Plateau's problem in metric spaces and applications

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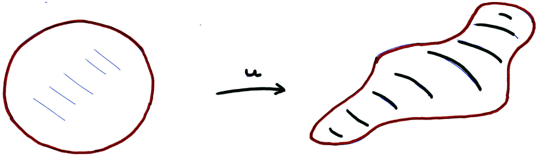
A Mathematical Tribute to Ennio De Giorgi
(19 - 23 September 2016)

Plateau's Problem: Find surface of least area with prescribed boundary.

Douglas, Radó, Morrey: Existence of disc-type surface



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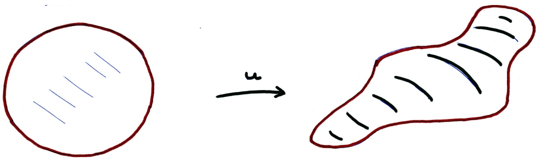

of least area with prescribed rectifiable Jordan boundary in Euclidean space and Riemannian manifolds.

Some generalizations:

- Two-dim. surfaces with fixed genus (Courant '37, Jost '85).
- Integral currents (Federer-Fleming '60, ...).
- Chains mod 2 (Fleming '66, ...).

Currents in metric spaces (De Giorgi, Ambrosio-Kirchheim '00):

- Cpt metric & some Banach spaces (Ambrosio-Kirchheim '00).
- Dual Banach and $\mathrm{CAT}(0)$-spaces (W.'05).
- Non-compact boundaries (Ambrosio-Schmidt '13, W. '14).

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Disc-type surfaces:

- Spaces of non-positive curvature ($\mathrm{CAT}(0)$) (Nikolaev '79).
- some Alexandrov spaces (Mese-Zulkowski '10).
- 3-dimensional Finsler manifolds (Overath-von der Mosel '14).

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Aim of talk:

- Existence and regularity of area min. discs in metric spaces.
- Applications to geometric problems..

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Applications of existence and regularity results for area minimizing discs in metric spaces:

Analysis in metric spaces:

- Parametrization problems for metric spaces.
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- Parametrization problems for metric spaces.
- Regularity of quasi-harmonic maps to metric spaces.

(Large scale) Geometry:

- Characterization of (large scale) non-positive curvature via isoperimetric inequality.

Geometric group theory:

- Hölder 1-connectedness of asymptotic cones (tangent cones at ∞) of spaces with quadratic Dehn function.
- Quasi-isometry invariance of Dehn function under very weak conditions.

Metric space valued Sobolev maps

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Definition via post-composition (Ambrosio '90, Reshetnyak '97):

A map $u\colon D \to X$ is in $W^{1,p}(D, X)$ if

- u measurable and essentially separably valued
- $\exists\, g \in L^p(D)$ such that $\varphi \circ u \in W^{1,p}(D)$ and $|\nabla(\varphi \circ u)| \leq g$ for all $\varphi \in \mathrm{Lip}_1(X)$.

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The Reshetnyak energy of u is

$$E_+^p(u) := inf \left\{ \|g\|^p_{L^p(D)} : g \text{ as above} \right\}.$$

The map u has a trace $\mathrm{tr}(u)\colon S^1 \to X$; it satisfies

$$\mathrm{tr}(u) \in L^p(S^1, X).$$

Existence of energy minimizers

For $\Gamma \subset X$ Jordan curve let

$$\Lambda(\Gamma) = \{v \in W^{1,2}(D, X) : \text{ tr}(v) \text{ weakly mon. param. of } \Gamma\}.$$

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Theorem (Lytchak-W. '15)

Let X be proper metric space and $\Gamma \subset X$ such that $\Lambda(\Gamma) \neq \emptyset$. Then there exists E_+^2-energy minimimizer in $\Lambda(\Gamma)$.

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In Riemannian manifolds energy minimizers are weakly conformal.

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Definition

A map $u \in W^{1,2}(D, X)$ is Q-quasi-conformal if for a.e. $z \in D$

$$\mathrm{md}_z\, u(v) \leq Q \cdot \mathrm{md}_z\, u(w)$$

for all $v, w \in S^1$.

The identity map from D to $(\mathbb{R}^2, \|\cdot\|_\infty)$ is $\sqrt{2}$-qc.

Energy minimizers are quasi-conformal

Theorem (Lytchak-W. '15)

If X is complete metric space and $u \in W^{1,2}(D, X)$ is such that

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- The classical proof using family $\{\psi_t\}$ of deformations of D and calculating
$$0 = \frac{d}{dt}\bigg|_{t=0} E^2(u \circ \psi_t) = \ldots$$
breaks down as soon as non-Euclidean norms appear.

Parametrized area of Sobolev maps

The parametrized Hausdorff area of $u \in W^{1,2}(D, X)$ is

$$\operatorname{Area}(u) = \int_D \mathbf{J}_2(\operatorname{md}_z u)\, dz,$$

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Area formula: If u has Lusin's property (N) then

$$\mathrm{Area}(u) = \int_X \# u^{-1}(x)\, d\mathcal{H}^2_X(x)$$
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Other natural choices of area from convex geometry work too: Gromov mass*, Holmes-Thompson, Ivanov area, etc.

Solution to Plateau's problem

Theorem (Lytchak-W. '15)

If X is proper and $\Gamma \subset X$ with $\Lambda(\Gamma) \neq \emptyset$ then there exists $u \in \Lambda(\Gamma)$ with

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- E_+^2-minimizers in $\Lambda(\Gamma)$ need not be minimizers of Area.
- Quasi-convex defs. of energy E induce quasi-convex defs. of area Area_E such that E-minimizers are Area_E-minimizers. Example: $\operatorname{Area}_{E_+^2} =$ Ivanov area.

Proof:

Use quasi-convexity of $\mathcal{H}^2$ (Burago-Ivanov '12) to prove lower semi-continuity of Area.

Use $\sqrt{2}$-qc. of energy minimizers to find for each $u \in \Lambda(\Gamma)$ some $v \in \Lambda(\Gamma)$ with

$$E_+^2(v) \leq 2\,\mathrm{Area}(v) \leq 2\,\mathrm{Area}(u).$$

Pick area-minimizing sequence in $\Lambda(\Gamma) \Rightarrow \exists$ new area-minimizing sequence with uniformly bounded energy.

Use Rellich compactness and Courant-Lebesgue Lemma and lower semi-continuity of Area. □

Regularity of area minimizers

Definition

A metric space X admits local quadratic isoperimetric inequality if $\exists C, r_0 > 0$ such that every Lip. curve c in X with $\ell(c) \leq r_0$ is the trace of some $u \in W^{1,2}(D, X)$ with

$$\mathrm{Area}(u) \leq C \cdot \ell(c)^2.$$

Examples:

- Homogeneously regular Riemannian manifolds (Morrey).
- Compact Lipschitz manifolds and Banach spaces.
- $\mathrm{CAT}(\kappa)$-spaces and compact Alexandrov spaces.
- Some Carnot-Carathéodory spaces (e.g. Heisenberg $\mathbb{H}^{n\geq 2}$).
- Spaces appearing in Analysis on metric spaces.
- Many interesting spaces appearing in geometric group theory.

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Let X be complete metric space, $\Gamma \subset X$ Jordan curve, $u \in \Lambda(\Gamma)$ such that

$$\operatorname{Area}(u) = \inf\{\operatorname{Area}(v) : v \in \Lambda(\Gamma)\}$$

and u is Q-qc. If X admits loc. quad. isop. then:

1. *u is loc. $W^{1,p>2}$, thus has Lusin (N).*
2. *u is locally α-Hölder and extends cont. to $\overline{D}$, $\alpha = \frac{1}{4\pi Q^2 C}$.*
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Remarks:

- Hölder exponent α is optimal (when $Q = 1$).
- Similar result for E_+^2-energy minimizers.
- Proof along lines of classical proof of Morrey.

Application to uniformization problems

Uniformization problem: X metric space homeomorphic to model space M. Find metric conditions on X such that there exists homeomorphism

$$\varphi\colon M \to X$$

which is biLipschitz, or quasisymmetric, or quasiconformal, etc.

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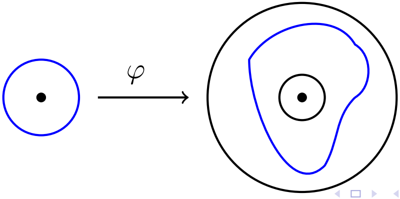
A homeomorphism $\varphi\colon M \to X$ is quasisymmetric if there exists homeomorphism η of $[0, \infty)$ such that

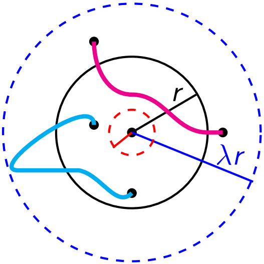
$$\frac{|z-a|}{|z-b|} \leq t \quad \Rightarrow \quad \frac{d(\varphi(z), \varphi(a))}{d(\varphi(z), \varphi(b))} \leq \eta(t).$$



Necessary conditions for biLipschitz

If X is biLipschitz to $\overline{D}$ or S^2 then

- X is Ahlfors 2-regular: $C^{-1}r^2 \leq \mathcal{H}^2(B(x, r)) \leq Cr^2$.
- X is linearly locally connected (LLC):



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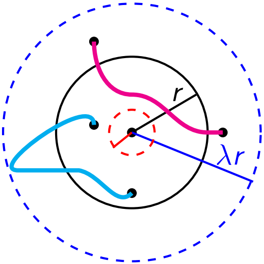

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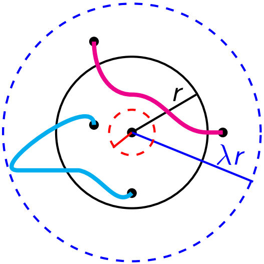

The LLC condition restricts geometry: no cusps, no neck-pinch.

Laakso '02: $\exists$ X Ahlfors 2-regular, LLC, homeomorphic to $\overline{D}$ such that X does not biLipschitz embed into any Hilbert space.

A new and natural proof of the Bonk-Kleiner theorem

Theorem (Lytchak-W. '16)

Let X be a geodesic metric space homeomorphic to $\overline{D}$ and such that $\ell(\partial X) < \infty$. If X is Ahlfors 2-regular and LLC then X is quasisymmetric to $\overline{D}$.

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Corollary (Bonk-Kleiner '02, Rajala '14)

A metric space homeomorphic to S^2 which is Ahlfors 2-regular and LLC is quasisymmetric to S^2.

Outline of proof of theorem

Step 1: X admits a quadratic isoperimetric inequality.

- Existence of $\frac{1}{n}$-thickening X_n of X which is Lipschitz 1-connected up to scale $\sim \frac{1}{n}$.
- Stability of quad. isop. inequality under GH-convergence.
- Existence of triangulation of Jordan domain $\Omega \subset X$ into N triangles of length $\leq \frac{1}{n}$, where

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Step 3: Well-known methods yield upgrade to quasisymmetric. $\square$

Application to (large scale) geometry

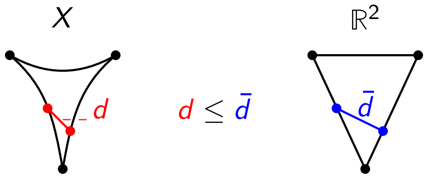
Isoperimetric characterization of non-positive curvature

Theorem (Lytchak-W. '16)

Let X be proper geodesic metric space. Then TFAE:

1. X *is* $\mathrm{CAT}(0)$.
2. X *admits quadratic isoperimetric inequality with constant* $\frac{1}{4\pi}$.

The $\mathrm{CAT}(0)$-condition:



Ex: Simply-connected Riemannian manifolds M with $\sec_M \leq 0$.

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- $X = (\mathbb{R}^2, \|\cdot\|)$ non-Euclidean $\Rightarrow \exists K \subset X$ convex with

$$\mathcal{H}^2(K) > \frac{1}{4\pi} \cdot \ell(\partial K)^2.$$

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$$\mathcal{H}^2(K) > \frac{1}{4\pi} \cdot \ell(\partial K)^2.$$

- X has isoperimetric constant $< \frac{1}{4\pi} \Rightarrow X$ metric tree (W. '08).

Isoperimetric characterization of non-positive curvature

Theorem (Lytchak-W. '16)

Let X be proper geodesic metric space. Then TFAE:

1. *X is $\mathrm{CAT}(0)$.*
2. *X admits quadratic isoperimetric inequality with constant $\frac{1}{4\pi}$.*

History:

- (1) $\Rightarrow$ (2) by Reshetnyak's majorization theorem.
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- X has isoperimetric constant $< \frac{1}{4\pi} \Rightarrow X$ metric tree (W. '08).
- (2) $\Rightarrow$ (1) when X is surface of bounded integral curvature (Reshetnyak '61).

Outline of proof of (2) $\Rightarrow$ (1)

Step 1: Let $\Gamma \subset X$ be geodesic triangle. Can reduce to case that Γ Jordan triangle.

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- is locally Lipschitz on D and continuous on $\overline{D}$.

Step 3: The conformal factor $f(z) := \mathrm{md}_z\, u(e_1)$ of u satisfies

$$\int_{B(z,r)} f^2 \leq \frac{1}{4\pi}\left(\int_{\partial B(z,r)} f\right)^2.$$

$\Rightarrow$ f has unique representatitve $\bar{f}$ which is log-subharmonic.

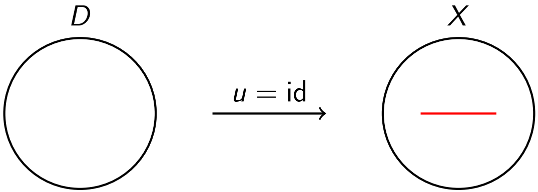
$\Rightarrow$ The metric space $Y := (D, d_{\bar{f}})$ with

$$d_{\bar{f}}(z,z') := \inf\left\{\int_\gamma \bar{f} : \gamma \text{ connects } z \text{ and } z' \text{ in } D\right\}$$

is homeomorphic to D and locally $\mathrm{CAT}(0)$.

Problematic example

Let $X = D$, with distance on red line scaled by factor < 1.



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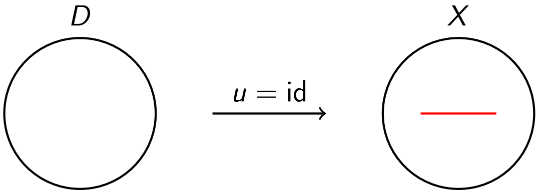

Then $Y = (D, |\cdot|_{\mathrm{Eucl}})$, so Y doesn't reflect the geometry of X.

Outline of proof of (2) $\Rightarrow$ (1)

Step 4: Consider the metric space $Z = (\overline{D}, d_u)_{/\sim}$ with

$$d_u(z, z) := \inf\{\ell(u \circ \gamma) : \gamma \text{ connects } z \text{ and } z' \text{ in } \overline{D}\}.$$

$\Rightarrow \exists \bar{u}\colon Z \to X$ 1-Lipschitz with $u = \bar{u} \circ P$, where $P =$ proj.

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Theorem (Lytchak-W. '16)

The space Z is geodesic and homeomorphic to $\overline{D}$ and $\bar{u}$ preserves lengths of all curves $P \circ \gamma$. For all $\Omega \subset Z$ Jordan domain we have

$$\mathcal{H}^2(\Omega) \leq \frac{1}{4\pi} \cdot \ell(\partial\Omega)^2.$$

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Step 5: Natural map $\iota \colon Y \to Z$ is 1-Lipschitz and preserves area

$\Rightarrow \iota$ must preserve lengths of curves (uses isop. ineq.)

$\Rightarrow Y$ and Z locally isometric, so $Z \setminus \partial Z$ locally $\mathrm{CAT}(0)$.

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Step 6: Show Z is $\mathrm{CAT}(0)$.

Large scale analog

Theorem (Lytchak-W. '16)

Let X be proper geodesic metric space. Suppose $\forall \varepsilon > 0 \; \exists \; r > 0$ such that every Lipschitz $c\colon S^1 \to X$ with $\ell(c) \geq r$ is trace of $u \in W^{1,2}(D, X)$ with

$$\mathrm{Area}(u) \leq \frac{1+\varepsilon}{4\pi} \cdot \ell(c)^2.$$

Then X is asymptotically $\mathrm{CAT}(0)$.

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Then X is asymptotically $\mathrm{CAT}(0)$.

Remarks:

- Asymptotically $\mathrm{CAT}(0)$:⇔ *every asymptotic cone is* $\mathrm{CAT}(0)$.
- Need metric space Plateau even when X Riemannian manifold.
- Uses existence of solutions to Plateau's problem in asymptotic cones of X (Guo-Lytchak-W. '16).

Gromov hyperbolicity

Theorem (W. '08)

If X is geodesic metric space and $\varepsilon, r > 0$ such that every Lip. loop c in X of length $\geq r$ is trace of some $u \in W^{1,2}(D, X)$ with

$$\mathrm{Area}(u) \leq \frac{1-\varepsilon}{4\pi} \cdot \ell(c)^2$$

then X is Gromov hyperbolic.

Remarks:

- Gromov '87: $\frac{1}{16\pi}$ for Riem mfds, $\frac{1}{4000}$ for metric spaces.
- Proof uses currents in metric spaces. If X proper then can use existence/regularity results above (Lytchak-W.-Young '16).

The end

Thank you for your attention!

Step 2: u is homeomorphism

Outline of proof:

(a) Show that u is monotone: $u^{-1}(x)$ connected for all $x \in X$.

(b) Use the upper bound

$$\mathcal{H}^2(B(x,r)) \leq Cr^2$$

to show that $u^{-1}(x)$ cannot be non-degenerate and so u is homeomorphism.

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For (a) we first show a topological result:

Theorem

Let Z be geodesic metric space homeomorphic to $\overline{D}$. Let $\varphi\colon \overline{D} \to Z$ be continuous, surjective, and such that

- *$\varphi|_{S^1}$ is a weakly monotone parametrization of ∂Z.*
- *for all biLipschitz curves $\Gamma \subset Z$ every component of $\varphi^{-1}(\Gamma)$ is a cell-like set.*

Then φ is a cell-like map and, in particular, monotone.

Step 2: u is homeomorphism

A subset $K \subset S^2$ is cell-like if and only if K and $S^2 \setminus K$ are connected.

We can show that the solution u to Plateau's problem satisfies the hypotheses above:

Proposition

For every biLipschitz curve $\Gamma \subset X$ every component of $u^{-1}(\Gamma)$ is a cell-like set.

As a consequence we obtain that u is monotone.