Free boundary regularity in the parabolic fractional obstacle problem

A. Figalli
ETH Zürich

A Mathematical Tribute to Ennio De Giorgi

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The classical obstacle problem

Given an obstacle $\phi : \Omega \to \mathbb{R}$ and a boundary datum $f : \partial\Omega \to \mathbb{R}$ with $f > \phi|_{\partial\Omega}$, the classical obstacle problem consists in minimizing the Dirichlet energy among all functions above the obstacle:

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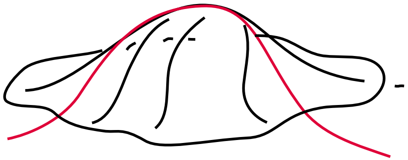

Properties of solutions

$$u \text{ minimizers} \quad \Longrightarrow \quad \begin{aligned} & u + \varepsilon v \ \text{ admissible} \\ & \text{if } \ \varepsilon > 0\,, \\ & v \geq 0\,, \quad v \in C_c^\infty(\Omega). \end{aligned}$$

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Hence,

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Moreover

$$\Delta u = 0 \qquad \text{on } \{u > \phi\}.$$

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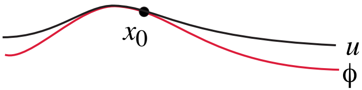

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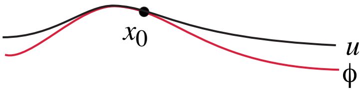

- $u(x_0)$ controls u in average from above
- $u \geq \phi \longrightarrow$ bound from below
- u harmonic if $\{u > \phi\}$

Theorem (Brezis - Kinderlehrer, 1974)

- $\phi \in C^{1,1} \implies u \in C^{1,1}$

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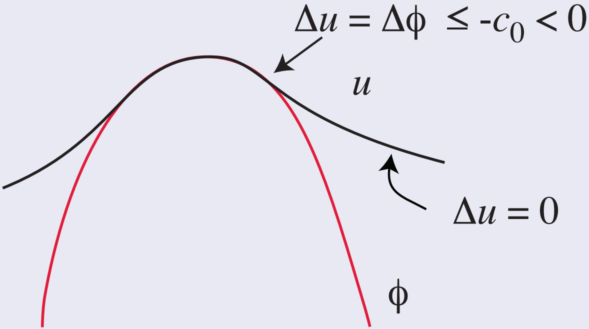

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Key fact: if $\Delta\phi < 0$, then

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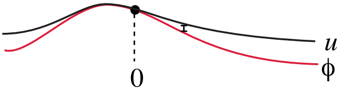

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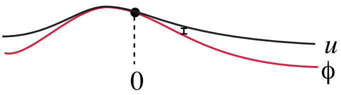

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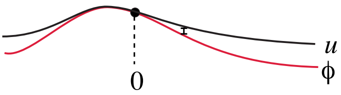

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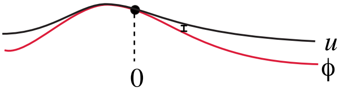

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$$v_r(x) := \frac{u(rx)-\phi(rx)}{r^2} \to \bar{v} \quad \text{as } r \to 0.$$

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- *The set of singular points is contained in* C^1 *hypersurfaces.*

Math finance

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$$\begin{cases} v \geq \psi \\ v > \psi \Rightarrow \mathcal{L}v = 0 \\ v(T) = \psi \ \leftarrow \ \text{payoff} \end{cases}$$

$\nwarrow$ expiration time

$$
\begin{aligned}
\mathcal{L}v = v_\tau &+ r\,v \\
&+ \sum_i (r - d_i)\, x_i\, v_{x_i} \\
&+ \frac{1}{2} \sum_{ij} \alpha_{ij}\, x_i\, x_j\, v_{x_i x_j} \\
&+ \int \left[v(\tau, x_1 e^{y_1}, \ldots) - v(\tau, x) - \sum_i (e^{y_i} - 1)\, v_{x_i} \right] d\mu(y).
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Consider the change of variables

$$x_i \longleftrightarrow \log x_i.$$

Then v solves a backward parabolic equation

$$v_\tau + r\,v + b \cdot \nabla v + Q_{ij}\, v_{x_i x_j} + K[v] = 0.$$

- $$\left.\begin{array}{c} K \equiv 0 \\ \psi \in C^\infty \\ Q \geq \lambda I \end{array}\right] \Longrightarrow v \in C^1_t C^{1,1}_x$$

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We need to gain regularity from K.

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Hypothesis

$K[v] = \Delta^s v + K'[v]$, K' *lower order w.r.t.* Δ^s, $s \in (0,1)$

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Then, we consider the model equation:

$$\begin{cases} u \geq \psi \\ u > \psi \Longrightarrow u_t = \Delta^s u \\ u = \psi \Longrightarrow \Delta^s u \leq 0 \\ u(0) = \psi. \end{cases}$$

Preliminaries on Δ^s

$$v : \mathbb{R}^n \to \mathbb{R}\,, \quad v \in C_c^\infty$$

$$\begin{aligned}\Delta^s v(x) &= \left(-|\xi|^{2s}\hat{v}\right)^\vee (x)\\ &= C_{n,s}\,\text{p.v.} \int \frac{v(x+y)-v(x)}{|y|^{n+2s}}\,dy.\end{aligned}$$

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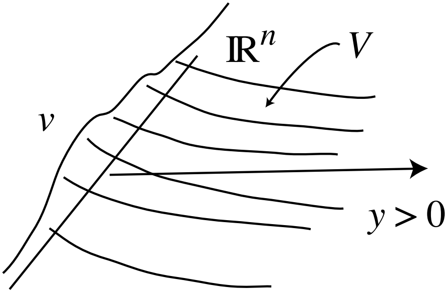
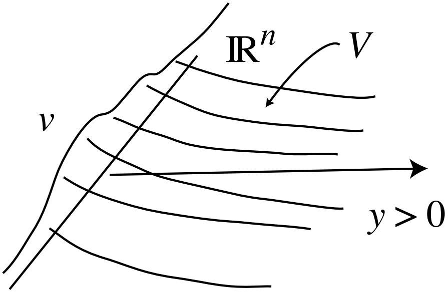
Other definition: Dirichlet-to-Neumann



$$\begin{cases} \Delta_{x,y} V = 0 \\ V(x,0) = v(x) \end{cases}$$



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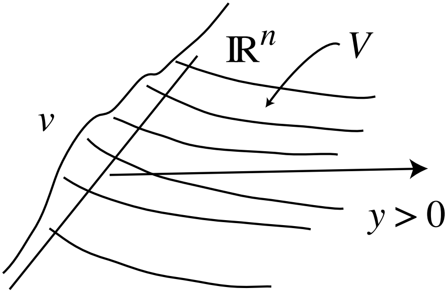
Then

$$\Delta^{1/2} v(x) = V_y(x,0)$$

In general (Caffarelli - Silvestre, 2007)

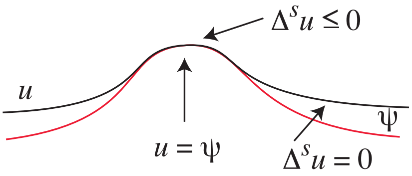


$$\begin{cases} L_a V := \operatorname{div}_{x,y}(y^a \nabla_{x,y} V) = 0 \\ V(x,0) = v(x) \end{cases}$$

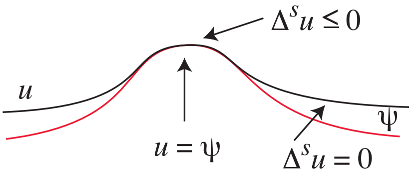
$$a = 1 - 2s$$

$$\Delta^s v(x) = \lim_{y \to 0^+} y^a V_y(x,y)$$

Regularity of u: the stationary case



Regularity of u: the stationary case



Theorem (Caffarelli - Salsa - Silvestre, 2008)

$$u \in C^{1,s} (\textit{equivalently, } \Delta^s u \in C^{0,1-s})$$

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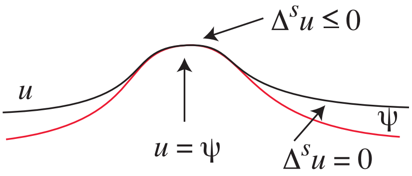

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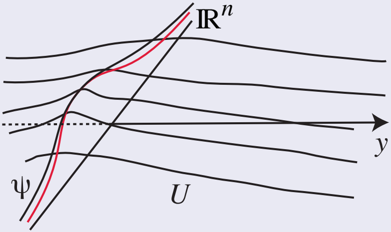
Remark

This result is optimal.

Remark: **Signorini Problem**



$$\min_{U(x,0)\geq\psi(x)} \int_{\mathbb{R}^{n+1}} |\nabla_{x,y}U|^2$$

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$$\begin{cases} \Delta_{x,y}U = 0 \\ U_y(x,0) = 0 \;\; if \;\; \{U > \psi\}. \end{cases}$$

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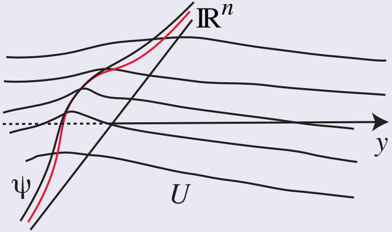

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Since $U_y(x,0) = \Delta_x^{1/2} U(x,0)$, *this is the obstacle problem for* $\Delta^{1/2}$.

Regularity of u: the parabolic case

Question: What regularity can we expect?

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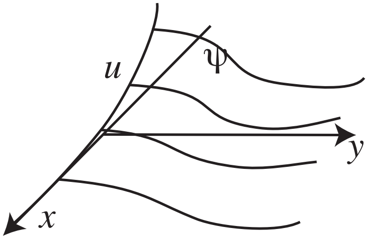
$$s = 1/2\,, \quad n = 1\,, \quad \psi = 0$$



$$\begin{cases} \Delta_{x,y} U = 0 \\ U_t(t,x,0) = U_y(t,x,0) \ \text{ if } \ U > 0 \end{cases}$$

$U_t = U_y$ has hyperbolic scaling $\rightsquigarrow$ look for travelling waves:

$$U(t,x,y) = W(at + x, y)$$

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$$\begin{cases} W_y = aW_x \quad \text{on} \quad \{y=0\} \cap \{W > 0\} \\ \Delta_{x,y} W = 0\ . \end{cases}$$

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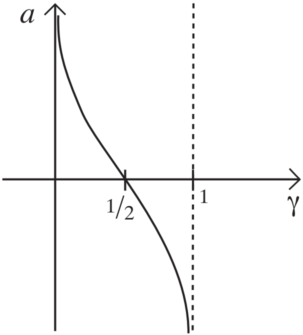
Try $\mathrm{Re}(z^\beta)$, $\mathrm{Im}(z^\beta)$,

$$z = x + iy$$

$$W = -\,\mathrm{Im}(z^{1+\gamma})\,, \qquad \gamma \in (0,1),$$

solves the equation for

$$a = \frac{1}{\tan(\gamma\pi)}.$$



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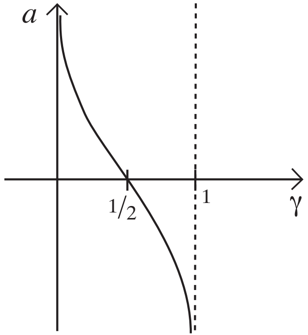
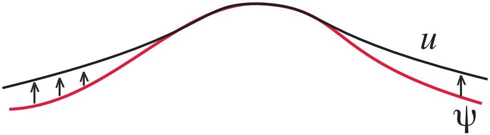


Hence the solution is only $C^{1,\gamma}$!!

Key observation:

$$u(0) = \psi \ , \quad u \geq \psi \Rightarrow u_t \geq 0$$



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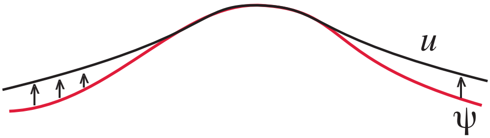

Hence only $a \leq 0$ is admissible $\rightsquigarrow$ $\gamma \geq 1/2$

$$\Downarrow$$

solutions should be $C^{1,1/2}$

(**Remark:** $a = 0 \leftrightarrow$ stationary case)

Theorem (Caffarelli - Figalli, 2013)

ψ *nice. Then*

- $s > 1/3 \to u_t,\ \Delta^s u \in C_t^{\frac{1-s}{2s}-} C_x^{1-s}$
- $s \leq 1/3 \to u_t,\ \Delta^s u \in \log \mathrm{Lip}_t C_x^{1-s}$

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$s = \frac{1}{2} \to C_t^{1/2-} C_x^{1/2}$ *almost optimal*

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Remark

Why $s = 1/3$?

$$u_t = \Delta^s u \longleftrightarrow (\lambda^{2s} t, \lambda x)$$

$$C_x^{1-s} \longleftrightarrow C_t^{\frac{1-s}{2s}} \quad \textit{if} \quad \frac{1-s}{2s} < 1$$

Free boundary regularity: the stationary case

- Stationary solutions:

$$n = 1\,, \quad \psi = 0\ :$$

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and the free boundary is $C^{1,\alpha}$ *in a neighborhood of* 0.

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Main tool: Almgren's frequency formula:

$$r \mapsto \Phi_u(r) := r \, \frac{\int_{B_r} |\nabla_{x,y} u|^2 y^a}{\int_{\partial B_r} u^2 y^a} \quad \text{is monotone non-decreasing.}$$

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Remark: $\Phi_u(0)$ captures the homogeneity of u at 0, and the monotonicity of Φ_u allows one to characterize blow-ups.

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- *The examples for $s = 1/2$ show that there is no gap in the homogeneity of solutions between $3/2$ and 2.*
- *The regime relevant for finance is $s > 1/2$.*

Hence, we focus on $s \in (1/2, 1)$.

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- $u\in C^{1+s}_{x,t}$ *near* (t_0,x_0) *and*

$$u(t,x)-\varphi(x)=c_0\big((x-x_0)\cdot e+a(t-t_0)\big)_+^{1+s} + o\big(|x-x_0|^{1+s+\alpha}+|t-t_0|^{1+s+\alpha}\big)$$

for some $c_0>0$, $e\in\mathbb{S}^{n-1}$, *and* $a>0$.

Idea of the proof

Step 1: u is semiconvex in (t,x), namely

$$\partial_{\xi\xi} u(t,x) \geq -C$$

for all $\xi \in \mathbb{R} \times \mathbb{R}^n$, $|\xi| = 1$.

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(along a suitable subsequence).

Also, by Step 1, $\bar{v}$ is convex in (t,x).

Furthermore, since (ii) fails, $\bar{v}|_{Q_R}$ grows less than $R^{2-\epsilon}$ for $R \gg 1$.

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(this uses $\partial_t \bar{v} \geq 0$ and $s > 1/2$).

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Step 4: using Step 3 and suitable barrier arguments (note that all partial derivatives solve an equation), we show that $u \in C^{1,s}_{t,x}$ and that the free boundary is Lipschitz in (t, x).

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Step 7: by Step 6 and the $C^{1,s}$ regularity of u, we conclude the expansion near the free boundary point.

THANKS FOR

YOUR

ATTENTION