ON A QUOTIENT OF THE ALGEBRA OF BRAIDS AND TIES AND ITS RELATIONSHIPS WITH THE YOKONUMA-HECKE ALGEBRA

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Introduction

This project continues the investigation on the algebra mentioned in the title started some years ago by Juyumaya with the partial support of the Fondecyt projects. It should benefit of the expertise of Papi in the representation theory of Hecke algebras and its relationships with the combinatorics of symmetric groups and more generally reflection groups [43], [44], [45].

The project is based on some natural questions derived from recent papers by Juyumaya on these algebras: it started conceptually during a visit of Papi at the University of Valparaiso in September/October 2016.

We briefly review in Sections 1 and 2 the basic setting and make explicit in Section 3 the goals of our project.

1. Yokonuma–Hecke algebra

1.1. The Yokonuma–Hecke algebras, called from now on simply Y–H algebras, were introduced by T. Yokonuma in [50] as centralizer of the permutation representations associated to a finite Chevalley group with respect to a maximal unipotent subgroup. Thus, they correspond to natural generalizations of the Hecke algebras; also, the Y–H algebra corresponds to a particular case of unipotent Hecke algebra, see [49]. After the paper of Yokonuma above only few papers were written on the Y–H algebras. The interest for the Y–H algebras was revived notably in the last years, principally due to the application of these algebras to the construction of invariants for framed links [34] and classical links [33]. Thus, these algebras become a subject of study not only in knot theory [9, 14, 46, 23] but in representation theory [12, 17, 15, 16] as well.

1.2. The Y–H algebras considered for application to knot theory correspond to those of type A and are denoted by $\mathrm{Y}_{d,n}(u)$ [26, 27, 29], cf. [25]. More precisely, let d and n be two positive integers and u an indeterminate. The Y–H algebra $\mathrm{Y}_{d,n}(u)$ is the associative unital $\mathbb{C}(u)$–algebra presented by braiding generators $g_1, \dots, g_{n-1}$ and framing generators $t_1, \dots, t_n$ subject to A–braided relations among the g_i's together with the following relations:

$$t_i^d = 1, \qquad t_i t_j = t_j t_i, \qquad t_j g_i = g_i t_{s_i(j)} \qquad \text{for all} \quad i, j$$

where $s_i(j)$ is the effect of the transposition $s_i := (i, i+1)$ on j. And the quadratic relations

$$g_i^2 = 1 + (u-1)e_i(1 + g_i) \tag{1}$$

with

$$e_{d,i} := \frac{1}{d}\sum_{s=0}^{d-1} t_{i-1}^k t_i^{-k} \qquad (1 \leq i \leq n-1). \tag{2}$$

[12, 8] introduce a variant of the above presentation for the Yokonuma–Hecke algebra, which plays an important role as we will see later. More precisely, set now the ground field $\mathbb{C}(q)$, with q an indeterminate s.t. $u = q^2$; the mentioned variant is obtained by replacing in the original presentation of Y–H algebra the braiding generators g_i's by the braiding generators $\tilde{g}_i$'s, where now the $\tilde{g}_i$'s satisfy the same relations of the g_i's but replace the quadratic relations (1) by the following quadratic relations:

$$\tilde{g}_i^2 = 1 + (q - q^{-1})e_i \tilde{g}_i. \tag{3}$$

We shall denote by $\mathrm{Y}_{d,n}(q)$ the Y–H algebra considered with this last presentation.

1.3. From the definition of $\mathrm{Y}_{d,n}(u)$ it follows that this can be regarded naturally as a deformation of the framed braid group [38]. Having in mind this fact and the construction of the Homflypt polynomial by the *Jones recipe*[1], in [29] we proved that $\mathrm{Y}_{d,n}(u)$ supports also a Markov trace tr, with the aim to imitate the Jones recipe to define an invariant of framed knots. Consequently, in [34] we constructed an invariant for framed links through the Jones recipe applied to $\mathrm{Y}_{d,n}(u)$ and certain specialization of the trace tr. Moreover, by using again the Jones recipe, in [33] a family of invariant $\{\Delta_d\}_{d\in\mathbb{Z}_{>0}}$ for classical link was defined and in [32] a family of invariant for singular knots was constructed as well.

[1]The Jones recipe means the remarkable construction of the Homflypt polinomial given in [24] which indeed yields a recipe, or mechanism, to construct invariant of like–knotted objects.

It is worth to note that by using purely algebraic arguments Jacon and Poulain D'Andecy have also proved in [23] that the invariant Θ_d and Homflypt are equivalent at level of knots. They obtained it as a consequence of the fact that Y–H algebra is isomorphic to a sum of matrices having as coefficients tensor products of Hecke algebra, cf. [17, Theorem 7].

2. The bt–algebra

2.1. With the denomination bt–algebra we shall refer to the algebra of braids and ties, denoted by $\mathcal{E}_n(u)$, introduced by F. Aicardi and J. Juyumaya, see e.g. [1, 47, 5, 3]. The original motivation to define this algebra was to find new representations of braid groups. The construction of the bt–algebra comes out by considering a presentation of the algebra generated abstractly by the braid generators g_i's of $\mathrm{Y}_{d,n}(u)$ and the idempotents e_i's defined in (2), for details of the construction, see [3, Subsection 3.2]. The bt–algebra is a new mathematical object that becomes interesting in knot theory [3, 2], Partition algebra [5, 30] and representation theory [47, 17]. Further, it is worth to note that recently in [40], I. Marin has attached to every Coxeter system an algebra which becomes the bt–algebra when the Coxeter system is finite of type A.

To continue explaining the proposal, we need to recall the definition of the bt–algebra. The bt–algebra $\mathcal{E}_n(u)$ is defined as the unital associative $\mathbb{C}(u)$–algebra presented by generators $T_1, \ldots, T_{n-1}$, $E_1, \ldots, E_{n-1}$ satisfying A–braid relations among the T_i's, $E_iE_j = E_jE_i$, $E_i^2 = E_i$, for all i, j; together with the following relations:

$$\begin{array}{rcll} E_iT_j & = & T_jE_i & \text{for} \quad |i-j|>1 \\ E_iT_i & = & T_iE_i & \\ E_jT_iT_j & = & T_iT_jE_i & \text{for} \quad |i-j|=1 \\ E_iE_jT_j & = & E_iT_jE_i = T_jE_iE_j & \text{for} \quad |i-j|=1 \\ T_i^2 & = & 1+(u-1)E_i(1+T_i). & \end{array}$$

2.2. The Markov trace supported by the bt–algebra is at 2–parameters, see [3, Theorem 3] and its existence attracted our attention to define certain invariants for classical links and singular links. Thus, by using this Markov trace in the Jones recipe applied to the bt–algebra, we have defined a 3–parameters invariant for classical links, denoted by $\overline{\Delta}$, and also an invariant for singular links. The invariant $\overline{\Delta}$ generalizes the invariant Δ_d, see [3, Subsection 5.3] for details.

3. **Goals**

We want to study the quotient of the algebra $\mathcal{E}_n(u)$ by the "Temperley-Lieb" ideal J generated by the elements

$$E_iE_{i+1}(1 + T_i + T_{i+1} + T_iT_{i+1} + T_{i+1}T_i + T_iT_{i+1}T_i), \quad i = 1, \ldots, n-1.$$

Let $\mathcal{TLE}_n(u)$ be this quotient algebra. The interest in its study is related to the fact that $\mathcal{TLE}_n(u)$ should be an algebra of relatively small size which supports a kind of Markov trace. We list a series of problems we plan to attack, in increasing order of difficulty

(1) Estimate $\dim \mathcal{TLE}_n(u)$.
(2) Calculate $\dim \mathcal{TLE}_n(u)$.
(3) Establish a graphic calculus in $\mathcal{TLE}_n(u)$.
(4) Find a nice basis of $\mathcal{TLE}_n(u)$.
(5) Understand the relationships with cellular algebras.
(6) Investigate the representation theory of $\mathcal{TLE}_n(u)$.
(7) Deepen the possibile connections with knots and 3-manifolds invariants.

Another possibility, in a slightly different direction, is to look for generalizations of quotients of analogues of $\mathcal{E}_n(u)$ for finite reflection groups.

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