Spaces with lower Ricci curvature bounds

State of the art and future challenges

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Completion / Compactification

A common practice in various fields of mathematic is to start studying a certain class of ‘smooth’ or ‘nice’ objects, and to close it w.r.t. some relevant topology.

In general, the study of the limit objects turns out to be useful to understand the properties of the original ones.

Gromov's plan

When the original class of objects are Riemannian manifolds with some curvature bounds, this program has been proposed by Gromov.

	Topology	Synthetic notion
Bounds from above/below on sectional curvature	Gromov-Hausdorff convergence	Alexandrov spaces
Bounds from below on the Ricci curvature	measured Gromov-Hausdorff convergence	$\mathrm{CD}^*(K, N)$ spaces / $\mathrm{RCD}^*(K, N)$ spaces
Non-negative scalar curvature	intrinsic flat convergence?	??

mGH convergence of compact spaces

Let $(X_n, \mathsf{d}_n, \mathfrak{m}_n)$, $n \in \mathbb{N} \cup \{\infty\}$ be compact and normalized, i.e. $\mathfrak{m}_n(X_n) = 1$.

$(X_n, \mathsf{d}_n, \mathfrak{m}_n)$ converges to $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$ in the mGH sense if there is (Y, d_Y) and isometric embeddings ι_n, ι_∞ of the X's into Y such that

$$(\iota_n)_\sharp \mathfrak{m}_n \qquad \text{weakly converges to} \qquad (\iota_\infty)_\sharp \mathfrak{m}_\infty$$

Gromov's precompactness theorem

Thm. (Gromov '80) Let $K \in \mathbb{R}$, $N \geq 1$ and $D \in [0, \infty)$.

The class of smooth, connected, complete Riemannian manifolds with $\mathrm{Ric} \geq K$, $\dim \leq N$ and $\mathrm{Diam} \leq D$ is precompact in the mGH topology

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A key result

Theorem (Sturm-VonRenesse '05 after Otto-Villani '00 and Cordero Erasquin-McCann-Schmuckenschläger '01)
Let M be a smooth connected complete Riemannian manifold. Then the following are equivalent:

i) The Ricci curvature of M is uniformly bounded from below by K
ii) The relative entropy functional is K-convex on the space $(\mathscr{P}_2(M), W_2)$

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Definition (Lott-Villani and Sturm '06) $(X, \mathsf{d}, \mathfrak{m})$ has Ricci curvature bounded from below by K if the relative entropy is K-convex on $(\mathscr{P}_2(X), W_2)$. Called $\mathrm{CD}(K, \infty)$ spaces, in short.

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More general $\mathrm{CD}^*(K, N)$ spaces can be introduced along the same lines

Γ-convergence of the entropies

Thm. (Lott-Sturm-Villani)
Let $(X_n, \mathsf{d}_n, \mathfrak{m}_n)$ be mGH-converging to $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$. Then:

- $\Gamma - \underline{\lim}$ inequality: for every sequence $n \mapsto \mu_n \in \mathscr{P}(X_n)$ weakly converging to $\mu_\infty \in \mathscr{P}(X_\infty)$ we have

$$\mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty) \leq \varliminf_{n\to\infty} \mathrm{Ent}_{\mathfrak{m}_n}(\mu_n).$$

- $\Gamma - \overline{\lim}$ inequality: for every $\mu_\infty \in \mathscr{P}(X_\infty)$ there is a sequence $n \mapsto \mu_n \in \mathscr{P}(X_n)$ weakly converging to μ_∞ such that

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Cor. The $\mathrm{CD}(K, \infty)$ condition is closed w.r.t. mGH convergence.

Finsler structures are included

Consider $\mathbb{R}^d$ with the Lebesgue measure and the distance coming from a norm.

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Then:

Thm. (Cordero Erasquin-Sturm-Villani '09) This is a $\mathrm{CD}(0, \infty)$ space (in fact $\mathrm{CD}^*(0, d)$).

Thm. (Cheeger-Colding '97) It cannot occur as mGH-limit of Riemannian manifolds with Ricci uniformly bounded from below and dimension uniformly bounded from above.

Some observations

- For a given Finsler manifold the following are equivalent:
  - The manifold is Riemannian
  - The Sobolev space $W^{1,2}$ is Hilbert
  - The heat flow is linear

- The heat flow can be seen as :
  - Gradient flow of the Dirichlet energy w.r.t. L^2
  - Gradient flow of the relative entropy w.r.t. W_2 (Jordan, Kinderleher, Otto '98 ..... Ohta, Sturm '09)

The idea

Restrict to the class of $\mathrm{CD}(K, \infty)$ spaces with linear heat flow.

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What we have to do to show this makes sense:

- understand who is the heat flow on $\mathrm{CD}(K,\infty)$ spaces
- show that such flow is stable w.r.t. mGH convergence.

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- show that such flow is stable w.r.t. mGH convergence.

Plan pursued in:
G. '09
G., Kuwada, Ohta '10
Ambrosio, G., Savaré '11 '11 '12

Sobolev functions on $(X, \mathsf{d}, \mathfrak{m})$

(1/2)

It is possible to define the space $W^{1,2}(X)$ of functions $f \in L^2(\mathfrak{m})$ having 'distributional differential' in $L^2(\mathfrak{m})$.

Works by Hajlasz, Cheeger, Shanmugalingham, Ambrosio, G., Savaré...

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Works by Hajlasz, Cheeger, Shanmugalingham, Ambrosio, G., Savaré...

For $f : X \to \mathbb{R}$ define $\mathrm{lip}(f) : X \to \mathbb{R}$ as

$$\mathrm{lip}(f)(x) := \varlimsup_{y \to x} \frac{|f(y) - f(x)|}{\mathsf{d}(y, x)}$$

and then

$$\mathrm{Ch}(f) := \frac{1}{2} \inf_{(f_n)} \varliminf_{n} \int \mathrm{lip}(f_n)^2 \, \mathrm{d}\mathfrak{m}$$

the inf is taken among all sequences $(f_n) \subset \mathrm{LIP}(X)$ L^2-converging to f.

Sobolev functions on $(X, \mathsf{d}, \mathfrak{m})$

(2/2)

We put $W^{1,2}(X) := \{\mathrm{Ch} < \infty\}$ and endow it with the norm

$$\|f\|^2_{W^{1,2}} := \|f\|^2_{L^2} + 2\mathrm{Ch}(f)$$

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For $f \in W^{1,2}(X)$, we define $|Df| \in L^2(\mathfrak{m})$ is defined as the L^2-limit of $(\mathrm{lip}(f_n))$ for an optimal sequence (f_n) in the definition of $\mathrm{Ch}(f)$.

This definition is well posed and

$$\mathrm{Ch}(f) = \frac{1}{2}\int |Df|^2 \,\mathrm{d}\mathfrak{m}$$

$|Df|$ plays the role of the modulus of distributional differential.

Heat flow as gradient flow

Theorem (Ambrosio-G.-Savaré '11 and G.-Kuwada-Ohta '10)
Let $(X, \mathsf{d}, \mathfrak{m})$ be $\mathsf{CD}(K, \infty)$ and $\mu = f\mathfrak{m} \in \mathscr{P}(X)$ with $f \in L^2(\mathfrak{m})$. Let:

- (μ_t) be the W_2-gradient flow of the relative entropy w.r.t. $\mathfrak{m}$ starting from μ
- (f_t) be the $L^2(\mathfrak{m})$-gradient flow of Ch starting from f.

Then

$$\mu_t = f_t\mathfrak{m} \qquad \forall t \geq 0.$$

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Definition (Ambrosio-G.-Savaré '11)

$$\begin{aligned} \mathrm{RCD}(K, \infty) :=& \mathrm{CD}(K, \infty) \ + \ \text{the heat flow is linear} \\ =& \mathrm{CD}(K, \infty) \ + \ W^{1,2}(X) \text{ is Hilbert} \end{aligned}$$

Theorem (Ambrosio-G.-Savaré '11) The $\mathrm{RCD}(K, \infty)$ condition is stable under mGH-convergence.

Laplacian

Let $(X, \mathsf{d}, \mathfrak{m})$ be such that $W^{1,2}(X)$ is Hilbert.

Then the object

$$\langle \nabla f, \nabla g \rangle := \frac{1}{2}\Big(|D(f+g)|^2 - |Df|^2 - |Dg|^2 \Big)$$

turns out to be a bilinear form on $W^{1,2}(X)$.

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Then we say that $f \in W^{1,2}(X)$ has a Laplacian in $L^2(\mathfrak{m})$ if there is $\Delta f \in L^2(\mathfrak{m})$ such that

$$\int g \Delta f \, \mathrm{d}\mathfrak{m} = -\int \langle \nabla f, \nabla g \rangle \, \mathrm{d}\mathfrak{m} \qquad \forall g \in W^{1,2}(X).$$

Bochner inequality

Theorem (Ambrosio-G.-Savaré '11) On a RCD(K,∞) space the Bocher inequality

$$\Delta\frac{|Df|^2}{2} \geq \langle \nabla f, \nabla\Delta f\rangle + K|Df|^2$$

holds in the weak sense, i.e.

$$\int \Delta g\frac{|Df|^2}{2}\,\mathrm{d}\mathfrak{m} \geq \int g\big(\langle \nabla f, \nabla\Delta f\rangle + K|Df|^2\big)\,\mathrm{d}\mathfrak{m}$$

for every $f \in D(\Delta)$ with $\Delta f \in W^{1,2}(X)$, $g \in L^\infty \cap D(\Delta)$ non-negative with $\Delta g \in L^\infty$.

The other way around: from Bochner to RCD

Under proper assumptions, the converse implication also holds:
Theorem (Ambrosio-G.-Savaré '12)
Let $(X, \mathsf{d}, \mathfrak{m})$ be such that

- $\mathfrak{m}(B_r(x)) \leq e^{Cr^2}$ for some $C > 0$
- $W^{1,2}$ is Hilbert
- The Bochner inequality

$$\Delta \frac{|Df|^2}{2} \geq \langle \nabla f, \nabla \Delta f \rangle + K|Df|^2$$

  holds in the weak sense
- every $f \in W^{1,2}(X)$ with $|Df| \leq 1$ $\mathfrak{m}$-a.e. has a 1-Lipschitz representative.

Then it is a $\mathrm{RCD}(K, \infty)$ space.

The finite dimensional case

Theorem With the same assumptions as before we also have that the $\mathrm{CD}^*(K, N)$ condition holds if and only if the Bochner inequality

$$\Delta \frac{|Df|^2}{2} \geq \frac{(\Delta f)^2}{N} + \langle \nabla f, \nabla \Delta f \rangle + K|Df|^2$$

holds.

Such spaces are called $\mathrm{RCD}^*(K, N)$ spaces.

By:
G. '12
Ambrosio-G.-Savaré '12
Erbar-Kuwada-Sturm '13
Ambrosio-Mondino-Savaré '15

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Non-exhaustive list of geometric properties of $\mathrm{RCD}^*(K, N)$ spaces

Abresch-Gromoll inequality (G.-Mosconi '12)

Splitting theorem (G. '13)

Maximal diameter theorem (Ketterer '13)

Rectifiability results (Mondino-Naber '14)

Obata theorem (Ketterer '14)

Levy-Gromov isoperimetric inequality (Cavalletti-Mondino '15)

Volume cone implies metric cone (De Philippis-G. '16)

Properties of fundamental group (Mondino-Wei '16)

Rigidity results for Perelman $\mathcal{W}$ entropy (Kuwada-Li '16)

Rigidity results for Bishop-type inequality (Kitabeppu '16)

Non-exhaustive list of geometric properties of $\mathsf{RCD}^*(K,N)$ spaces

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The splitting theorem

Theorem (G. '13, after Cheeger-Gromoll '71 and Cheeger-Colding '97)
Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}^*(0, N)$ space containing a (straight) line, i.e. a map $\gamma : \mathbb{R} \to X$ such that

$$\mathsf{d}(\gamma_s, \gamma_t) = |s - t| \qquad \forall t, s \in \mathbb{R}.$$

Then there is a space $(X', \mathsf{d}', \mathfrak{m}')$ such that

$$(X, \mathsf{d}, \mathfrak{m}) \quad \text{is isomorphic to} \quad (X' \times \mathbb{R}, \mathsf{d}' \otimes \mathsf{d}_{\mathrm{Eucl}}, \mathfrak{m}' \times \mathcal{L}^1)$$

where

$$(\mathsf{d}' \otimes \mathsf{d}_{\mathrm{Eucl}})\big((x', t), (y', s)\big) := \sqrt{\mathsf{d}'(x', y')^2 + |t - s|^2}$$

Moreover:

- If $N \geq 2$ then $(X', \mathsf{d}', \mathfrak{m}')$ is an $\mathsf{RCD}^*(0, N - 1)$ space
- If $N \in [1, 2)$ then X' contains only one point

Very very rough idea of the proof

Try to imitate the proof in the smooth setting by:

- building calculus tools in the non-smooth world
- finding ‘shortcuts’ in the original argument to lower the regularity required for the proof to work

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The isometry in the end is built using the following fact:
Let X_1, X_2 be two $\mathrm{RCD}(K, \infty)$ spaces and $T : X_1 \to X_2$ invertible. Then T is, up to a modification in a negligible set, a measure preserving isometry iff

$$\|f \circ T\|_{W^{1,2}(X_1)} = \|f\|_{W^{1,2}(X_2)} \qquad \forall f : X_2 \to \mathbb{R} \text{ Borel}$$

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Compare with: Let X_1, X_2 be two metric spaces and $T : X_1 \to X_2$ invertible. Then T is an isometry iff

$$\mathrm{Lip}_{X_1}(f \circ T) = \mathrm{Lip}_{X_2}(f) \qquad \forall f : X_2 \to \mathbb{R}.$$

The maximal diameter theorem

Theorem (Ketterer '14, after Cheng '75 and Cheeger-Colding '97)
Let $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}^*(N-1, N)$ space having two points at distance π.
Then X is a spherical subspension over a space $(X', \mathrm{d}', \mathfrak{m}')$ which

- is a $\mathrm{RCD}^*(N-2, N-1)$ space if $N \geq 2$
- is either one or two points if $N \in [1, 2)$

Key lemma

Let $(X, \mathsf{d}, \mathfrak{m})$ be the cone built over $(X', \mathsf{d}', \mathfrak{m}')$.

Then X is $\mathsf{RCD}^*(0, N+1)$ if and only if X' is $\mathsf{RCD}^*(N-1, N)$.

Key lemma

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Proved by Eulerian calculus, i.e. by looking at the Bochner inequality.
Hard part: dealing with the vertex

Levy-Gromov inequality: the smooth case

Theorem Let M be a smooth Riemannian manifold of dimension n and with $\mathrm{Ric} \geq n-1$.
Then for every $E \subset M$ sufficiently smooth we have

$$\frac{|\partial E|}{|M|} \geq \frac{|\partial B|}{|S^n|} =: f_n\Big(\frac{|E|}{|M|}\Big)$$

where $B \subset S^n$ is the spherical cap with $|B| = |E|$.

If equality holds for some E with $0 < |E| < |M|$ then $M = S^n$.

Levy-Gromov inequality: the non-smooth case

For $E \subset X$ Borel define the outer Minkowski content as

$$M^+(E) := \liminf_{\varepsilon \downarrow 0} \frac{\mathfrak{m}(E^\varepsilon) - \mathfrak{m}(E)}{\varepsilon}$$

Thm. (Cavalletti-Mondino '15) Let $(X, \mathsf{d}, \mathfrak{m})$ be $\mathrm{RCD}^*(N-1, N)$ with $\mathfrak{m}(X) = 1$.
Then for every $E \subset X$ we have

$$M^+(E) \geq f_N\big(\mathfrak{m}(E)\big)$$

If equality holds for some E with $0 < \mathfrak{m}(E) < 1 = \mathfrak{m}(X)$, then X is a spherical subspension.

Key lemma: needle decomposition

Let $(X, \mathsf{d}, \mathfrak{m})$ be $\mathsf{RCD}^*(K, N)$ and ρ_0, ρ_1 probability densities.
Then $\mathfrak{m}$ can be disintegrated into a family of measures $\mathfrak{m}_\alpha$ such that for every α:

- Each $\mathfrak{m}_\alpha$ is concentrated on a geodesic γ_α
- $(\gamma_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ is a $\mathsf{RCD}^*(K, N)$ space
- $\int \rho_0 \, \mathrm{d}\mathfrak{m}_\alpha = \int \rho_1 \, \mathrm{d}\mathfrak{m}_\alpha = 1$

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Such disintegration is found by considering the L^1 (!) optimal transport from ρ_0 to ρ_1.

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Such disintegration is found by considering the L^1 (!) optimal transport from ρ_0 to ρ_1.
Given this, one picks $\rho_0 := \mathfrak{m}(E)^{-1}\chi_E$ and $\rho_1 := \mathfrak{m}(E^c)^{-1}\chi_{E^c}$ to reduce the study to the one-dimensional case

Inspired by works of Klartag '14 in the smooth setting. The metric construction has roots in works by Bianchini-Cavalletti '13 and Cavalletti '14

The almost rigidity statement

Taking into account:

i) The theory of BV functions on mms (Miranda '03, Ambrosio-Di Marino '12)

ii) The link between $M^+(E)$ and $\|D\chi_E\|_{\mathrm{TV}}$ (Ambrosio-Di Marino-G. '16)

iii) Mosco convergence of BV energies for sequences of RCD spaces mGH-converging (Ambrosio-Honda '16)

we deduce:

Corollary For every N, ε there is δ such that if $(X, \mathsf{d}, \mathfrak{m})$ is $\mathrm{RCD}(N-1-\delta, N+\delta)$ and $E \subset X$ is such that

$$\mathfrak{m}(E) = \frac{1}{2} \qquad M^+(E) \leq f_N(\mathfrak{m}(E)) + \delta$$

then X is ε-close to a spherical subspension in the mGH-topology.

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Among others, two lines of research

Sturm '15 (inspired by McCann-Topping): definition of *super*-Ricci flow on RCD spaces for which there is existence and compactness results.

G. '14 (inspired by Weaver): set up of tensor calculus on RCD spaces, with objects like vector fields, covariant derivative and measure-valued Ricci tensor well defined.

Thank you