Another triumph for De Giorgi's Γ-convergence

Haïm Brezis, Pisa Sept. 2016

PLAN

1. Some background.
2. A suggestive computation. Pointwise convergence of $\Lambda_\delta(u)$
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1. Some background

Starting point comes from a paper Bourgain - Brezis - Mironescu, 2001.

Let $\Omega \subset \mathbb{R}^N$ smooth bounded, $N \geq 1$ (possibly $\Omega = \mathbb{R}^N$).

Let (ρ_ε) be a family of radial mollifiers i.e., $\rho_\varepsilon \in L^1_{loc}(0,\infty;\mathbb{R})$, (no smoothness required)

$$\rho_\varepsilon \geq 0 \text{ on } (0,\infty),$$

$$\int_0^\infty \rho_\varepsilon(r) r^{N-1} dr = 1,$$

and $\rho_\varepsilon \to \delta_0$, i.e. $\int_0^a \rho_\varepsilon(r) r^{N-1} dr \underset{\varepsilon\to 0}{\to} 1$, $\forall a > 0$.

Given $u \in L^1(\Omega)$ set

$$J_\varepsilon(u) = \int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|} \rho_\varepsilon(|x-y|) dx dy$$

Thm (BBM 2001, refinement by J. Davila 2002)
Given $u \in L^1(\Omega)$, then

$$u \in BV(\Omega) \iff \liminf_{\varepsilon_n \to 0} J_{\varepsilon_n}(u) < \infty.$$

In that case

$$\lim_{\varepsilon\to 0} J_\varepsilon(u) = K_N \int_\Omega |Du|, \quad \forall u \in BV(\Omega),$$

where $K_N = 2|B^{N-1}| = \int_{S^{N-1}} |\sigma\cdot e| d\sigma \quad \forall e \in S^{N-1}$.

Mode of convergence is robust.
A. Ponce (2004) proved that

$$J_\varepsilon(\cdot) \xrightarrow[\varepsilon \to 0]{} K_N |\cdot|_{BV}$$

in the sense of Γ-convergence in L^1.

Typical examples

Ex 1 $\rho_\varepsilon(r) = \frac{N}{\varepsilon^N} \mathbb{1}_{(0,\varepsilon)}$ gives

$$\frac{1}{\varepsilon^N} \iint_{\substack{\Omega\ \Omega \\ |x-y|<\varepsilon}} \frac{|u(x)-u(y)|}{|x-y|} dx\,dy \to \frac{K_N}{N} \int_\Omega |\nabla u| \quad \forall u \in BV.$$

Ex 2 $\rho_\varepsilon(r) = \frac{\varepsilon}{r^{N-\varepsilon}} \mathbb{1}_{(0,1)}$ gives

$$\varepsilon \int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|^{N+1-\varepsilon}} dx\,dy \to K_N \int_\Omega |\nabla u| \quad \forall u \in BV.$$

In particular if $u = \mathbb{1}_E$, then $\varepsilon \int_E \int_{CE} \frac{dx\,dy}{|x-y|^{N+1-\varepsilon}} \to \frac{K_N}{2} \operatorname{Per} E$

Appears under the name "Nonlocal perimeter"
in papers by Ambrosio - De Philippis - Martinazzi (2011)
Caffarelli - Valdinocci (2011), Fusco (2015) etc..

Around 2002 I played (for fun!) with another type of nonlocal functionals and I discovered that they also converge to the BV-norm.

Given $u \in L^1(\Omega)$, $\delta > 0$, set

$$L_\delta(u) = \delta \iint_{\substack{\Omega\ \Omega \\ |u(x)-u(y)|>\delta}} \frac{1}{|x-y|^{N+1}}\, dx\, dy .$$

Note that J_ε and L_δ are quite different nonlocal functionals

- J_ε is <u>convex</u> in u
- L_δ is <u>not</u> convex in u

(and this will be a source of tremendous difficulties).

Comment (J. M. Morel):

- J_ε is reminiscent of Riemann integral: "slicing in x"

- L_δ is reminiscent of Lebesgue integral: "slicing in u"

Fact (H.B. 2003). $\forall u \in C^1(\bar{\Omega})$

$$\lim_{\delta \to 0} L_\delta(u) = K_N \int_\Omega |\nabla u|.$$

See later a slightly more general version with a sketch of proof. ["A suggestive computation"].

In view of BBM it was natural to ask

Question (H.B. 2004) Assume $u \in L^1(\Omega)$. Is it true that

$$u \in BV(\Omega) \iff \liminf_{\delta_n \to 0} L_{\delta_n}(u) < \infty \quad ?$$

If so, is it true that

$$\lim_{\delta \to 0} L_\delta(u) = K_N \int_\Omega |\nabla u| \quad \forall u \in BV(\Omega) ?$$

This question generated an abundance of results — both positive and negative!

Pathology 1 (Ponce, 2005)

$\exists u \in W^{1,1}(\Omega)$ s.t. $\lim_{\delta \to 0} L_\delta(u) = +\infty$.

Hence $\Rightarrow$ in my question has a negative answer.

How about $\Leftarrow$? Turned out to be hard, but true!

Thm (Bourgain – H.M. Nguyen, 2006)
Let $u \in L^1(\Omega)$ be such that

$$\liminf_{\delta_n \to 0} L_{\delta_n}(u) < \infty$$

then $u \in BV(\Omega)$.
Moreover

$$\int_\Omega |\nabla u| \leq C_\Omega \liminf_{\delta_n \to 0} L_{\delta_n}(u) .$$

For the second component of my question

$$\liminf_{\delta_n \to 0} L_{\delta_n}(u) < \infty \overset{?}{\Longrightarrow} \lim_{\delta \to 0} L_\delta(u) = K_N \int |\nabla u|$$

the answer is ~~also~~ negative. The construction of a counterexample is new (B – Nguyen, 2016). See later Pathology 2.

In view of this "erratic" behavior of pointwise convergence of $L_\delta(u)$ [<u>Pointwise</u> convergence means analysis of the asymptotic behavior of $L_\delta(u)$ as $\delta \to 0$ for <u>fixed</u> u] I suggested to study $\lim_{\delta \to 0} L_\delta$ in the sense of Γ-convergence?

<u>Question</u> (H. B. 2006) Is it true that

$$L_\delta(\cdot) \xrightarrow[\delta \to 0]{} K_n \, | \cdot |_{BV}$$

in the sense of Γ convergence in L^1 ?

<u>Thm</u> (Nguyen, Duke 2011)

$$L_\delta \xrightarrow[\delta \to 0]{} L_0$$

in the sense of Γ-convergence in L^1, where

$$L_0(u) = \begin{cases} K_N' \int |\nabla u| & \text{if } u \in BV \\ +\infty & \text{otherwise} \end{cases}$$

for some constant $0 < K_N' < K_N$ (!!)

<u>Rem</u> This result is much stronger than Bourgain-Nguyen, but some ingredients from the proof of BN enter here.

A very lucky coincidence happened in 2011. At the Technion I met (casually) colleagues from the Computer Science Dept. involved in Image Processing (F. Bruckstein, R. Kimmel). They mentioned to me that nonlocal functionals of the form

$$\int_\Omega\int_\Omega \varphi\left(\frac{|u(x)-u(y)|}{\delta}\right) w(|x-y|)\,dx\,dy$$

were popular in Image Processing.
[$\delta>0$ small parameter, $\varphi(t)$ function to be discussed later, $w(|x-y|)$ weight function].
They also told that J.M. Morel is a leading expert ... J.M. Morel had been my PhD student, but we had lost contact. Quickly reconnected ... This information caught my attention because $\frac{1}{\delta}L_\delta$ has the form above with

$$\varphi(t)=\begin{cases}0 & 0<t\le 1\\ 1 & t>1\end{cases}\qquad w(|x-y|)=\frac{1}{|x-y|^{N+1}}.$$

I proposed to Nguyen that we study together such functionals.
Outcome is a long paper Aug. 2016 now on Arxiv.
[worked on/off 2012 – 2016].
Longest section concerns Γ-convergence (~30 p.).

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Non-local functionals related to the total variation and connections with Image Processing

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Abstract

We present new results concerning the approximation of the total variation, $\int_\Omega |\nabla u|$, of a function u by non-local, non-convex functionals of the form

$$\Lambda_\delta u = \int_\Omega \int_\Omega \frac{\delta\varphi\big(|u(x) - u(y)|/\delta\big)}{|x-y|^{d+1}}\,dx\,dy,$$

as $\delta \to 0$, where Ω is a domain in $\mathbb{R}^d$ and $\varphi : [0, +\infty) \to [0, +\infty)$ is a non-decreasing function satisfying some appropriate conditions. The mode of convergence is extremely delicate and numerous problems remain open. De Giorgi's concept of Gamma-convergence illuminates the situation, but also introduces mysterious novelties. The original motivation of our work comes from Image Processing.

Key words: Total variation, bounded variation, non-local functional, non-convex functional, Gamma-convergence, Sobolev spaces.

Mathematics Subject Classification: 49, 26B30, 46E35, 28A75.

Contents

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1 Introduction

Throughout this paper, we assume that $\varphi : [0, +\infty) \to [0, +\infty)$ is continuous on $[0, +\infty)$ except at a finite number of points in $(0, +\infty)$ where it admits a limit from the left and from the right. We also assume that $\varphi(0) = 0$ and that $\varphi(t) = \min\{\varphi(t+), \varphi(t-)\}$ for all $t > 0$, so that φ is lower semi-continuous. We assume that the domain $\Omega \subset \mathbb{R}^d$ is either bounded and smooth, or that $\Omega = \mathbb{R}^d$. The case $d = 1$ is already of great interest; many difficulties (and open problems!) occur even when $d = 1$.

Given a measurable function u on Ω, and a small parameter $\delta > 0$, we define the following non-local functionals:

$$\Lambda(u) := \int_\Omega \int_\Omega \frac{\varphi(|u(x) - u(y)|)}{|x - y|^{d+1}} \, dx \, dy \quad \text{and} \quad \Lambda_\delta(u) := \delta \Lambda(u/\delta). \tag{1.1}$$

Sometimes, it is convenient to be more specific and to write $\Lambda_\delta(u, \varphi, \Omega)$ or $\Lambda_\delta(u, \Omega)$ instead of $\Lambda_\delta(u)$.

2. A suggestive computation. Pointwise convergence of $\Lambda_\delta(u)$.

Let $\Omega \subset \mathbb{R}^N$ smooth bounded, $N \geq 1$. The case $N=1$ is already very interesting [still many problems open when $N=1$]. The case $N=2$ is important because it appears in Image Processing.

The basic notations/assumptions:

Fix a function $\varphi : [0,\infty) \to [0,\infty)$.

Given $u \in L^1(\Omega)$ consider the nonlocal functional

$$\Lambda(u) = \int_\Omega \int_\Omega \varphi(|u(x)-u(y)|) \frac{1}{|x-y|^{N+1}} \, dx\, dy \leq \infty$$

and the scaled version

$$\Lambda_\delta(u) = \delta \int_\Omega \int_\Omega \varphi\left(\frac{|u(x)-u(y)|}{\delta}\right) \frac{1}{|x-y|^{N+1}} \, dx\, dy = \delta \Lambda\left(\frac{u}{\delta}\right),$$

with $\delta > 0$ small parameter.

<u>Class $\mathcal{A}$</u> consists of all functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying:

(1) φ is nondecreasing on $[0,\infty)$.

(2) φ is continuous on $[0,\infty)$ except at a finite number of points where

$$\varphi(t) = \varphi(t-).$$

(3) $\varphi(t) \leq a t^2 \quad \forall t \in [0,1]$ and some $a > 0$

(4) $\varphi(t) \leq b \quad \forall t \geq 0$ and some $b > 0$

(5) $\displaystyle K_N \int_0^\infty \frac{\varphi(t)}{t^2}\, dt = 1.$

Assumption (5) is a <u>normalization</u> condition - its role will become clear later.

Typical examples are

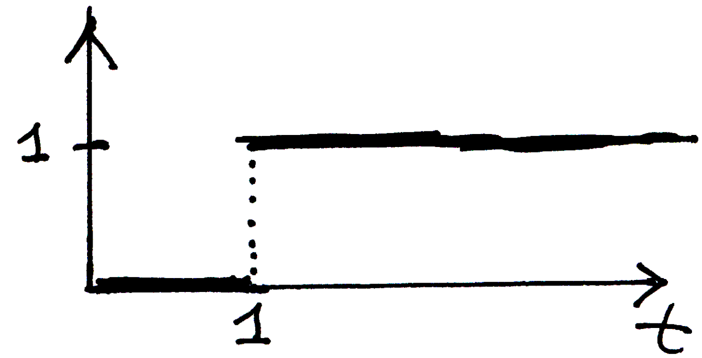
$$\varphi_1(t) = C_1 \tilde{\varphi}_1(t)$$
$$\varphi_2(t) = C_2 \tilde{\varphi}_2(t)$$
$$\varphi_3(t) = C_3 \tilde{\varphi}_3(t)$$

where $\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3$ are defined below and $C_1, C_2\, C_3$

are adjusted so that $\varphi_1, \varphi_2, \varphi_3$ satisfy (5).

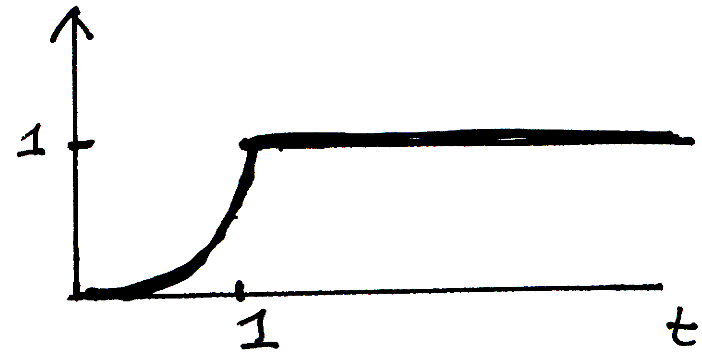
Ex 1

$$\tilde{\varphi}_1(t) = \begin{cases} 0 & t \leq 1 \\ 1 & t > 1 \end{cases}$$



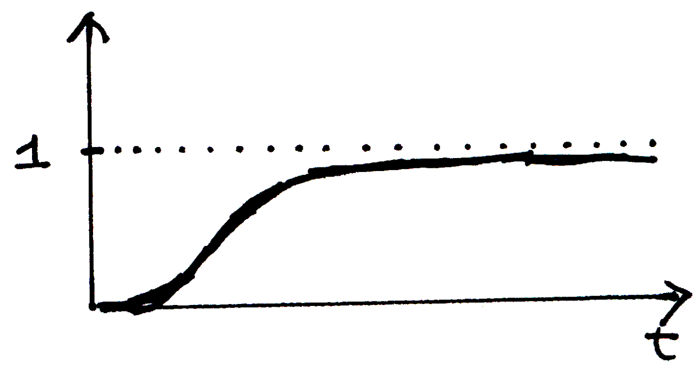
Ex 2

$$\tilde{\varphi}_2(t) = \begin{cases} t^2 & t \leq 1 \\ 1 & t > 1 \end{cases}$$



Ex 3

$$\tilde{\varphi}_3(t) = 1 - e^{-t^2}$$



$\tilde{\varphi}_1$ is the case studied by Nguyen.

$\tilde{\varphi}_2, \tilde{\varphi}_3$ occur in Image Processing.

Prop Assume $\varphi \in \mathcal{A}$. Then

(6) $$\lim_{\delta \to 0} \Lambda_\delta(u) = \int_\Omega |\nabla u| \quad \forall u \in C^1(\overline{\Omega}).$$

[Normalization condition enters to have 1 here]

More generally

(7) $$\lim_{\delta \to 0} \Lambda_\delta(u) = \int_\Omega |\nabla u| \quad \text{if } u \in W^{1,p}(\Omega) \text{ for some } 1 < p < \infty.$$

(8) If $u \in W^{1,1}(\Omega)$ one can only assert that

$$\liminf_{\delta \to 0} \Lambda_\delta(u) \geq \int_\Omega |\nabla u|.$$

Proof of (6) is fairly easy (see below).
Proof of (7) is much more complicated.
(Uses theory of maximal functions].
Proof of (8) relies on Fatou.

Sketch of the proof of (6)

"A suggestive computation".

$$\Lambda_\delta(u) = \underbrace{\delta \iint_{|x-y|<\delta^\alpha} \cdots}_{=A_\delta} + \underbrace{\delta \iint_{|x-y|\geq\delta^\alpha}}_{=B_\delta}$$

where $0<\alpha<\frac{1}{N+1}$ is fixed.

Clearly $B_\delta \to 0$ as $\delta \to 0$.

Changing variables $y \longmapsto x+h$

$$A_\delta = \delta \iint_{|h|<\delta^\alpha} \varphi\left(\frac{|u(x+h)-u(x)|}{\delta}\right)\frac{1}{|h|^{N+1}}\,dx\,dh.$$

Taylor's expansion (OK since $|h|<\delta^\alpha \to 0$)

$$A_\delta \simeq \delta \iint_{|h|<\delta^\alpha} \varphi\left(\frac{|h\cdot\nabla u(x)|}{\delta}\right)\frac{1}{|h|^{N+1}}\,dx\,dh.$$

[Cheating but this can be fixed]

Use polar coordinates $r=|h|$ $\sigma=\frac{h}{|h|}\in S^{N-1}$

Then

$$A_\delta \simeq \delta \int_\Omega dx \int_{S^{N-1}} d\sigma \int_0^{\delta^\alpha} \varphi\left(\frac{r|\sigma\cdot\nabla u(x)|}{\delta}\right)\frac{1}{r^2}\,dr$$

Change of variable $r \mapsto s = \dfrac{r\,|\sigma\cdot\nabla u(x)|}{\delta}$.

$$A_\delta \simeq \int_\Omega dx \int_{S^{N-1}} d\sigma \int_0^{\frac{|\sigma\cdot\nabla u(x)|}{\delta^{1-\alpha}}} \frac{\varphi(s)}{s^2}\,|\sigma\cdot\nabla u(x)|\,ds,$$

As $\delta \to 0$,

$$A_\delta \to \int_\Omega dx \int_{S^{N-1}} |\sigma\cdot\nabla u(x)|\,d\sigma \int_0^\infty \frac{\varphi(s)}{s^2}\,ds.$$

But

$$\int_{S^{N-1}} |\sigma\cdot\nabla u(x)|\,d\sigma = |\nabla u(x)| \int_{S^{N-1}} |\sigma\cdot e|\,d\sigma \qquad e\in S^{N-1}$$

$$= K_N\,|\nabla u(x)|.$$

From the normalization condition (5) we have

$$A_\delta \xrightarrow[\delta\to 0]{} \int_\Omega |\nabla u(x)|\,dx.$$

Some pathologies

Pathology 1 (similar to Ponce, 2005)

$\forall \varphi \in \mathcal{A} \quad \exists u \in W^{1,1}(\Omega)$ s.t.

$$\lim_{\delta \to 0} \Lambda_\delta(u) = +\infty$$

Pathology 2 $\forall \varphi \in \mathcal{A}, \exists u \in W^{1,1}(\Omega)$ s.t.

$$\limsup_{\delta \to 0} \Lambda_\delta(u) = +\infty$$

$$\liminf_{\delta \to 0} \Lambda_\delta(u) = \int_\Omega |\nabla u|$$

Pathology 3 $\forall \varphi \in \mathcal{A}, \exists u \in BV(\Omega) \ (u \notin W^{1,1})$

s.t.

$$\liminf_{\delta \to 0} \Lambda_\delta(u) < \int_\Omega |\nabla u| .$$

Seems hopeless to have a satisfactory theory of pointwise convergence for Λ_δ.

Γ-convergence saves the situation !!

3. Γ-convergence of Λ_δ

The main result:

THEOREM 1 $\forall \varphi \in \mathcal{A}$, $\exists$ constant $k = k(\varphi)$ (k is independent of Ω) s.t.

(9) $$0 < k \leq 1,$$

and

(10) $$\Lambda_\delta \xrightarrow[\delta \to 0]{} \Lambda_0 \text{ in the sense of } \Gamma\text{-convergence in } L^1(\Omega)$$

where

$$\Lambda_0(u) = \begin{cases} k \int_\Omega |\nabla u| & \text{if } u \in BV(\Omega), \\ +\infty & \text{if } u \in L^1(\Omega), u \notin BV(\Omega). \end{cases}$$

Recall definition of Γ-convergence

(G1) $\forall (u_\delta) \subset L^1(\Omega)$ s.t. $u_\delta \xrightarrow[\delta \to 0]{} u$ in $L^1(\Omega)$ we have

$$\liminf_{\delta \to 0} \Lambda_\delta(u_\delta) \geq \Lambda_0(u).$$

(G2) $\forall u \in L^1(\Omega)$, $\exists (\tilde{u}_\delta) \subset L^1(\Omega)$ s.t.

$$\tilde{u}_\delta \to u \quad \text{in } L^1(\Omega)$$

and

$$\limsup_{\delta \to 0} \Lambda_\delta(\tilde{u}_\delta) \leq \Lambda_0(u).$$

Comments

1) Proof of Thm 1 is long and intricate. Total $\simeq 50$ p. [30 p. in new paper + 20 p. of results coming from Nguyen, Duke 2011]

Please help to simplify it !!

2) The fact that $k > 0$ is already nontrivial. A special consequence is a stronger version of the question I raised in 2004:

Cor Assume $(u_{\delta_n}) \subset L^1(\Omega)$ and $u \in L^1(\Omega)$ satisfy

$$u_{\delta_n} \to u \quad \text{in } L^1(\Omega) \text{ as } \delta_n \to 0,$$

and

$$\liminf_{\delta_n \to 0} \Lambda_{\delta_n}(u_n) < \infty$$

Then

$$u \in BV(\Omega) \text{ and } \int_\Omega |\nabla u| \leq \frac{1}{k} \liminf_{\delta_n \to 0} \Lambda_{\delta_n}(u_n).$$

My original question was for $u_{\delta_n} \equiv u \in L^1$ and $\varphi = \varphi_1$. Settled positively by Bourgain - Nguyen (2006).

3) The occurence of the constant $0 < k \leq 1$ in Thm 1 is still <u>mysterious</u>!

Not clear why it may happen that $k(\varphi) < 1$ for <u>some</u> φ's (possibly <u>every</u> $\varphi \in \mathcal{A}$?)

<u>Open problem</u> Is it true that

$$k(\varphi) < 1 \quad \forall \varphi \in \mathcal{A}$$

Open even for $N=1$, $\varphi = \varphi_2$ or $\varphi = \varphi_3$.

The existence of k is established via a "<u>semi-explicit</u>" construction (see later) which yields <u>very little information</u> about k. In fact the <u>exact</u> value of k is <u>not</u> known even when $N=1$, $\varphi = \varphi_1$

At this time, all we know is

<u>Prop 2</u> (Nguyen, Duke, 2011). For any $N \geq 1$

$$k(\varphi_1) < 1 .$$

<u>Sketch of proof</u> (bizarre!)

Take e.g. $N=1$, $\Omega = (0,1)$, $\varphi = \varphi_1$

Set $U(x) = x$, so that by Prop. 1

$$\lim_{\delta \to 0} \Lambda_\delta(U) = \int_0^1 |U'| = 1.$$

A brute force construction yields a family (U_δ) s.t.

$$U_\delta \to U \quad \text{in } L^1 \text{ (even in } L^\infty\text{)}$$

and

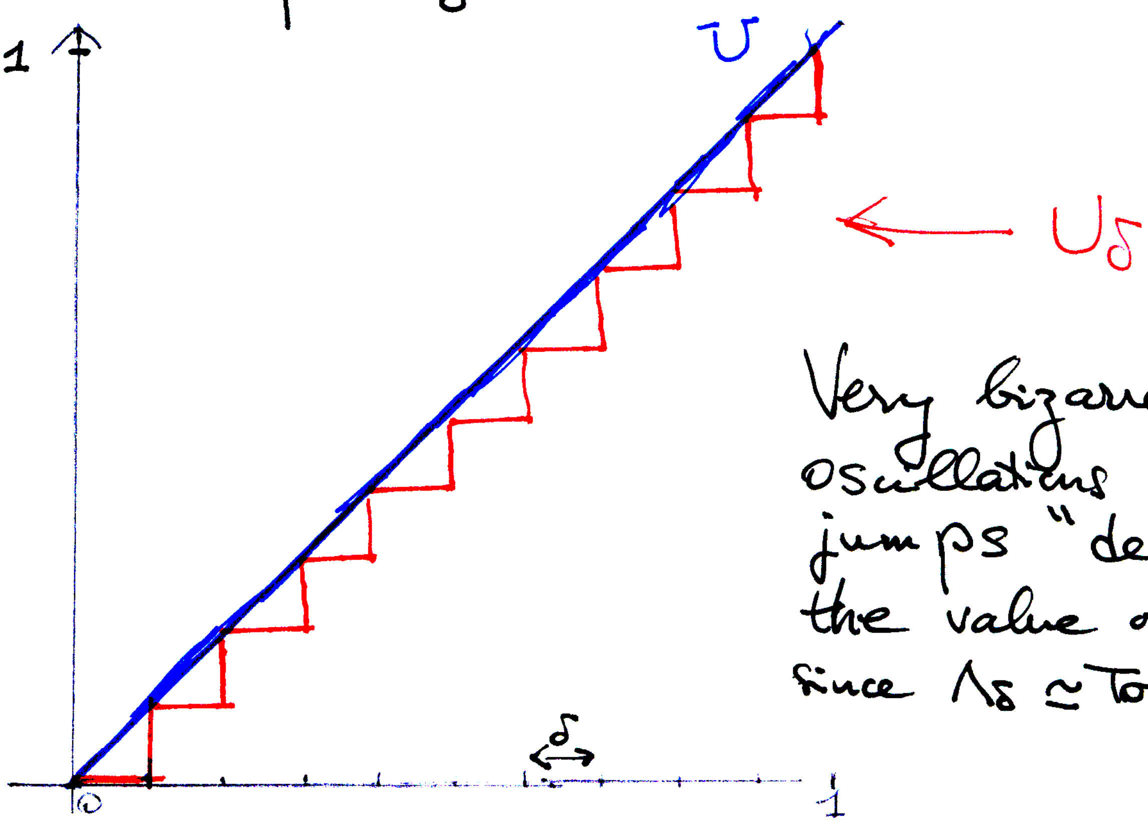
$$\liminf_{\delta \to 0} \Lambda_\delta(U_\delta) < 1 .$$

From (G1) we have

$$k \underbrace{\int |U'|}_{=1} \leq \liminf_{\delta \to 0} \Lambda_\delta(U_\delta) < 1$$

and thus $k < 1$.

Construction of U_δ



Very bizarre that oscillations and jumps "decrease" the value of Λ_δ since $\Lambda_\delta \simeq$ Total variation

So far this argument works only for $\varphi=\varphi_1$
We are unable to adapt it to φ_2 or φ_3.
It might well be that $k(\varphi_2)=1$ or $k(\varphi_3)=1$?

Structure of the proof of Thm 1.
Take $\Omega = Q$ = unit cube in $\mathbb{R}^N$.
<u>Fix</u> a special simple function U :

$$U(x_1, x_2, .., x_N) = \frac{1}{\sqrt{N}} (x_1 + x_2 + .. + x_N)$$

so that $\int_Q |\nabla U| = 1$.

<u>Define</u>

$$k(\varphi) = \inf \left\{ \liminf_{\delta\to 0} \Lambda_\delta(u_\delta);\ u_\delta \underset{\delta\to 0}{\to} U \text{ in } L^1 \right\}.$$

taken over all sequence u_δ s.t. $u_\delta \to U$ in L^1

<u>Prove</u> that (G.1) and (G.2) in Thm 1
hold ~~for~~ with <u>this</u> k !

4. Variational problems with roots in Image Processing

A fundamental challenge in Image Processing is to improve images of poor quality.
Denoising is an immense subject.
Excellent survey by Buades - Coll - Morel SIAM Review 2010.
A prevailing strategy is to use a variational formulation associated with a filter F. Given an image $f(x)$ of poor quality, consider

$$(11) \quad \operatorname*{Min}_{u} \left\{ \lambda \int_{\Omega} |f-u|^2 + F(u) \right\}$$

where $\Omega \subset \mathbb{R}^2$ (we'll work with $\Omega \subset \mathbb{R}^N$)
$\lambda > 0$ is the fidelity parameter (at our disposal). F is an appropriate functional: many filters are being tested - still an empirical science!
The minimizers of (11) are the denoised images.

Two popular filters:

A) <u>Rudin - Osher - Fatemi</u> (ROF, 1992)

$$F(u) = \int_\Omega |\nabla u|$$

(ROF) $\quad \underset{u}{\mathrm{Min}} \left\{ \lambda \int_\Omega |f-u|^2 + \int_\Omega |\nabla u| \right\}$

Convex functional. $\forall f \in L^2 \; \exists$ unique minimizer $u_0 \in BV \cap L^2$.

B) <u>Neighborhood filters</u>

(Lee, Yaroslavsky, 1985, SUSAN filter, 1997 Bilateral filters, 2008, Non-local means, 2011 ...)

uses $\quad F(u) \simeq \int_\Omega \int_\Omega \varphi\left(\frac{|u(x)-u(y)|}{\delta}\right) w(|x-y|)\, dx\, dy$

where $\delta > 0$ is a small parameter, $\varphi = \varphi_3$ for example and $w(\cdot)$ is weight function (many choices have been used).

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SIAM Review, 2010

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problem (2.2) for a given value of λ. The Euler–Lagrange equation associated with the minimization problem is given by

$$(u(\mathbf{x}) - v(\mathbf{x})) - \frac{1}{2\lambda} curv(u)(\mathbf{x}) = 0$$

(see [78]). Thus, we have the following theorem.

THEOREM 2.5. *The method noise of the total variation minimization* (2.2) *is*

$$u(\boldsymbol{x}) - TVF_\lambda(u)(\boldsymbol{x}) = -\frac{1}{2\lambda} curv(TVF_\lambda(u))(\mathbf{x}).$$

As in the anisotropic case, straight edges are maintained because of their small curvature. However, details and texture can be oversmoothed if λ is too small, as is shown in Figure 2.

2.4. Neighborhood Filters. The previous filters are based on a notion of spatial neighborhood or proximity. *Neighborhood filters* instead take into account grayscale values to define neighboring pixels. In the simplest and more extreme case, the denoised value at pixel i is an average of values at pixels which have a grayscale value close to $u(i)$. The grayscale neighborhood is therefore

$$B(i, h) = \{j \in I \mid u(i) - h < u(j) < u(i) + h\}.$$

This is a fully nonlocal algorithm, since pixels belonging to the whole image are used for the estimation at pixel i. This algorithm can be written in a more continuous form,

$$NF_h u(\mathbf{x}) = \frac{1}{C(\mathbf{x})} \int_\Omega u(\mathbf{y}) e^{-\frac{|u(\mathbf{y}) - u(\mathbf{x})|^2}{h^2}} d\mathbf{y},$$

(at the level of the Euler eq. $2\lambda(u-f) + F'(u) = 0$)

where $\Omega \subset \mathbb{R}^2$ is an open and bounded set, and $C(\mathbf{x}) = \int_\Omega e^{-\frac{|u(\mathbf{y})-u(\mathbf{x})|^2}{h^2}} d\mathbf{y}$ is the normalization factor.

The Yaroslavsky neighborhood filters [97, 95] consider mixed neighborhoods $B(i,h) \cap B_\rho(i)$, where $B_\rho(i)$ is a ball of center i and radius ρ. So the method takes an average of the values of pixels which are both close in grayscale and spatial distance. This filter can be easily written in a continuous form as

$$YNF_{h,\rho}(\mathbf{x}) = \frac{1}{C(\mathbf{x})} \int_{B_\rho(\mathbf{x})} u(\mathbf{y}) e^{-\frac{|u(\mathbf{y}) - u(\mathbf{x})|^2}{h^2}} d\mathbf{y},$$

where $C(\mathbf{x}) = \int_{B_\rho(\mathbf{x})} e^{-\frac{|u(\mathbf{y})-u(\mathbf{x})|^2}{h^2}} d\mathbf{y}$ is the normalization factor. More recent versions, namely, the *SUSAN filter* [83] and the *bilateral filter* [86], weigh the distance to the reference pixel $\mathbf{x}$ instead of considering a fixed spatial neighborhood,

In the next theorem we compute the asymptotic expansion of the Yaroslavky neighborhood filter when $\rho, h \to 0$.

THEOREM 2.6. *Suppose $u \in C^2(\Omega)$, and let $\rho, h, \alpha > 0$ such that $\rho, h \to 0$ and $h = O(\rho^\alpha)$. Let us consider the continuous function $\tilde{g}$ defined by $\tilde{g}(t) = \frac{1}{3}\frac{te^{-t^2}}{E(t)}$, for $t \neq 0$, $\tilde{g}(0) = \frac{1}{6}$, where $E(t) = 2\int_0^t e^{-s^2}\, ds$. Let $\tilde{f}$ be the continuous function defined by $\tilde{f}(t) = 3\tilde{g}(t) + \frac{3\tilde{g}(t)}{t^2} - \frac{1}{2t^2}$, $\tilde{f}(0) = \frac{1}{6}$. Then, for $\boldsymbol{x} \in \Omega$,*

1. *if $\alpha < 1$, $YNF_{h,\rho} u(\boldsymbol{x}) - u(\boldsymbol{x}) \simeq \frac{\Delta u(\boldsymbol{x})}{6}\rho^2$;*

We suggest to use

$$\lambda \simeq \frac{1}{\delta}, \quad \varphi \in \mathcal{A}, \quad w(|x-y|) = \frac{1}{|x-y|^{N+1}}$$

Then denoising mechanism becomes

$$(NF)_\delta \qquad \operatorname*{Min}_{u \in L^2} \left\{ \int_\Omega |f-u|^2 + \Lambda_\delta(u) \right\}.$$

THEOREM 2 Assume $\varphi \in \mathcal{A}$ and also

(11) $\quad \varphi(t) > 0 \quad \forall t > 0 \quad [\varphi = \varphi_2 \text{ or } \varphi_3 \text{ OK}]$

Then, given any $f \in L^2$

(i) $\forall \delta > 0 \ \exists u_\delta$ minimizer of $(NF)_\delta$ (but no uniqueness)

(ii) As $\delta \to 0$, $u_\delta \to u_0$ in L^2 where u_0 is the unique minimizer of

$$(ROF_k) \qquad \operatorname*{Min}_{u \in L^2} \left\{ \int_\Omega |f-u|^2 + k \int |\nabla u| \right\}$$

and $k > 0$ is the constant which comes from the Γ-convergence Thm 1

Comments

1) The convergence part (ii) is totally new and came as a surprise to the experts in Image Processing!!
Concretely they still prefer to use $(NF)_\delta$ with $\delta > 0$ small (and not $\delta \to 0$) but it is gratifying to know that $(NF)_\delta \to (ROF)$ as $\delta \to 0$.

[Compare with various convergence results in Fluid Mechanics. Boltzmann, Navier-Stokes, Euler..]

Part (ii) does not require assumption (11); it suffices to assume that u_δ is an "almost minimizer".

2) The existence of a minimizer - part (i) - for $(NF)_\delta$ with fixed $\delta > 0$ is also totally new. [But it is of limited interest in Image Processing because people are satisfied with almost minimizers].
Recall that Λ_δ is not convex and standard tools of Functional Analysis cannot be applied. Use instead new compactness results.

Basic ingredients for the proof of Thm 2

Two compactness lemmas:

- For fixed $\delta > 0$.
- For $\delta \to 0$.

Lemma 1 Assume $\varphi \in \mathcal{A}$ and (11).
For every fixed $\delta > 0$ (e.g. $\delta = 1$) the set

$$\left\{ u \in L^1(\Omega) ; \ \|u\|_{L^1} \leq C \text{ and } \Lambda_\delta(u) \leq C \right\}$$

is compact in $L^1(\Omega)$.

Lemma 1 is used to prove part (i) in Thm 2.

Lemma 2 Assume $\varphi \in \mathcal{A}$. Let (u_{δ_n}) be a sequence such that $\delta_n \to 0$,

$$\|u_{\delta_n}\|_{L^1} \leq C \quad \text{and} \quad \Lambda_{\delta_n}(u_{\delta_n}) \leq C.$$

Then (u_{δ_n}) is relatively compact in $L^1(\Omega)$.

Part (ii) in Thm 2 is an immediate consequence of Thm 1 (Γ-convergence of Λ_δ) and Lemma 2.

The proofs of Lemmas 1 and 2 rely on some new unusual inequality due to H.M. Nguyen (CVPDE, 2011) with roots in Bourgain-Nguyen (2006).

Let B_1 = unit ball in $\mathbb{R}^N$.

Lemma 3 $\exists C_N$ such that $\forall u \in L^1(B_1)$

$$\int_{B_1}\int_{B_1} |u(x)-u(y)|\,dx\,dy \leq C_N \Big[1 + \underset{[|u(x)-u(y)|>1]}{\int_{B_1}\int_{B_1}} \frac{1}{|x-y|^{N+1}}\,dx\,dy\Big].$$

Proof of Lemma 3 is nontrivial. Better proof? Better inequality? e.g.

$$\underset{[|u(x)-u(y)|>1]}{\int_{B_1}\int_{B_1}} |u(x)-u(y)|\,dx\,dy \leq C_N \underset{[|u(x)-u(y)|>1]}{\int_{B_1}\int_{B_1}} \frac{1}{|x-y|^{N+1}}\,dx\,dy \;?$$