# Stability results for the nonlocal Mullins-Sekerka and for the Hell-Shaw flow

Nicola Fusco

A Mathematical Tribute to Ennio De Giorgi
Pisa, September 19-23, 2016

Copolymer = a polymer derived from two or more monomeric structures

Diblock Copolymer = structure given by two different chemical blocks of polymers

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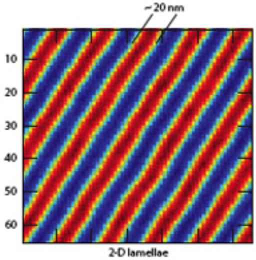
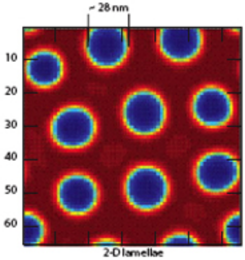
Microphase separation

Formation of nanostructures

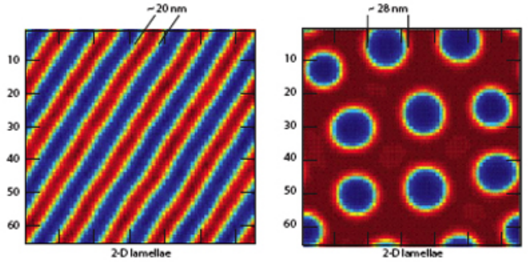
The relative lenghts of each block $\Longrightarrow$ different morphologies



Lamellae



Spheres

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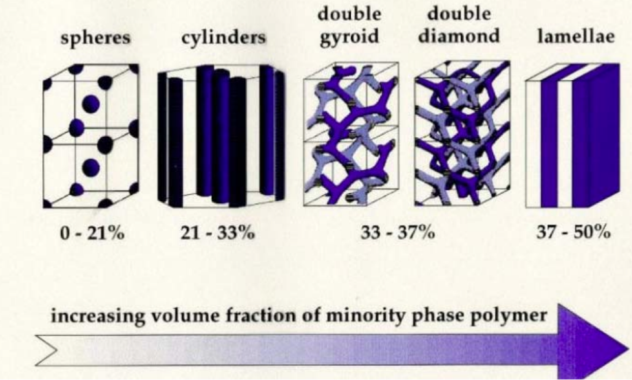

Denote $u : \Omega \to [-1, 1]$ the function describing the density

$$u(x) = \begin{cases} 1 & \text{on phase } A \\ -1 & \text{on phase } B \end{cases}$$

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$$\mathcal{E}_\varepsilon(u) = \underbrace{\varepsilon \int_\Omega |\nabla u|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega (1 - u^2)^2 \, dx}_{\text{attractive short range interaction}} + \underbrace{\gamma_0 \int_\Omega |\nabla(\Delta^{-1} u)|^2 \, dx}_{\text{repulsive long range interaction}}$$

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Letting $\varepsilon \to 0$, the functionals $\mathcal{E}_\varepsilon$ $\Gamma$-converge (Ren-Wei, 2003) to

$$\mathcal{E}(u) = \frac{1}{2}|\nabla u|(\Omega) + \frac{3\gamma_0}{16}\int_\Omega |\nabla(\Delta^{-1}u)|^2\,dx$$

where

$$u \in BV(\{-1, 1\}), \quad u = u_E := \chi_E - \chi_{\Omega\setminus E}, \quad |\nabla u|(\Omega) = 2P(E;\Omega)$$

Open problem: are minimizers of

$$J(E) = P(E;\Omega) + \gamma \int_{\Omega} |\nabla(\Delta^{-1} u_E)|^2 \, dx \quad \text{(almost) periodic?}$$

Known in one dimension (Müller, 1993); if $n \geq 2$ partial answer: Alberti-Choksi-Otto 2009, Spadaro 2009

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$$\begin{cases} -\Delta_y G(x,y) = \delta_x - 1 & \text{in } \mathbb{T}^n \\ \int_{\mathbb{T}^n} G(x,y)\, dy = 0 \end{cases} \qquad u(x) = \begin{cases} 1 & \text{if } x \in E \\ -1 & \text{if } x \in \mathbb{T}^n \setminus E \end{cases}$$

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Choksi-Sternberg, 2007: calculated $J''$ at critical points

Ren-Wei, 2002–2008: critical spheres, cylinders and lamellae with $J'' > 0$ minimize the energy with respect to some special variations

$$J(E) = P_{\mathbb{T}^n}(E) + \gamma \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} G(x,y)u_E(x)u_E(y)\,dxdy$$

can be also written as

$$J(E) = P_{\mathbb{T}^n}(E) + \gamma \int_{\mathbb{T}^n} |\nabla v_E|^2\,dx \qquad \text{(Pattern Formation)}$$

where $\begin{cases} -\Delta v_E = u_E - m & \text{in } \mathbb{T}^n \\ \int_{\mathbb{T}^n} v_E = 0 \end{cases}$ $\qquad m = |E| - |\mathbb{T}^n \setminus E|$

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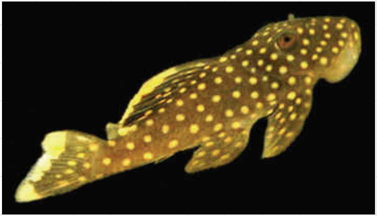

Hypostomus Plecostomus

Distichodus Sexfasciatus

Distance between (equivalence classes) of sets:

$$d(E,F) = \min_{\tau} |E \triangle (F+\tau)|$$

$E \subset \mathbb{T}^n$ is a (strict) local minimizers if $\exists \delta > 0$ s.t.

$$J(F) > J(E)$$

whenever $F \subset \mathbb{T}^n$ with $0 < d(E,F) < \delta$, and $|F| = |E|$

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Minimizers of $J(E)$ under a volume constraint satisfy

$$(*) \qquad H_{\partial E}(x) + 4\gamma v_E(x) = \lambda \qquad \text{on } \partial E$$

where $H_{\partial E} =$ sum of principal curvatures

while a solution $E \in C^2$ of equation $(*)$ is a critical configuration

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Theorem

*If $E \subset \mathbb{T}^n$ is a local minimizer of $J$, then $\partial E \setminus \Sigma$ is $C^{3,\alpha}$, for any $\alpha < 1$, and $\Sigma$ is a closed set such that $\dim_{\mathcal{H}}(\Sigma) \leq n - 8$.*
*In fact, $\partial E \setminus \Sigma$ is $C^\infty$ (Julin-Pisante, 2014).*

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Let $E \in C^2$, and fix a $C^2$ vector field $X : \mathbb{T}^n \mapsto \mathbb{T}^n$. Then, let us consider

$\Phi : \mathbb{T}^n \times (-1, 1) \mapsto \mathbb{T}^n$ the associated flow

$$\frac{\partial \Phi}{\partial t} = X(\Phi), \qquad \Phi(x, 0) = x$$

and set $E_t := \Phi(\cdot, t)(E)$

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Then we set $\displaystyle J''(E)[X] := \frac{d^2}{dt^2} J(E_t)\Big|_{t=0}$

Theorem (Choksi-Sternberg 2007)

*If $E$ is a critical point and $X$ is as above, then*

$$J''(E)[X] = \int_{\partial E}\Big(|D_\tau(X\cdot\nu)|^2 - |B_{\partial E}|^2(X\cdot\nu)^2\Big)\,d\mathcal{H}^{n-1}$$
$$+\,8\gamma\int_{\partial E}\int_{\partial E} G(x,y)\big(X\cdot\nu\big)(x)\big(X\cdot\nu\big)(y)\,d\sigma_x\,d\sigma_y$$
$$+\,4\gamma\int_{\partial E}\partial_\nu v_E(X\cdot\nu)^2\,d\sigma$$

$|B_{\partial E}|^2 =$ sum of the squares of principal curvatures

Since the second variation depends only on $X \cdot \nu$,

we define for a $C^2$ critical point $E$ and for $\varphi \in H^1(\partial E)$

$$
\begin{aligned}
\partial^2 J(E)[\varphi] = & \int_{\partial E} \left( |D_\tau \varphi|^2 - |B_{\partial E}|^2 \varphi^2 \right) d\mathcal{H}^{n-1} \\
& + 8\gamma \int_{\partial E} \int_{\partial E} G(x,y)\varphi(x)\varphi(y) d\mathcal{H}^{n-1}(x)\, d\mathcal{H}^{n-1}(y) \\
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Space of admissible variations:

$$\widetilde{H}^1(\partial E) := \left\{ \varphi \in H^1(\partial E) : \underbrace{\int_{\partial E} \varphi = 0}_{\text{volume pres.}}, \quad \underbrace{\int_{\partial E} \varphi \nu_E = 0}_{\text{translation inv.}} \right\}$$

Theorem (Acerbi-F.-Morini 2013)

*Let $E$ be a $C^2$ critical configuration such that*

$$\partial^2 J(E)[\varphi] > 0 \qquad \forall \varphi \in \widetilde{H}^1(\partial E).$$

*Then, $E$ is a strict local minimizer. Precisely, there exists $\delta > 0$, s.t. for every set of finite perimeter $F \subset \mathbb{T}^n$, with $d(E,F) < \delta$*

$$(**) \qquad J(F) \geq J(E) + C_0 d(E,F)^2$$

## Consequences: $\gamma = 0 \implies$ quantitative isop. ineq.

### Corollary

*Let $E \subset \mathbb{T}^n$ be smooth open set with constant mean curvature. If*

$$\int_{\partial E} \left( |D_\tau \varphi|^2 - |B_{\partial E}|^2 \varphi^2 \right) d\mathcal{H}^{n-1} > 0 \qquad \forall\, \varphi \in T^\perp(\partial E) \setminus \{0\}\,,$$

*there exist $\delta$, $C > 0$ s.t. for $F \subset \mathbb{T}^n$, with $|F| = |E|$ and $d(E, F) < \delta$*

$$P_{\mathbb{T}^n}(F) \geq P_{\mathbb{T}^n}(E) + C[d(E, F)]^2\,.$$

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The local minimality w.r.t. $L^\infty$ perturbations (B.White, 1994) or w.r.t. $L^1$ perturbations ($\Rightarrow n \leq 7$, Morgan-Ros, 2010)

# Application: Global and local minimality of lamellae

$$(\mathcal{P}) \qquad \text{Min} \ \left\{ J(E) = P_{\mathbb{T}^n}(E) + \gamma \int_{\mathbb{T}^n} |\nabla v_E|^2 \, dx \,, \qquad |E| = d \right\}$$

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For $0 < d < 1$, $k \geq 1$, set

$$L_k = \mathbb{T}^{n-1} \times \cup_{i=1}^{k} \left[ \tfrac{i-1}{k}, \tfrac{i-1}{k} + \tfrac{d}{k} \right]$$

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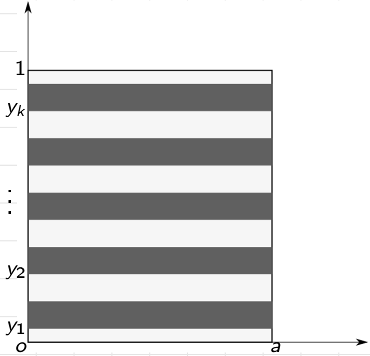
### Theorem (Acerbi-F.-Morini, 2013)

*If $L_1$ is the unique global minimizer of the periodic isoperimetric problem in $\mathbb{T}^n$, then it is also the unique global minimizer of $(\mathcal{P})$, provided $\gamma$ is sufficiently small.*
*Moreover, let $d$ and $\gamma > 0$. Then there exists $k_0$ such that if $k \geq k_0$ the set $L_k$ is a strict local minimizer for $(\mathcal{P})$.*

# Critical 2d $k$-lamellar patterns

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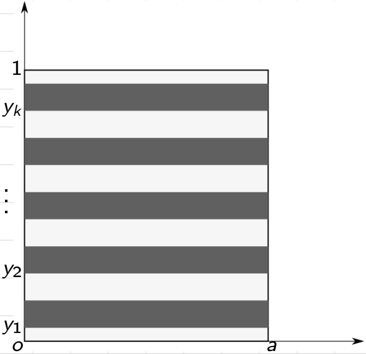


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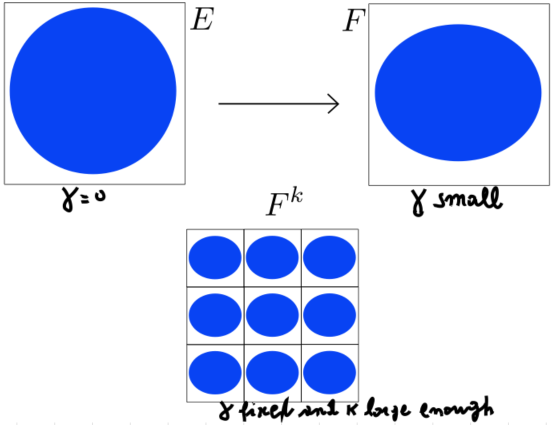
**Theorem (Morini-Sternberg, 2014)**

*For any positive integer $k$, if*

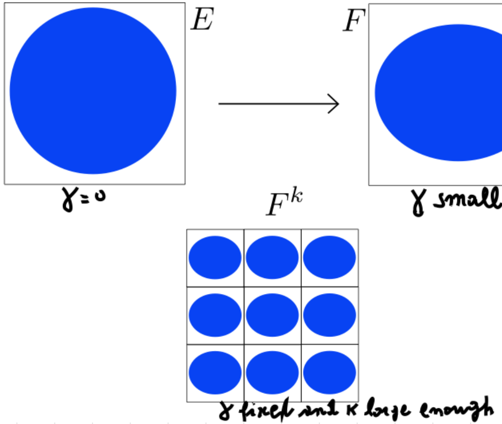
$$a < \pi\sqrt{\frac{k}{2\gamma}},$$

*then the $k$-lamellar critical configuration $u_k$ is an isolated $L^1$-local minimizer in $\Omega_a := (0, a) \times (0, 1)$.*

# Application: Periodic local minimizers



# Application: Periodic local minimizers




Cristoferi (2015): for every critical set $E$ that has positive second variation for the perimeter, there exists a set $F$ locally minimizing for $J$, which closely resemble a rescaled version of $E$.

# Extension: Global minimality of a single droplet

## Theorem (Cicalese-Spadaro, 2013)

*If $\Omega$ is $C^2$ and bounded and*

$$\gamma r^3 |\log r| << 1 \ \ (n = 2), \qquad \gamma r^3 << 1 \ \ (n \geq 3),$$

*then the unique global minimizer is a convex set $E$ such that*

$$\partial E = \{x + (r + \varphi(\omega))\omega : \omega \in \mathbb{S}^{n-1}\}, \qquad \|\varphi\|_{C^1(\mathbb{S}^{n-1})} \leq c(n)\gamma r^{n+3}$$

*Moreover $E$ is a ball iff $\Omega$ is a ball.*

$$J(E) = P(E;\Omega) + \gamma \int_\Omega |\nabla v_E|^2 \, dx, \qquad |E| = d = \omega_n r^n < |\Omega|$$

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Julin-Pisante, 2014: Local minimality (with a quantitative estimate) for critical points of in a smooth open set under Neumann boundary condition

# Evolutionary counterpart: the nonlocal Mullins-Sekerka flow

A smooth flow of sets $(E_t)_t \subset\subset \Omega$ is a solution to the nonlocal Mullins-Sekerka flow if

$$
\begin{cases}
V_t = [\partial_\nu w_t] & \text{on } \partial E_t, \\
\Delta w_t = 0 & \text{in } \Omega \setminus \partial E_t, \\
w_t = H_{\partial E_t} + 4\gamma v_{E_t} & \text{on } \partial E_t, \\
-\Delta v_{E_t} = u_{E_t} - \fint_\Omega u_{E_t}\,, & \text{in } \Omega,
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We have set $V_t =$ normal velocity at time $t$

$$[\partial_\nu w_t] = \text{jump of the normal derivative of } w_t$$

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$$\Downarrow$$

The flow is volume preserving.

# The nonlocal Mullins-Sekerka flow

- It arises as the sharp interface limit of the Ohta-Kawaski equation

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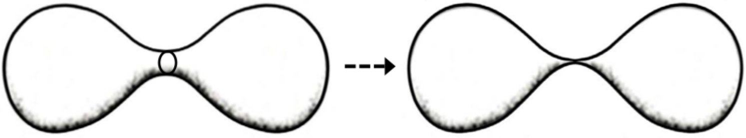
# Features of the nonlocal Mullins-Sekerka flow

- Singularities may appear



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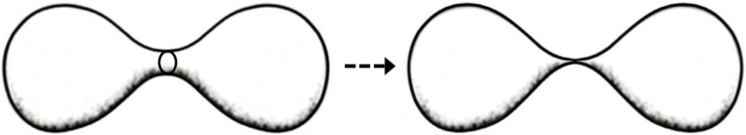

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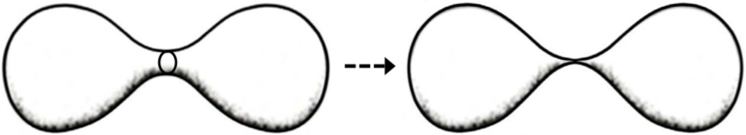

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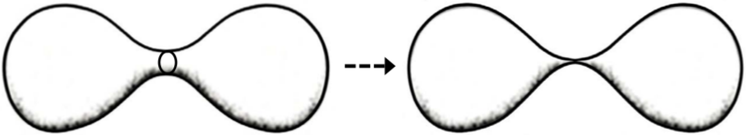

- Lack of a comparison principle
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- Local-in-time existence theory by Escher-Nishiura, 2002

# Nonlinear stability

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$$|E_0| = |F|\,, \quad \operatorname{dist}_{C^1}(E_0, F) \le \delta_0\,, \quad \textit{and} \quad \int_{\mathbb{T}^3} |Dw_{E_0}|^2\,dx \le \delta_0\,,$$

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Previous related results: exponential stability of spheres for the Hele-Shaw flow (Chen 1993, Escher-Simonett 1998)

# Nonlinear stability: ingredients of the proof

**Step 1** (Energy identity)

$$\frac{d}{dt}\int_{\mathbb{T}^3} |Dw_t|^2\,dx = -2\partial^2 J(E_t)\,[V_t] + R(E_t)$$

where

$$R(E_t) = \int_{\partial E_t} (\partial_{\nu_t} w_t^+ + \partial_{\nu_t} w_t^-)[\partial_{\nu_t} w_t]^2\,d\mathcal{H}^2\,,$$

**Step 2** (Stopping time) Let

$$\bar{t} := \sup\Big\{ t > 0 :\ \mathrm{dist}_{C^1}(E_t, F) < 2\delta_0 \quad \text{and} \quad \int_{\mathbb{T}^3} |Dw_t|^2\,dx < 2\delta_0 \qquad \text{for all } t \in (0, \bar{t}), \Big\}$$

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**Step 3** ($\bar{t}$ =maximal time of existence $T^*$)

Assume by contradiction that

$$\int_{\mathbb{T}^3} |Dw_{\bar{t}}|^2 \, dx = 2\delta_0 \qquad \text{for some } \bar{t} < T^*$$

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The last inequalities follow from delicate boundary estimates for

$$-\Delta w = f \mathcal{H}^2 \llcorner \partial E$$

# Boundary estimates

## Proposition (Boundary estimates for harmonic functions)

*Let $E \subset \mathbb{T}^3$ be of class $C^{1,\alpha}$, $f \in C^\alpha(\partial E)$ (with zero average) and $u \in H^1(\mathbb{T}^3)$ be the solution of*

$$-\Delta u = f\mathcal{H}^2 \llcorner \partial E$$

*with zero average in $\mathbb{T}^3$. Then, for every $1 < p < \infty$ there exists a constant $C$, which depends on the $C^{1,\alpha}$ bounds on $\partial E$ and on $p$, such that:*

(i) $\|\partial_{\nu_E} u^+\|_{L^2(\partial E)} + \|\partial_{\nu_E} u^-\|_{L^2(\partial E)} \leq C\|u\|_{H^1(\partial E)}$;

(ii) $\|\partial_{\nu_E} u^+\|_{L^p(\partial E)} + \|\partial_{\nu_E} u^-\|_{L^p(\partial E)} \leq C\|f\|_{L^p(\partial E)}$.

(iii) *Moreover, if $f \in H^1(\partial E)$, then for every $1 \leq p < +\infty$*

$$\|f\|_{L^p(\partial E)} \leq C\|f\|_{H^1(\partial E)}^{\frac{p-1}{p}} \|u\|_{L^2(\partial E)}^{\frac{1}{p}}.$$

# Nonlinear stability: ingredients of the proof

Combining (1) and (2) with energy identity

$$\frac{d}{dt}\int_{\mathbb{T}^3} |Dw_t|^2\,dx = -2\partial^2 J(E_t)\,[V_t] + R(E_t)$$
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**Step 4** $T^* = +\infty$ and conclusion

Thank you for your attention!