Equivariant wave maps on rotationally symmetric manifolds

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A Mathematical Tribute to Ennio De Giorgi
Scuola Normale Superiore
Pisa, September 19-23, 2016

DG's interest in linear hyperbolic equations

Origin of his interest was (probably) his investigation on the problem of uniqueness for evolution equations

- Colombini, De Giorgi, Spagnolo: Sur les équations hyperboliques avec des coefficients qui ne d épendent que du temps, Ann. Sc. Norm. Sup. Pisa Cl. Sci. 1979
- De Giorgi: Some open problems in the theory of partial differential equations, Hyperbolic Equations (Padova, 1985)
- De Giorgi: Congetture riguardanti alcuni problemi di evoluzione, A celebration of J. F. Nash Jr, Duke Math. J. 1995

DG and non-linear hyperbolic equations

The 3D supercritical wave equation:

$$\boxed{\Box u + u^7 = 0}$$

that is to say, $u(t,x) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ solves

$$u_{tt} - \Delta u + u^7 = 0, \qquad u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x)$$

The critical power in 3D is $p = 5$, so any $p > 5$ is supercritical

De Giorgi's immediate reaction:

"Well... I think a radial solution should blow up at the origin" (that's all)

Still an open problem – numerics suggest blow up

Actually, one can prove global existence and stability if one removes say a sphere around the origin (exterior problem with Dirichlet boundary conditions)

References

This talk is partly based on joint works with Qidi Zhang (Shanghai)

1. D'A-Zhang: *IMRN* 2016
2. D'A-Zhang: work in progress
3. D'A.: *CMP* 2015

The problem

Wave maps are maps $u : M \to N$ where

1. the target manifold (N, g) is a complete manifold
2. the base manifold (M, h) is a complete Lorentzian manifold (with signature $(+, -, \dots, -)$)
3. u is a critical point for the functional on M with density $L(u) = Tr_h(u^* g)$ (trace with respect to the metric h of the pullback through u of the metric g)

If M is Riemannian we obtain the usual harmonic maps

In local coordinates (x^α) on M and (u^a) on N:

$$L(u) = h^{\alpha\beta} g_{ab} \frac{\partial u^a}{\partial x^\alpha} \frac{\partial u^b}{\partial x^\beta}$$

Wave maps arise in different contexts:

1. Field theory (nonlinear sigma models)
2. General relativity (Einstein equations under suitable assumptions)
3. They are an evolutionary version of harmonic maps
4. Special case of Yang-Mills

It is also a test ground for more complex geometric equations

Standard case: M is the flat Minkowski space-time $\mathbb{R}_t \times \mathbb{R}^n$

In local coordinates on the target we get a system of quasilinear wave equations

$$\Box u^a + \Gamma^a_{bc}(u)\partial_\alpha u^b \partial^\alpha u^c = 0$$

Embedding $N \subseteq \mathbb{R}^K$ isometrically (via the Nash Theorem) gives an equivalent formulation, in the smooth case

$$u : \mathbb{R}^n \times \mathbb{R}_t \to \mathbb{R}^K, \qquad u(t,x) \in N, \qquad \Box u \perp N$$

Example (Target: sphere)

Wave maps $u : \mathbb{R}_t \times \mathbb{R}^n \to \mathbb{S}^k$ satisfy the equation

$$\Box u + (|u_t|^2 - |\nabla u|^2)u = 0$$

Example (Target: hypersurface $F(u) = 0$)

Wave maps $u : \mathbb{R}_t \times \mathbb{R}^n \to S$ where $S = \{u \in \mathbb{R}^k : F(u) = 0\}$

$$\Box u + D^2 F(\partial_\alpha u, \partial^\alpha u) \frac{\nabla F(u)}{|\nabla F(u)|^2} = 0$$

Results for a flat base manifold

An extensive theory is available for wave maps $u : \mathbb{R}_t \times \mathbb{R}^n \to N$

The natural setting is the Cauchy problem for data

$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x)$$

with $(u_0, u_1) \in H^s \times H^{s-1}$ and $u \in C([0,T]; H^s(\mathbb{R}^n))$

The critical space is $H^{\frac{n}{2}}$, determined by scaling

$$u(t,x) \mapsto u(\lambda t, \lambda x)$$

One expects:

- $s < \frac{n}{2}$: the problem is ill-posed
- $s > \frac{n}{2}$: the problem is well-posed
- $s = \frac{n}{2}$: the problem is critical

In the critical case the geometry of the problem and the structure of the data may play an essential role

Local theory meets the expectations:

- The problem is locally well posed for $s > \frac{n}{2}$ $(n \geq 3)$
  Klainerman-Machedon 93, Klainerman-Rodnianski 01
  Null form structure is essential
  Bilinear methods for the wave equations were used for the first time to solve this problem
- The problem is ill posed for $s < \frac{n}{2}$
  D'A-Georgiev 04
  Construction of unstable families of solutions taking values in a geodesic in N $(s = \frac{n}{2})$
  D'A-Georgiev 03
  Non-uniqueness of weak solutions with target the sphere $(s < \frac{n}{2})$

Global theory is more delicate:

- Global existence with small data in $H^{\frac{n}{2}}$ holds for all reasonable targets
  Tao 01, Shatah-Struwe 02, Tataru 04
- Blow-up in finite time can occur for large data
  Shatah-TahvildarZadeh 94: $N =$ sphere, $n \geq 3$
  Cazenave-Shatah-TahvildarZadeh 98: $N =$ certain manifolds with negative curvature, $n \geq 7$

Note: no blow-up examples are known in dimension $n \leq 6$ for negatively curved targets

The 2D case is especially interesting:

Krieger, Schlag, Tataru 06-10:
Blow up for wave maps $u : \mathbb{R}_t \times \mathbb{R}^2 \to \mathbb{S}^2$
Global existence for wave maps $u : \mathbb{R}_t \times \mathbb{R}^2 \to \mathbb{H}^2$

Blow up classification (Raphaël-Rodnianski 12), soliton resolution (Côte 15)....

The equivariant Ansatz

Special case: equivariant wave maps

N is an ℓ-dimensional rotationally symmetric manifold

$$N = \{(\phi, \chi) : \ \phi \in [0, \phi^*), \chi \in S^{\ell-1}\}, \qquad \phi^* \le \infty$$

with metric

$$d\phi^2 + g(\phi)^2 d\chi^2$$

where $d\chi^2$ is the standard metric on $\mathbb{S}^{\ell-1}$. In the coordinates $(\phi; \chi_1, \dots, \chi_{\ell-1})$, denoting by χ_{ij} the coefficients of the metric $d\chi^2$, the only nonzero Christoffel are

$$\Gamma^{\phi}_{\chi_i \chi_j} = -g'(\phi) g(\phi) \chi_{ij}, \qquad \Gamma^{\chi_i}_{\chi_j \phi} = \frac{g'(\phi)}{g(\phi)} \delta_{ij}, \qquad \Gamma^{\chi_i}_{\chi_j \chi_s} = \gamma^i_{js},$$

where γ^i_{js} are the Christoffel symbols for the metric χ_{ij}.

Example

The sphere $\mathbb{S}^n$ is $[0, \pi) \times \mathbb{S}^{n-1}$ with metric

$$d\phi^2 + \sin^2(\phi) d\chi^2_{\mathbb{S}^{n-1}}$$

Example

The real hyperbolic space $\mathbb{H}^n$ is $[0, +\infty) \times \mathbb{S}^{n-1}$ with metric

$$d\phi^2 + \sinh^2(\phi) d\chi^2_{\mathbb{S}^{n-1}}$$

Equivariant wave maps satisfy the Ansatz

$$u(t,x) = (\phi, \chi), \qquad \phi = \phi(t,r), \qquad \chi = \chi(\omega),$$

where (r, ω) are the spherical coordinates on R^n.

The equations for ϕ, χ decouple: $\chi : S^{n-1} \to S^{\ell-1}$ must be a harmonic map, which forces ℓ to be of the form $\ell = k(k+n-2)$, while ϕ must satisfy the equivariant wave map equation

$$\phi_{tt} - \phi_{rr} - \frac{n-1}{r}\phi_r + \frac{k(k+n-2)}{r^2} g'(\phi) g(\phi) = 0.$$

A semilinear, radial wave equation, with critical space $H^{\frac{n}{2}}$

This was the model originally studied by Shatah 86, Christodoulou-TahvildarZadeh 93, Shatah-TahvildarZadeh 94, Grillakis ∼94

- Global existence for small $H^{\frac{n}{2}}$ data
  Later extended to general WM by Tao and Tataru
- Blow up of large solutions if the target is a sphere, $n \geq 3$, and for some target manifolds with negative curvature, $n \geq 7$
- Global existence of large solutions for the class of 'open' manifolds satisfying the Grillakis condition

$$g'(\phi) + \phi g(\phi) > 0$$

later relaxed by Struwe 03 to

$$\int_0^{+\infty} g(\phi) d\phi = +\infty$$

Curved backgrounds

Cauchy problem on nonflat base manifolds:

- Shatah-TahvildarZadeh 97: wave maps $u : \mathbb{R}_t \times \mathbb{S}^2 \to \mathbb{S}^2$
- Choquet-Bruhat 00: wave maps on Robertson-Walker spacetimes
- Lawrie 13: wave maps $u : \mathbb{R}_t \times \mathbb{R}^4 \to N$, with a metric h on $\mathbb{R}^4$ which is a small perturbation of the flat metric
- Lawrie-Oh-Shahshahani 14, 15: global small wave maps $u : \mathbb{R}_t \times \mathbb{H}^d \to N$
- Anderson-Gudapati-Szeftel 15: equivariant Einstein-wave maps on M^{1+2} with (Lorentzian) metric $\check{g} + r^2 d\theta^2$

Our goal:

- **global existence of small (equivariant) WM between two rotationally symmetric manifolds**

Consider equivariant wave maps $u : \mathbb{R}_t \times M^n \to N^\ell$, where $M^n = [0, +\infty) \times \mathbb{S}^{n-1}$ with metric

$$dr^2 + h(r)^2 d\omega^2_{\mathbb{S}^{n-1}}$$

$N^\ell = [0, \phi^*) \times \mathbb{S}^{\ell-1}$ (here $\phi^* \leq \infty$) with metric

$$d\phi^2 + g(\phi)^2 d\chi^2_{\mathbb{S}^{\ell-1}}$$

Again $\ell = k(k + n - 2)$ for some $k \geq 1$, and the equation for the radial component of u is

$$\boxed{\phi_{tt} - \phi_{rr} - (n-1)\frac{h'(r)}{h(r)}\phi_r + \ell\frac{g(\phi)g'(\phi)}{h(r)^2} = 0}$$

Summary of main results

We prove:

- **global existence of small equivariant wave maps**
- **the initial data are in the critical space $H^{\frac{n}{2}}$**
- **arbitrary target manifold**
- **the base manifold belongs to a suitable class of manifolds** which we call admissible, dimension ≥ 3
- **local existence with large $H^{\frac{n}{2}}$ data**
- **higher regularity**
- **unconditional uniqueness** for slightly smoother data

Work in progress:

- the 2D case (same conditions)
- global existence with small **nonradial** data, i.e. without the equivariant Ansatz but still on rotationally symmetric manifolds
  (Strichartz estimates are ok - checking the geometry)

Admissible manifolds: definition

The assumptions on M^n are expressed in terms of the radial component $h(r)$ of the metric $dr^2 + h(r)^2 d\omega^2$

Define

$$H(r) := h^{\frac{1-n}{2}} (h^{\frac{n-1}{2}})''$$

We say that M^n is admissible if $n \geq 3$ and:

1. $H(r) = h_\infty + O(r^{-2})$ for r large, with $h_\infty \geq 0$
2. $H^{(j)}(r) = O(r^{-1})$ and $(h^{-\frac{1}{2}})^{(j)} = O(r^{-\frac{1}{2}-j})$ for r large, $1 \leq j \leq [\frac{n-1}{2}]$
3. For some $c, \delta > 0$ we have $h(r) \geq cr$ for all r and the function $P(r) = r[H(r) - h_\infty] + \frac{1-\delta}{4r}$ satisfies

$$P(r) \geq 0 \geq P'(r). \tag{1}$$

Examples of admissible manifolds

Example (Hyperbolic spaces $\mathbb{H}^n$)

The real hyperbolic manifolds $\mathbb{H}^n$ are admissible for $n \geq 3$:

$$h(r) = \sinh(r), \qquad H(r) = \tfrac{(n-1)^2}{4} + \tfrac{(n-1)(n-3)}{4} \tfrac{1}{\sinh^2 r}$$

$$P(r) = \tfrac{(n-1)(n-3)}{4} \tfrac{r}{\sinh^2 r} + \tfrac{1-\delta}{4r} \qquad \Longrightarrow \qquad P \geq 0 \geq P'$$

Example (Perturbations of $\mathbb{H}^n$)

More generally are admissible the rotationally invariant perturbations of $\mathbb{H}^n$, $n \geq 3$, with a metric

$$h_\epsilon(r) = \sinh r + \mu(r)$$

where $\mu : \mathbb{R}^+ \to \mathbb{R}$ is such that for small $\epsilon > 0$

$$|\mu(r)| + |\mu'(r)| + |\mu''(r)| + |\mu'''(r)| \leq \epsilon \langle r \rangle^{-3} \sinh r \quad \text{for all} \quad r > 0$$

and

$$|\mu^{(j)}(r)| \lesssim r^{-1} e^r \quad \text{for} \quad r >> 1, \quad j \leq [\tfrac{n-1}{2}] + 2.$$

Example (Perturbations of Minkowski space)

Asymptotically flat spaces of dimension $n \geq 3$, with a radial component of the metric of the form $h_\epsilon(r) = r + \mu(r)$, with $\mu : \mathbb{R}^+ \to \mathbb{R}$ such that for small $\epsilon > 0$

$$|\mu(r)| + r|\mu'(r)| + r^2|\mu''(r)| + r^3|\mu'''(r)| \leq \epsilon r \quad \text{for all} \quad r > 0$$

and

$$|\mu^{(j)}(r)| \lesssim r^{1-j} \quad \text{for} \quad r >> 1, \quad j \leq [\tfrac{n-1}{2}] + 2.$$

Example (Metrics with prescribed growth)

It is easy to construct examples of admissible manifolds with exponential growth, for instance

$$h(r) = r(1 + \chi(r)e^r)$$

where $\chi(r)$ is a cutoff equal to 0 near 0 and equal to 1 for large r

The following class of metrics

$$h(r) = r(1 + \chi(r)\sqrt{r})^M$$

with polinomial growth, produces an admissible manifold for any $M \geq 0$, where $\chi(r)$ is a cutoff equal to 0 near 0

We have also general results of the form:

Perturbations of admissible manifolds are admissible

Conditions of the type

$$|\partial_r^j (h - h_\epsilon)| \leq \epsilon h(r) \cdot r^{-j}, \qquad j = 1, 2, 3$$

Statement of main result

We consider the Cauchy problem

$$\phi_{tt} - \phi_{rr} - (n-1)\frac{h'(r)}{h(r)}\phi_r + k(k+n-2)\frac{g(\phi)g'(\phi)}{h(r)^2} = 0 \qquad (2)$$

with data

$$\phi(0,r) = \phi_0, \qquad \phi_t(0,r) = \phi_1 \qquad (3)$$

where ϕ represents the radial component of an equivariant wave map from the manifold M^n with metric $dr^2 + h(r)^2 d\omega$ to the manifold N^ℓ, $\ell = k(k+n-2)$

Recall that

$$h_\infty = \lim_{r\to\infty} h^{\frac{1-n}{2}}(h^{\frac{n-1}{2}})''$$

We use the notation

$$\|v\|_{H^s} = \|(1 - \Delta_M)^{\frac{s}{2}} v\|_{L^2(M)}$$

where M is the Laplace-Beltrami operator on M. We define also the weighted space $H^s_q(w)$ of radial functions on M^n with norm

$$\|\phi\|_{H^s_q(w)} := \|w^{-1}(|x|)\phi(|x|)\|_{H^s_q(\mathbb{R}^{n+2k})}, \quad w(r) := r^{k+\frac{n-1}{2}} h(r)^{-\frac{n-1}{2}}$$

Theorem ($h_\infty > 0$)

Let $n \geq 3$, $k \geq 1$ and assume the manifold M is admissible. Assume also that $h_\infty > 0$. Then for all data $(\phi_0, \phi_1) \in H^{\frac{n}{2}} \times H^{\frac{n}{2}-1}$ with sufficiently small norm, problem (2), (3) has a global solution in $L^\infty_t H^{\frac{n}{2}}$, continuous in time. In addition, the solution belongs to the space $L^p_t H_q^{\frac{n-1}{2}}(w)$ for suitable values of $p, q > 2$ and uniqueness holds in this space.

The formulation of the result is almost identical in the case $h_\infty = 0$: the initial data are in the space

$$\|(-\Delta_M)^{\frac{1}{4}}(1-\Delta_M)^{\frac{n-1}{2}}\phi_0\|_{L^2} + \|(-\Delta_M)^{-\frac{1}{4}}(1-\Delta_M)^{\frac{n-1}{2}}\phi_1\|_{L^2}$$

The couple (p,q) must be chosen as

$$p = \frac{4(m+1)}{m+3}, \qquad q = \frac{4m(m+1)}{2m^2-m-5}, \qquad m = n+2k.$$

Unconditional uniqueness

Solutions enjoy the additional property

$$\phi \in L^p H^s_q(w)$$

via Strichartz estimates. This condition is required for uniqueness

Unconditional uniqueness:
Is the solution unique in the space $L^\infty_t H^{\frac{n}{2}}$?

For a flat Minkowski base manifold, UU was proved in Masmoudi-Planchon 12

We conjecture that UU holds in our more general situation
Main obstruction is a suitable nonlinear estimate in negative Sobolev norms

Partial workaround:

Theorem (Higher regularity and UU)

If the data are in $H^s \times H^{s-1}$ for some $\frac{n}{2} < s < \frac{n}{2} + k$ then the solution is in $L^\infty_t H^s$. Moreover, uniqueness holds in this space if $s \geq \frac{n}{2} + \frac{1}{n+2k+1}$.

Note that $\frac{1}{n+2k+1} \leq \frac{1}{6}$

Structure of the proof

- Smoothing estimate for the equivariant WM equation
- Reduction to a wave equation with potential
- Well posedness for a critical semilinear wave equation with singular potential
- Higher regularity and unconditional uniqueness
- Back to WM equation. Equivalence of Sobolev norms
- Construction of manifolds

Reduction to a WE with potential

In the coordinates

$$\phi(r) = w(r) \cdot \psi(r), \qquad w(r) = \frac{r^{\frac{m-1}{2}}}{h(r)^{\frac{n-1}{2}}}$$

with $m = n + 2k$, the equivariant wave maps equation becomes

$$\psi_{tt} - \psi_{rr} - \frac{m-1}{r}\psi_r + V(r)\psi + \frac{r^{m-1}}{h(r)^{n+1}}\psi^3 \Gamma\left(\frac{r^{\frac{m-1}{2}}}{h^{\frac{n-1}{2}}}\psi\right) = 0$$

where $\ell g(s) g'(s) = \ell s + s^3 \Gamma(s)$ and

$$V(r) = \frac{n-1}{2}\left[\frac{h''}{h} + \frac{n-3}{2}\left(\frac{h'^2}{h^2} - \frac{1}{r^2}\right)\right] + k(k+n-2)\left(\frac{1}{h^2} - \frac{1}{r^2}\right).$$

This is a cubic semilinear wave equation for which the critical space is $H^{\frac{n}{2}}$

Additional difficulty: singular coefficients in the nonlinear term

Standard tool: global Strichartz estimates

Existence follows by a fixed point argument in suitable spaces

Strichartz estimates for the WE

Recall the Strichartz estimates

$$\| |D|^{\frac{1}{q}-\frac{1}{p}} e^{it|D|} f \|_{L^p_t L^q_x} \lesssim \|f\|_{L^2}$$

with

$$\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}, \qquad 2 \le p \le \infty, \qquad 2 \le q \le \frac{2(n-1)}{n-3} < \infty.$$

and, for $u = \int_0^t |D|^{-1} e^{i(t-s)|D|} F(s) ds$,

$$\| |D|^{\frac{1}{q}-\frac{1}{p}+\frac{1}{\tilde{q}}-\frac{1}{\tilde{p}}} u \|_{L^p L^q} \lesssim \|F\|_{L^{\tilde{p}'} L^{\tilde{q}'}}$$

(p, q) and $(\tilde{p}, \tilde{q})$ as above

Strichartz estimates for the perturbed WE

We need Strichartz estimates for the perturbed equation

$$\Box u + Vu = F$$

with V of critical decay $V \sim |x|^{-2}$

Related results:

Beals, Beals-Strauss 92: $V \in \mathscr{S}$, $V \geq 0$
Rodnianski-Schlag 04: almost critical potentials $\sim \langle x \rangle^{-2-\epsilon}$
Burq-Planchon-Stalker-TahvildarZadeh 04: non-repulsive potentials with critical decay $\sim |x|^{-2}$
D'A-Fanelli 08: small magnetic potentials
D'A 15: large magnetic potentials

- We first prove a smoothing estimate for the resolvent

$$\Delta_M v + zv = f, \qquad z \in \mathbb{C} \setminus \mathbb{R}$$

of the form

$$\| |x|^{-1} v \|_{L^2(M)} \leq 4\delta^{-1} \| |x| f \|_{L^2(M)}$$

- This implies the same estimate for the resolvent of $-\Delta + V(r)$
- the classical theory of smoothing operators (Kato 66, Kato-Yajima 89) gives for the Schrödinger flow

$$\| |x|^{-1} e^{it(\Delta - V)} f \|_{L^2 L^2} \lesssim \| f \|_{L^2}$$

- in D'A 15 the following appendix to Kato theory is proved: for any selfadjoint $H \geq 0$ on an Hilbert $\mathcal{H}$ and any closed, densely defined A

$$\|Ae^{itH}f\|_{L^2\mathcal{H}} \lesssim \|f\|_{\mathcal{H}} \quad \Longrightarrow \quad \|Ae^{it\sqrt{H}}f\|_{L^2\mathcal{H}} \lesssim \|H^{1/4}f\|_{\mathcal{H}}$$

  i.e. the smoothing estimate extends to the wave flow
- Strichartz estimates follow by a perturbation argument

Conclude by a fixed point argument for the semilinear wave equation in the space X with norm

$$\|u(t,x)\|_X := \|u\|_{L^{2a'}_t H_b^{\frac{n-1}{2}}} + \|u\|_{L^\infty_t H^{\frac{n}{2}}}.$$

when $h_{\infty>0}$, and

$$\|u(t,x)\|_X := \|u\|_{L^{2a'}_t H_b^{\frac{n-1}{2}}} + \||D|^{\frac{1}{2}} u\|_{L^\infty_t H^{\frac{n-1}{2}}}.$$

when $h_\infty = 0$. Here

$$2a' = \frac{4(m+1)}{m+3}, \qquad b = \frac{4m(m+1)}{2m^2 - m - 5}, \qquad m = 2k + n$$

(note that $m \geq 5$)

Addiditional difficulty: singular coefficients in the nonlinear term

$$\alpha(r)^2 Z(\beta(r)\psi)\psi^3$$

with

$$r|\alpha^{(j)}(r)| + |\beta^{(j)}(r)| \lesssim r^{1-j}$$

This requires ad-hoc radial Sobolev estimates: if

$$|\gamma^{(j)}(r)| \lesssim r^{\frac{m}{p}-\frac{m}{q}-s-j}, \qquad j = 0, \dots, [\sigma]+1$$

then

$$\|\gamma(|x|)u\|_{\dot{H}^{\sigma}_{q}} \lesssim \|u\|_{\dot{H}^{s+\sigma}_{p}}.$$

for $\frac{1}{p} - \frac{1}{q} \leq s < m/p - \sigma$, $1 < p \leq q < \infty$, $\sigma \geq 0$

(see D'A-Luca' 13)