{2\rho}} |D[v^{\frac{\epsilon}{p}}\varphi]|^p dx$$

$$\le \gamma' \rho^{N-p} \Big(\frac{\mu_{2\rho}^+ - \mu_\rho^+}{\frac{1}{4}\omega_{2\rho}} \Big)^\epsilon .$$

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if and only if (Wiener)

$$\int_\rho^1 [\delta(t)]^{\frac{1}{\epsilon}} \frac{dt}{t} \quad \longrightarrow \infty \quad \text{ as } \;\rho \to 0.$$

Lemma Left to be Proven

$$\int_{B_{2\rho}} |D[v^{\frac{\epsilon}{p}}\varphi]|^p \, dx \le \gamma(\epsilon) \int_{B_{2\rho}} v^{\epsilon} |D\varphi|^p dx \qquad (**)$$

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Proposition

There exists $p_o \in (1, p)$, that depends only on $N, p, Q, C_1/C_o$, such that for all $p_o \le q < p$,

$$\int_{B_{2\rho}} v^{-q} |Dv|^p \zeta^p dx \le \gamma \int_{B_{2\rho}} v^{p-q} |D\zeta|^p dx$$

for a constant $\gamma > 1$ that depends only on $N, p, Q, q, p_o, C_1/C_o$.

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Starting Point

$$\int_{B_{2\rho}} |D(v-k)_-|^p \varphi^p dx \le \gamma_o \int_{B_{2\rho}} (v-k)_-^p |D\varphi|^p dx,$$

for all $k \ge 0$, and $\varphi \in W_o^{1,p}(B_{2\rho})$.

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$$\int_{B_{2\rho}} |D(v-k)_-|^p v^{\sigma p} \zeta^p dx \le \gamma \int_{B_{2\rho}} (v-k)_-^p \left[\sigma^p v^{(\sigma-1)p} |Dv|^p \zeta^p + v^{\sigma p} |D\zeta|^p \right] dx.$$

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Multiply by $k^{-\sigma p - q - 1}$ and integrate in dk over $(0,\infty)$.

Integrals in Terms of Distribution Functions

Choose $\varphi = v^\sigma \zeta$ where $\sigma \in (0,1)$ is to be chosen, to get

$$\int_{B_{2\rho}} |D(v-k)_-|^p v^{\sigma p} \zeta^p dx \le \gamma \int_{B_{2\rho}} (v-k)_-^p \left[\sigma^p v^{(\sigma-1)p} |Dv|^p \zeta^p + v^{\sigma p} |D\zeta|^p\right] dx.$$

Multiply by $k^{-\sigma p - q - 1}$ and integrate in dk over $(0,\infty)$. Interchanging the order of integration by Fubini's theorem, the left-hand side equals

$$\int_0^\infty \int_{B_{2\rho}} |D(v-k)_-|^p v^{\sigma p} \zeta^p k^{-\sigma p - q - 1} dx dk = \frac{1}{\sigma p + q} \int_{B_{2\rho}} |Dv|^p v^{-q} \zeta^p dx.$$

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To conclude the proof choose $\sigma \in (0,1)$ such that

$$\gamma \frac{\sigma p + q}{q - (1-\sigma)p} \sigma^p = \frac{1}{2}, \quad \text{and} \quad (1-\sigma)p < q < p.$$