De Giorgi and Geometric Measure Theory

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Plan of the talk

1. Introduction
2. Geometric measure theory
3. Sets of finite perimeter
4. Rectifiable and integral currents
5. Higher dimensional Plateau problem
6. Regularity results
7. Regularity results in codimension 1 and Bernstein's Theorem
8. Remembrances of De Giorgi

1. Introduction

Main goal: outline De Giorgi's seminal contributions to geometric measure theory during the 1950s and 1960s.

These contributions have had a profound effect on the field.

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2. Geometric measure theory

k-dimensional measure and integration in euclidean $\mathbb{R}^n$ for $k < n$.

Important aspects include:

(a) Theory of k-dimensional Hausdorff measure $\mathcal{H}^k(K)$ of $K \subset \mathbb{R}^n$. Rectifiable sets differ in arbitrarily small $\mathcal{H}^k$ measure from finite unions of pieces of C^1 k-dimensional submanifolds of $\mathbb{R}^n$. They are the "well behaved" sets with $\mathcal{H}^k(K) < \infty$.

(b) Theories of k-dimensional integration without smoothness assumptions:

- Sets of finite perimeter (De Giorgi)
- Rectifiable and integral currents (Federer-Fleming)

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(c) Geometric problems in the Calculus of variations
Higher dimensional Plateau problem.

References

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De Giorgi	Selected papers (2006)
Morgan	Beginners guide to GMT (3rd Ed., 2000)
Fleming	GMT at Brown in the 1960s (2015)
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The classical Gauss-Green Theorem:

$E \subset \mathbb{R}^n$ bounded open set with smooth boundary B;
$\zeta \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$:

$$\int_E \operatorname{div} \zeta(x)\, d\mathcal{H}^n(x) = \int_B \zeta(y) \cdot \nu(y)\, d\mathcal{H}^{n-1}(y), \tag{3.1}$$

($\nu(y)$ = exterior unit normal at $y \in B$).

3. Sets of Finite Perimeter

How to make sense of the Gauss-Green formula without smoothness assumptions on the topological boundary?

De Giorgi's program (1954 - 1955).

(a) Require only that E is a "set of finite perimeter" $P(E)$.
(b) In (3.1), replace B by a set $B_r \subset B$ called the "reduced boundary."
(c) Show that B_r is a k-rectifiable set and that there is an "approximate normal" unit vector $\nu(y)$ at each $y \in B_r$.

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De Giorgi's program: (a)

Consider the indicator function

$$\mathbf{1}_E(x) = \begin{cases} 1 & \text{for } x \in E \\ 0 & \text{for } x \notin E \end{cases}$$

Let $\Phi = -\operatorname{grad} \mathbf{1}_E$ (in the Schwarz distributional sense).

Definition (Sets of finite perimeter)

E is a set of finite perimeter if Φ is a measure ($\mathbb{R}^n$-valued).

Definition (Perimeter)

$P(E) = \mu(\mathbb{R}^n)$ with μ total variation measure of Φ.

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Approximation of E in $\mathcal{H}^n$ measure by polygonal domains E_j, $j = 1, 2, \ldots$.

Theorem (Equivalence with Caccioppoli's definition)

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De Giorgi's program: (b) and (c)

$I(y,\rho)$ spherical ball in $\mathbb{R}^n$, center y and radius ρ.

Definition (Reduced boundary)

y belongs to the reduced boundary B_r of E if

$$\lim_{\rho \downarrow 0} \frac{\mathcal{H}^n(I(y,\rho) \cap E)}{\mathcal{H}^n(I(y,\rho) \setminus E)} = 1 \, .$$

The approximate exterior unit normal ν is the Radon-Nykodim derivative of Φ with respect to μ:

$$\nu := \frac{d\Phi}{d\mu} \, .$$

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3. Sets of Finite Perimeter

De Giorgi's program completed by his fundamental

Theorem (De Giorgi's structure Theorem)

- *B_r is rectifiable*
- $\mu(K) = \mathcal{H}^{n-1}(B_r \cap K)$ *for any (Borel)* K*;*
- ν *is normal to* B_r $\mathcal{H}^{n-1}$*-almost everywhere;*
- *the classical Gauss-Green formula holds.*

4. Rectifiable and integral currents

Federer-Fleming (1960)
De Rham's theory of currents (1955)

$\mathcal{D}_k$ = space of smooth forms ω of degree k with compact spt.

Definition (Currents)

A current T of dimension k is a linear functional on $\mathcal{D}_k$, continuous in the Schwarz topology.

Definition (Boundary and mass)

$$\partial T(\omega) := T(d\omega)\,.$$

$$\mathbf{M}(T) = \sup\{T(\omega) : \|\omega\| \leq 1\}$$

($\|\cdot\|$ a (suitable) sup norm).

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From Sets of Finite Perimeter to Currents.

For $k = n$, identify a set E of finite perimeter with a current U of dimension n:

$$U(f\,dx_1 \wedge \ldots \wedge dx_n) = \int_E f(x)\,d\mathcal{H}^n(x)\,.$$

Theorem

$T = \partial U$ corresponds to (integration over) the reduced boundary B_r, oriented by the approximate normal ν. In particular

$$\begin{aligned} \mathbf{M}(U) &= \mathcal{H}^n(E) \\ \mathbf{M}(\partial U) &= \mathcal{H}^{n-1}(B_r) = P(E)\,. \end{aligned}$$

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4. Rectifiable and integral currents

If E_j are polygonal domains approximating E as above (and U_j denote the corresponding currents), then, as $j \uparrow \infty$:

$$\mathbf{M}(U_j - U) \to 0 \qquad \text{(strong convergence)}$$

$$T_j = \partial U_j \rightharpoonup T = \partial U \qquad \text{weakly.}$$

Oriented cells as currents

$S \subset \mathcal{M}$ k-dimensional C^1 submanifold of $\mathbb{R}^n$.

$v_1(y), \ldots, v_k(y)$ a basis for the tangent space to $\mathcal{M}$

NB: order of this basis = orientation

T_S current of dimension k induced by S:

$$T_S(\omega) = \int_S \omega \qquad \forall \omega \in \mathcal{D}_k$$

$$\mathbf{M}(T_S) = \mathcal{H}^k(S)\,.$$

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If S has piecewise smooth boundary C (and T_C denotes the corresponding current), then the k-dimensional version of Stokes formula gives

$$\partial T_S = T_C$$

Definition (Rectifiable currents)

A current T is rectifiable if

- For every $\varepsilon > 0$ $\exists T_\varepsilon$ finite union of oriented cells s.t.

$$\mathbf{M}(T_\varepsilon - T) < \varepsilon\,.$$

To each rectifiable current T corresponds a rectifiable set K and positive integer valued "multiplicity function" $\Theta(x)$ such that

$$\mathbf{M}(T) = \int_K \Theta(x)\, d\mathcal{H}^k(x)\,.$$

$\mathbf{M}(T)$ is also called the k-area of T.
The support of T is denoted by $\mathrm{spt}(T)$.

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T is an integral current if both T and ∂T are rectifiable.

For integral currents we let

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Theorem (Closure theorem, Federer-Fleming 1960)

IF T_j is a sequence of integral currents such that

(a) $\operatorname{spt}(T_j)$ *is contained in a fixed compact set K;*

(b) $\mathbf{N}(T_j) = \mathbf{M}(T_j) + \mathbf{M}(\partial T_j)$ *is uniformly bounded;*

(c) $T_j \rightharpoonup T$ *as $j \uparrow \infty$ (weak convergence);*

THEN T is an integral current.

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5. Higher dimensional Plateau problem

Problem (Plateau)

Find a surface S of least area with given boundary C.

Classical Plateau: 2-dimensional surfaces S in $\mathbb{R}^3$.

Douglas and Rado independently gave solutions (1930s) for surfaces of the "topological type of a circular disk".

Douglas received the Fields medal in 1936

Parametric representation of a surface S:
$D \subset \mathbb{R}^2$ a circular disk, $f : D \to \mathbb{R}^3$.
Restriction of f to boundary of D a parametric representation of C.

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Problem (Parametric Plateau)

Find f which minimizes the area integral on D.

The Douglas-Rado result was extended by Douglas and Courant to surfaces bounded by a finite number of curves and of prescribed Euler characteristic.

These results in dimension $k = 2$ dependend on conformal representations of surfaces, and also on prescribing the topological type of the comparison surfaces.

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5. Higher dimensional Plateau problem

Several different formulations:

(a) Reifenberg (1960).
"Surface" is a closed set with $\mathcal{H}^k(S) < \infty$.
$B \subset S$ the "boundary of S" – defined in terms of Čech homology groups.

Theorem

Given B a set S^* which minimizes $\mathcal{H}^k(S)$ exists.
S^* is topologically a k-disk near $\mathcal{H}^k$–a.e. nonboundary point of S.

Reifenberg's methods – invented "from scratch" and required amazing ingenuity.

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(b) Oriented Plateau problem (FF 1960)

Problem

Given an integral current B of dimension $k-1$ with $\partial B = 0$, find an integral current T^* minimizing the k-area $\mathbf{M}(T)$ among all integral currents of dimension k with $\partial T = B$.

$\mathbf{M}(T)$ is lower semicontinuous under weak convergence.

Closure theorem implies existence of minimizing T^*.

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(c) Nonoriented versions

Reifenberg (see part (a))

Ziemer, Fleming – formulation in terms of Whitney's flat chains with coefficients in $\mathbb{Z}_2$

Almgren – varifold formulation.

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Reifenberg (see part (a))

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Almgren – varifold formulation.

Consider the oriented Plateau problem.

T^* minimizes $\mathbf{M}(T)$ among integral currents of dimension k with fixed $\partial T = S$.

Problem

Show that $\operatorname{spt}(T^*) \setminus \operatorname{spt}(\partial T^*)$ *is locally a smooth manifold of dimension* k*, except at points of a lower dimensional singular set.*

6. Regularity results

Federer (1965) Mass minimality of complex subvarieties of $\mathbb{C}^m$ gives a rich class of examples in $\mathbb{R}^{2m}$ with singular sets of real dimension $k-2$.

Example

$$\{(\zeta_1, \zeta_2) : \zeta_1\zeta_2 = 0\} \subset \mathbb{C}^2 = \mathbb{R}^4$$

defines a mass minimizing 2-dimensional current which is the union of two planes of real dimension 2 intersecting at the origin.
The origin is the only singularity
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De Giorgi's almost everywhere regularity result (1961)

Given an open $A \subset \mathbb{R}^n$, a set E of finite perimeter has minimal boundary in A if

$$P(E) \leq P(\tilde{E}) \quad \text{for all } \tilde{E} \text{ s.t. } E \setminus A = \tilde{E} \setminus A.$$

Theorem

If B_r is the reduced boundary of E and E has minimal boundary in A, then

- *$B_r \cap A$ is locally a smooth hypersurface;*
- $\mathcal{H}^{n-1}(A \cap (\overline{B_r} \setminus B_r)) = 0.$

The proof was an amazing "tour de force".

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Almost everywhere regularity for his solution to the Plateau problem, based on an “epiperimetric inequality”.

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Reifenberg died in 1964 in a mountaineering accident.

6. Regularity results

Almgren's work on regularity.

Varifold formulation. 1968 Annals of Math paper – a major advance.

Results for a broad class of variational problems with integrands satisfying an ellipticity condition.

Almgren's massive "Big regularity paper", appeared in book form in 2000.

The goal: show that singular sets for higher dimensional Plateau problem indeed have Hausdorff dimension at most $k - 2$.

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7. Codimension 1 and Bernstein's Theorem

Problem

Are there no singular points when $k = n - 1$?

Closely related question:

Problem

If a (hyper)cone in $\mathbb{R}^n$ minimizes $(n-1)$-area, must it be a hyperplane?

The answer to both questions is

- Yes if $n \leq 7$ (Fleming if $n = 3$,
  De Giorgi, Almgren, Simons if $4 \leq n \leq 7$);
- No if $n \geq 8$ (Bombieri-De Giorgi-Giusti, Inv. Math. 1969).

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The counterexample to regularity for $n = 8$:

$\mathbb{R}^8 = \mathbb{R}^4 \times \mathbb{R}^4 \qquad x = (x', x'')$

$$C := \{(x', x'') : |x'| = |x''|\}.$$

Theorem (Bombieri-De Giorgi-Giusti)

C locally minimizes area.

Federer (1970) For $n \geq 8$ and $k = n - 1$, the singular set of the oriented Plateau problem has Hausdorff dimension at most $n - 8$.

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7. Codimension 1 and Bernstein's Theorem

Theorem (Classical version of Bernstein's Theorem)

If f is a smooth solution of the minimal surface equation in all of $\mathbb{R}^2$, then f is an affine function.

A GMT proof (Fleming 1962):

(i) Cones in $\mathbb{R}^3$ which locally minimize the area must be planes.

(ii) If T locally minimizes area and T_r is the part of T in the ball $\mathcal{B}_r(0)$, then $r^{-k}\mathbf{M}(T_r)$ is a nondecreasing function of r (the monotonicity formula).

(iii) Let f be a solution of the minimal surface equation on $\mathbb{R}^2$ and T the current induced by its graph; from (i) and (ii) it is shown that $r^{-2}\mathbf{M}(T_r) = \pi$.

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IF Bernstein Theorem false on $\mathbb{R}^m$

THEN $\exists$ nonplanar area minimizing cones in $\mathbb{R}^m$.

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C. Pauc told Fleming in 1957 about De Giorgi's papers on sets of finite perimeter.

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J. P. Cecconi host
Workshop conducted in a "lingua mista".

De Giorgi visited Brown University in 1964.
The new result on the Bernstein Theorem for $m = 3$ was proved during his voyage (by ship) across the Atlantic Ocean.

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Pisa, June 1965

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Fred Almgren, Wendell Fleming and Dan Fleming

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Wendell Fleming, Fred Almgren and Ennio De Giorgi

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Thank you

for your attention!