A Mathematical Tribute to Ennio De Giorgi

The thresholding scheme for mean curvature flow and de Giorgi's ideas for minimizing movements

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The thresholding scheme

Merriman & Bence & Osher '92:

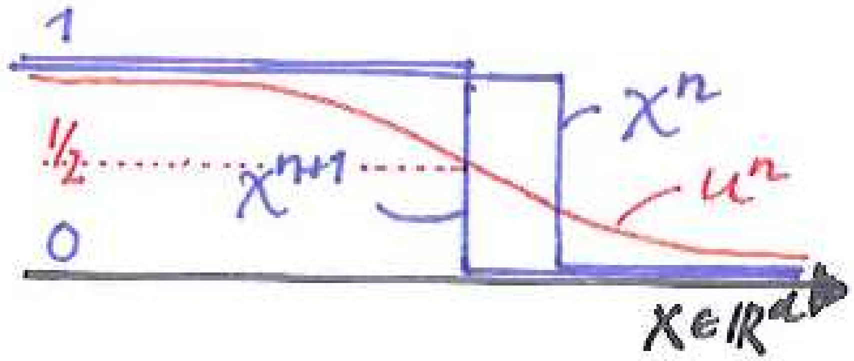
Computational scheme for flow by mean curvature (MCF)

Here just time discretization; time-step size h; $\chi \in \{0,1\}$

$$\chi^{n-1} \overset{\text{convolution}}{\rightsquigarrow} u^n := G_h * \chi^{n-1} \overset{\text{thresholding}}{\rightsquigarrow} \chi^n := I(u^n \geq \tfrac{1}{2})$$

G_h heat kernel at time h

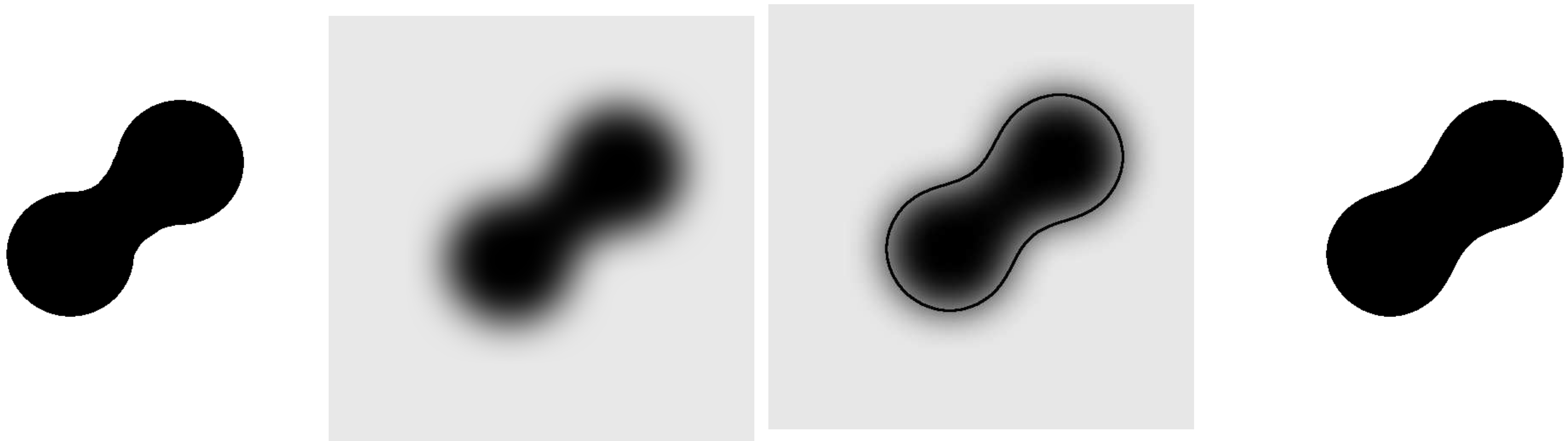
$=$ Gaussian of variance h



(conceptually robust, transfers to metric measure spaces)

Easy to implement

$$\chi^{n-1} \overset{\text{convolution}}{\rightsquigarrow} u^n := G_h * \chi^{n-1} \overset{\text{thresholding}}{\rightsquigarrow} \chi^n := I(u^n \geq \tfrac{1}{2})$$



χ^{n-1} $\qquad$ u^n $\qquad$ $\{u^n = \frac{1}{2}\}$ $\qquad$ χ^n

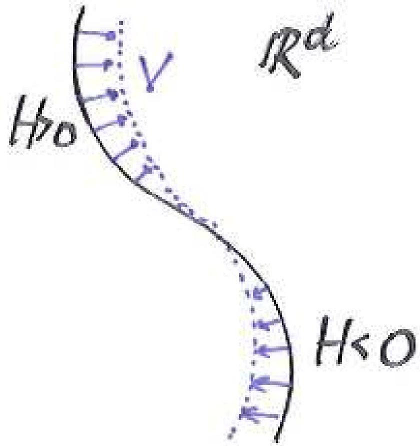
Low complexity: Fast Fourier Transform for convolution

Convergence in the two-phase case

$$\chi^{n-1} \overset{\text{convolution}}{\rightsquigarrow} u^n := G_h * \chi^{n-1} \overset{\text{thresholding}}{\rightsquigarrow} \chi^n := I(u^n \geq \tfrac{1}{2})$$

Thresholding satisfies comparison principle:

$$\chi^{n-1} \leq \tilde{\chi}^{n-1} \Longrightarrow u^n \leq \tilde{u}^n \Longrightarrow \chi^n \leq \tilde{\chi}^n$$



Evans '93: convergence to MCF

in sense of viscosity solution

Straightforward extension to multi-phase version

N phases, eg $\chi = \{\chi_i\}_{i=1,\cdots,N}$ with $\sum_{i=1}^{N} \chi_i = 1$

$\chi^{n-1} \rightsquigarrow u^n, u_i^n := G_h * \chi_i^{n-1} \rightsquigarrow \chi^n, \chi_i^n := I(u_i^n \geq u_j^n \; \forall j)$

Application to grain growth:

eg Elsey & Esedoglu & Smereka '11

Long-time existence of multi-phase MCF:

Kim & Tonegawa via Brakke's notion of MCF '15

Two issues

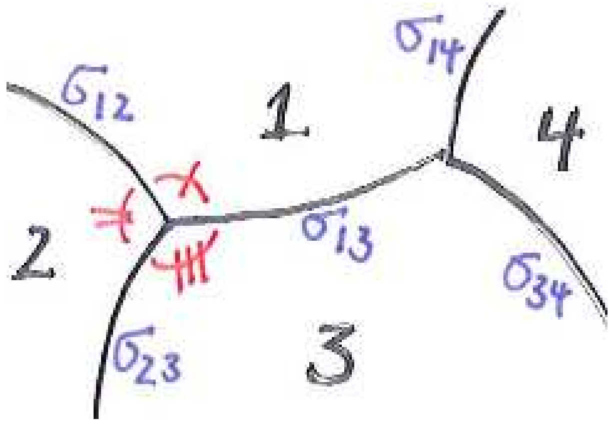
1) Generalization to $\binom{N}{2}$

surface tensions σ_{ij}

(Esedoglu & O. '14)

interfacial energy depends

on misorientation of grains



2) (conditional) convergence (Laux & O. '15, '16)

Both based on **minimizing movement interpretation**

of thresholding (EO'14)

Thresholding as minimizing movement (EO'14)

a) $E_h(\chi) := \sum_{i\neq j} \frac{1}{\sqrt{h}}\int \chi_i\, G_h * \chi_j$

Γ-converges to $c_0 \sum_{i\neq j} \frac{1}{2}\int |\nabla\chi_i| + |\nabla\chi_j| - |\nabla(\chi_i+\chi_j)|$

$= c_0 \sum_{i\neq j}$ area of interface between phase i and phase j

$= c_0$ total interfacial energy

b) $-E_h(\chi-\chi') = \sum_i \frac{1}{\sqrt{h}}\int (\chi_i-\chi_i')\, G_h * (\chi_i-\chi_i')$

$= \sum_i \frac{1}{\sqrt{h}}\int |G_{\frac{h}{2}} * (\chi_i-\chi_i')|^2$ is a distance^2 of χ and χ'

c) thresholding means that χ^n minimizes

$2E_h(\chi;\chi^{n-1}) = -E_h(\chi-\chi^{n-1}) + E_h(\chi) + const$,

which is of form $\frac{1}{2h}$distance$^2(\chi,\chi^{n-1})$ + energy(χ)

Scheme preserves comparison and *gradient flow structure*

Natural generalization to $\{\sigma_{ij}\}$ (EO'14)

a) $E_h(\chi) := \sum_{i,j} \sigma_{ij} \frac{1}{\sqrt{h}} \int \chi_i \, G_h * \chi_j$

Γ-converges to $c_0 \sum_{i,j} \sigma_{ij} \frac{1}{2} \int |\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)|$

$= c_0$ total interfacial energy (eg Ambrosio&Braides'90)

provided $\{\sigma_{ij}\}$ satisfies triangle inequality

New element in proof: monotonicity $E_{kh}(\chi) \le E_h(\chi)$

b) $-E_h(\chi - \chi')$ is a distance^2 of χ and χ'

provided $\{\sigma_{ij}\}$ negative definite on $\delta\chi = (\delta\chi_i)_i$ with $\sum_i \delta\chi_i = 0$.

$\Longleftrightarrow$ ℓ^2-embeddability, ok for Read-Shockley, ok for $N \le 4$

c) χ^n minimizes $-E_h(\chi - \chi^{n-1}) + E_h(\chi)$ turns into

$\chi^{n-1} \rightsquigarrow u_i^n := \sum_j \sigma_{ij} G_h * \chi_j^{n-1} \rightsquigarrow \chi_i^n := I(u_i^n \le u_j^n \; \forall j)$

Thresholding scheme of same complexity!

Convergence of multi-phase thresholding

Holds for any number of phases N provided

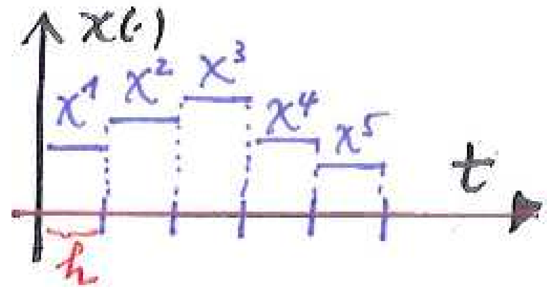
$\{\sigma_{ij}\}_{i,j=1,\cdots,N}$ positive definite & strict triangle inequality

State here for $N=2$ where $E_h(\chi) = \frac{1}{\sqrt{h}}\int_{[0,1)^d}(1-\chi)G_h * \chi$

χ^0 initial data with $\{E_h(\chi^0)\}_{h\downarrow 0}$ bdd

ie $\int_{[0,1)^d}|\nabla\chi^0| < \infty$,

χ_h piecewise constant interpolation of $\{\chi^n\}_n$



Have 2 *conditional* convergence results:

to BV solution and to Brakke-type solution

Convergence to BV solution (LO'15)

Theorem 1 Suppose $\chi_h \to \chi$ in $L^1((0,1)\times[0,1)^d)$ and

$$\int_0^1 E_h(\chi_h(t))dt \to c_0\int_0^1\int_{[0,1)^d}|\nabla\chi|dt.$$

Then there exists $V \in L^2(|\nabla\chi|dt)$ such that

for all $\zeta \in C_0^\infty((0,1)\times[0,1)^d)$

$$\int_0^1\int \partial_t\zeta\chi + \zeta V|\nabla\chi|dt = 0 \text{ (normal velocity } = V)$$

and for all $\xi \in C_0^\infty([0,1]\times[0,1)^d,\mathbb{R}^d)$

$$\int_0^1\int(\nabla\cdot\xi - \nu\cdot D\xi\nu + 2V\nu\cdot\xi)|\nabla\chi|dt = 0 \text{ (mean curv. } = -2V)$$

A *conditional* convergence result

Suppose $\chi_h \to \chi$ in $L^1((0,1)\times[0,1)^d)$ and

$$\int_0^1 E_h(\chi_h(t))dt \to c_0 \int_0^1 \int_{[0,1)^d} |\nabla\chi| dt.$$

Then $\exists\, V \in L^2(|\nabla\chi|dt)$ s. t. $\forall\, \zeta \in C_0^\infty((0,1)\times[0,1)^d)$, $\xi \in C_0^\infty([0,1]\times[0,1)^d, \mathbb{R}^d)$

$$\int_0^1 \int \partial_t\zeta\chi + \zeta V |\nabla\chi| dt = 0$$

$$\int_0^1 \int (\nabla\cdot\xi - \nu\cdot D\xi\nu + 2V\nu\cdot\xi)|\nabla\chi| dt = 0$$

Same assumption and notion of solution as in Luckhaus & Sturzenhecker '95 on

minimizing movement scheme for MCF introduced by Almgren & Taylor & Wang '93,

but more robust proof (no minimal surface regularity theory)

Gradient flow comes with energy (in)equality

$H :=$ mean curvature, $V =$ normal velocity

Seek energy inequality $\int (2V)^2 |\nabla\chi| = \int H^2 |\nabla\chi| \leq 2\frac{d}{dt}\int |\nabla\chi|$

Build-in into both notions of solutions of

De Giorgi $\frac{1}{2}\int H^2|\nabla\chi| + \frac{1}{2}\int (2V)^2|\nabla\chi| \leq 2\frac{d}{dt}\int|\nabla\chi|$

Brakke $\int (\zeta H^2 + \nu\cdot\nabla\zeta H)|\nabla\chi| \leq 2\frac{d}{dt}\int \zeta|\nabla\chi|$ for $\zeta \geq 0$

use an idea related to de Giorgi's to establish Brakke's

Convergence to Brakke-type solution (LO'16)

Theorem 2 Suppose $\chi_h \to \chi$ in $L^1((0,1)\times[0,1)^d)$ and

$$\int_0^1 E_h(\chi_h(t))dt \to c_0\int_0^1\int_{[0,1)^d}|\nabla\chi|dt.$$

Then there exists $H \in L^2(|\nabla\chi|dt)$ such that

for all $\xi \in C^\infty((0,1)\times[0,1)^d,\mathbb{R}^d)$

$$\int_0^1\int(\nabla\cdot\xi - \nu\cdot D\xi\nu - \nu\cdot\xi H)|\nabla\chi|dt = 0 \text{ (mean curv. } = H)$$

and for all $\zeta \in C^\infty((0,1)\times[0,1)^d,[0,\infty))$

$$\int_0^1\int(-2\partial_t\zeta + \zeta H^2 + \nu\cdot\nabla\zeta H)|\nabla\chi|dt \le 0$$

$(2\text{normal velocity} = -H)$

Contains correct inequality $2\frac{d}{dt}\int|\nabla\chi| \le -\int H^2|\nabla\chi|$

"Brakke-type" because
Brakke's inequality is expressed in BV-framework instead of varifold-framework

Comparison of both notions

BV solution (phase configuration) evokes smooth formulas

$\frac{d}{dt}\int_\Omega \zeta = \int_\Gamma \zeta V$ and $\int_\Gamma (\nabla\cdot\xi - \nu\cdot D\xi\nu) = \int_\Gamma \xi\cdot\nu H$

$\exists\ V$ s. t. $\forall\ \zeta \in \mathbb{R}, \xi \in \mathbb{R}^d$

$$\int_0^1 \int \partial_t\zeta\chi + \zeta V|\nabla\chi| dt = 0$$

$$\int_0^1 \int (\nabla\cdot\xi - \nu\cdot D\xi\nu + 2V\nu\cdot\xi)|\nabla\chi| dt = 0$$

Brakke's solution (phase *boundary* config.) replaces first by

$\frac{d}{dt}\int_\Gamma \zeta = \int_\Gamma (\zeta H V + \nabla\zeta\cdot\nu V)$

$\exists\ H$ s. t. $\forall\ \xi \in \mathbb{R}^d, \zeta \geq 0$

$$\int_0^1 \int (\nabla\cdot\xi - \nu\cdot D\xi\nu - \nu\cdot\xi H)|\nabla\chi| dt = 0$$

$$\int_0^1 \int (-2\partial_t\zeta\chi + \zeta H^2 + \nu\cdot\nabla\zeta H)|\nabla\chi| dt \leq 0$$

De Giorgi's abstract framework, cf. AGS '08

Minim. movem.: χ^n minimizes $\frac{1}{2h}\text{distance}^2(\cdot,\chi^{n-1})+E$.

Easy estimate $E(\chi^n)+\frac{1}{2}\sum_{k=1}^{n}\frac{1}{h}\text{distance}^2(\chi^k,\chi^{k-1})\le E(\chi^0)$

turns into suboptimal $E(\chi(nh))+\frac{1}{2}\int_0^{nh}|\dot\chi|^2dt\le E(\chi^0)$

Variational interpolation: $v(t)$, $t\in((n-1)h,nh]$, minimizes

$$\frac{1}{2(t-(n-1)h)}\text{distance}^2(\cdot,\chi^{n-1})+E,$$

metric slope $|\partial E|(\chi):=\limsup_{v\to\chi}\frac{E(\chi)-E(v)}{\text{distance}(v,\chi)}$. Get

$\frac{1}{2}|\partial E|^2(\chi^n)+\frac{1}{2h}\int_{(n-1)h}^{nh}|\partial E|^2(v(t))dt\le E(\chi^{n-1})-E(\chi^n)$,

converges to desired $|\partial E|^2(\chi)\le-\frac{d}{dt}E(\chi)$

Localization of minimizing movements

Seek: $\int(\zeta H^2+\nu\cdot\nabla\zeta H)|\nabla\chi|\le 2\frac{d}{dt}\int\zeta|\nabla\chi|$ for $\zeta\ge 0$

Have: χ^n minimizes among all $v\in[0,1]$

$$\frac{1}{\sqrt{h}}\int(1-v)G_h*v+\frac{1}{\sqrt{h}}\int(G_{\frac{h}{2}}*(v-\chi^{n-1}))^2$$
$$=E_h(v)+\frac{1}{2h}\mathrm{distance}^2(v,\chi^{n-1})$$

Localization: χ^n minimizes among all $v\in[0,1]$

$$\frac{1}{\sqrt{h}}\int\zeta(1-v)G_h*v+(v-\chi^{n-1})\Big([\zeta,G_h*](1-\chi^{n-1})+[\zeta,G_{\frac{h}{2}}*]G_{\frac{h}{2}}*(v-\chi^{n-1})\Big)$$
$$+\frac{1}{\sqrt{h}}\int\zeta(G_{\frac{h}{2}}*(v-\chi^{n-1}))^2=:\tilde{E}_h(v,\chi^{n-1})+\frac{1}{2h}\widetilde{\mathrm{distance}}^2(v,\chi^{n-1})$$

Via De Giorgi's framework to Brakke's inequality

Localization of minimizing movement's interpretation:
χ^n minimizes among all $v \in [0,1]$

$$\frac{1}{\sqrt{h}}\int \zeta(1-v)G_h * v + (v-\chi^{n-1})\Big([\zeta, G_h*](1-\chi^{n-1}) + [\zeta, G_{\frac{h}{2}}*]G_{\frac{h}{2}} * (v-\chi^{n-1})\Big)$$
$$+\frac{1}{\sqrt{h}}\int \zeta(G_{\frac{h}{2}} * (v-\chi^{n-1}))^2 =: \tilde{E}_h(v,\chi^{n-1}) + \frac{1}{2h}\widetilde{\text{distance}}^2(v,\chi^{n-1})$$

De Giorgi's abstract framework (Ambrosio&Gigli&Savaré'08)

$$\frac{1}{2}|\partial \tilde{E}_h(\cdot,\chi^{n-1})|^2(\chi^n) + \frac{1}{2h}\int_{(n-1)h}^{nh} |\partial \tilde{E}_h(\cdot,\chi^{n-1})|^2(v_h(t))dt$$
$$\leq \tilde{E}_h(\chi^{n-1},\chi^{n-1}) - \tilde{E}_h(\chi^n,\chi^{n-1})$$

Here $|\partial E|(\chi) := \limsup_{v\to\chi} \frac{(E(\chi)-E(v))_+}{\text{distance}(v,\chi)}$ ("metric slope")

and $v_h((n-1)h+t) := \text{argmin}\Big(E(v) + \frac{1}{2t}\text{distance}^2(v,\chi^{n-1})\Big)$
("variational interpolation" of $v_h((n-1)h) = \chi^{n-1}$ and $v_h(nh) = \chi^n$)

Characterize effect of localization

$$\tilde{E}_h(v,\chi^{n-1}) := \frac{1}{\sqrt{h}}\int \zeta(1-v)G_h * v + \frac{1}{\sqrt{h}}\int (v-\chi^{n-1})[\zeta, G_h *](1-\chi^{n-1}) + h.o.t.,$$

$$\frac{1}{2h}\widetilde{\text{distance}}^2(v,\chi^{n-1}) = \frac{1}{\sqrt{h}}\int \zeta(G_{\frac{h}{2}} * (v-\chi^{n-1}))^2$$

$$\frac{1}{2}|\partial \tilde{E}_h(\cdot,\chi^{n-1})|^2(\chi^n) + \frac{1}{2h}\int_{(n-1)h}^{nh} |\partial \tilde{E}_h(\cdot,\chi^{n-1})|^2(v_h(t))dt$$

$$\leq \tilde{E}_h(\chi^{n-1},\chi^{n-1}) - \tilde{E}_h(\chi^n,\chi^{n-1})$$

a) $|\partial E|(v) \geq \sup_\xi \dfrac{\text{diff}\,E(v).\xi}{2\sqrt{h}\int \zeta(G_{h/2} * (\xi\cdot\nabla v))^2}$ where

$\text{diff}\,E(v).\xi$ inner variation of E in v along vector field ξ,

b) $\text{diff}\,\tilde{E}_h(v_h,\chi^{n-1}).\xi = \text{diff}\,E_h(v_h).(\zeta\xi) + h.o.t.,$

c) $\frac{1}{\sqrt{h}}\int (\chi^n - \chi^{n-1})[\zeta, G_h *](1-\chi^{n-1})$

$= -\text{diff}\,\frac{1}{2h}\text{distance}^2(\cdot,\chi^{n-1})(\chi^n).\nabla\zeta + h.o.t. = \text{diff}\,E_h(\chi^n).\nabla\zeta + h.o.t.$

Reduce to 2 kernels and convergence assumption

a) $|\partial\tilde{E}_h(\cdot,\chi^{n-1})|^2(v_h) \geq \sup_\xi \frac{\text{diff}\,\tilde{E}_h(v_h,,\chi^{n-1}).\xi}{2\sqrt{h}\int\zeta(G_{h/2}*(\xi\cdot\nabla v))^2}$

b) $\text{diff}\,\tilde{E}_h(v_h,\chi^{n-1}).\xi = \text{diff}\,E_h(v_h).(\zeta\xi)+h.o.t.,$

c) $\frac{1}{\sqrt{h}}\int(\chi^n-\chi^{n-1})[\zeta,G_h*](1-\chi^{n-1}) = \text{diff}\,E_h(\chi^n).\nabla\zeta+h.o.t.$

a') $\sqrt{h}\int\zeta\big(G_{h/2}*(\xi\cdot\nabla v_h)\big)^2$
$= \int\zeta\xi\otimes\xi : (1-v_h)\sqrt{h}\nabla^2 G_h * v_h + h.o.t.$

b') $\text{diff}\,E_h(v_h).\xi$
$= \int\nabla\cdot\xi(1-v_h)\frac{1}{\sqrt{h}}G_h*v_h - D\xi : (1-v_h)\sqrt{h}\nabla^2 G_h*v_h + h.o.t.$

a'') $\int\zeta(1-v_h)\frac{1}{\sqrt{h}}G_h*v_h \to c_0\int\zeta|\nabla\chi|$ by convergence assumption

b'') $\int A : (1-v_h)\sqrt{h}\nabla^2 G_h*v_h \to c_0\int\nu\cdot A\nu|\nabla\chi|$ "

Additional steps in BV-convergence result

Challenge: limit in first variation of *metric term*

$$\text{diff}\frac{1}{2h}\text{dist}^2(\chi^n,\chi^{n-1}).\xi = 2\int\frac{\chi^n-\chi^{n-1}}{h}\,\xi\cdot\sqrt{h}\nabla G_h*\chi^n + \text{h.o.t.}$$

Morally speaking: Weak $\times$ weak convergence

$$\partial_t^h\chi_h := \frac{\chi^n-\chi^{n-1}}{h} \rightharpoonup V|\nabla\chi| \quad \text{and} \quad \sqrt{h}\nabla G_h*\chi^n \rightharpoonup c_0\nu$$

Strategy: Control $\int\partial_t^h\chi_h\,\xi\cdot\sqrt{h}\nabla G_h*\chi^n - \int\partial_t^h\chi_h c_0\xi\cdot e$

by **dissipation** $\alpha^2 := \sqrt{h}\int(G_{\frac{h}{2}}*(\partial_t^h\chi_h))^2$ and

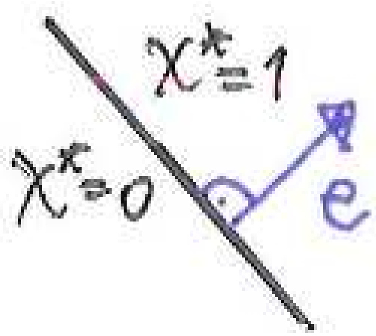
by **excess** $\epsilon^2 := E_h(\chi_h) - E_h(\chi^*)$ (in a localized version)

where χ^* corresponds to a half space.

Then use that $\epsilon^2 \longrightarrow c_0(\int|\nabla\chi| - \int|\nabla\chi^*|)$

by convergence assumption,

localized limit is small by BV-structure.



BV-convergence result less soft

Challenge in the above:

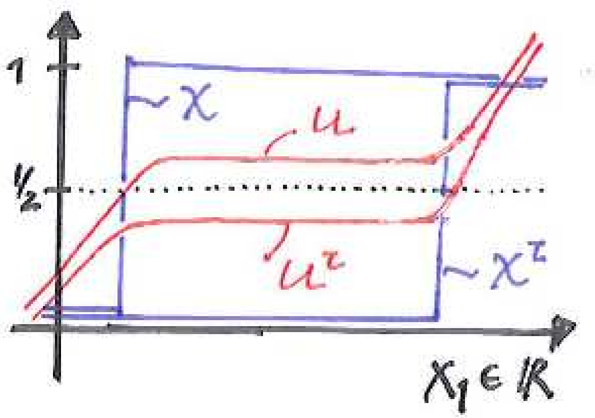
Control $\int|\partial_t^\tau\chi_h:=\frac{\chi_h(\cdot+\tau)-\chi}{\tau}|$ for time scales down to $\tau=o(\sqrt{h})$

by $\alpha^2:=\sqrt{h}\int(G_{\frac{h}{2}}*(\partial_t^h\chi_h))^2$ and $\epsilon^2:=E_h(\chi_h)-E_h(\chi^*)$,

Via estimate: $\int|\partial_t^\tau\chi_h|\lesssim\frac{\alpha^2}{s^2}\frac{\tau}{\sqrt{h}}+\frac{\sqrt{h}}{\tau}(s+\epsilon^2)$ for all $s\ll 1$

Use thresholding $\chi_h(\cdot+\tau)-\chi_h:=\chi_h^\tau-\chi_h=I(u_h^\tau>\frac{1}{2})-I(u_h>\frac{1}{2})$

get $\int|\chi^\tau-\chi|\lesssim\frac{1}{s^2}\int(u^\tau-u)^2+s\sqrt{h}+\int_{\{\frac14\le u\le\frac34\}}(\sqrt{h}\partial_1 u-\frac{1}{100})_-^2$



use $\int(u_h^\tau-u_h)^2\lesssim\frac{\tau^2}{\sqrt{h}}\alpha^2$ and

$\int_{\{\frac14\le u_h\le\frac34\}}(\sqrt{h}\partial_1 u_h-\frac{1}{100})_-^2\lesssim\sqrt{h}\epsilon^2$

Future questions on thresholding

Convergence to Brakke's notion for varifolds
without convergence assumption?
(learn from recent result by Kim&Tonegawa?)

Treatment of anisotropy (Chambolle, Esedoglu)