NEW DEVELOPMENTS IN THE THEORY OF MODULAR FORMS OVER FUNCTION FIELDS

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Titles and Abstracts

Julio Cesar Bueno De Andrade (University of Exeter, UK): *Universality of L-functions in Function Fields.*

In this talk, I will discuss Dirichlet L-functions associated with Dirichlet characters in Fq[x] and will prove that they are universal. That is, given a modulus of high enough degree, L-functions with characters to this modulus can be found that approximate any given nonvanishing analytic function arbitrarily closely. This is the function field analogue of Voronin's universality theorem.

Dirk Basson (University of Stellenbosch, South Africa): *Hecke operators in higher rank.*

Recently, F. Breuer, R. Pink and I have made available preprints on a foundational theory for Drinfeld modular form of rank greater than 2. In this talk I wish to give an overview of this theory with a focus on the Hecke operators that act on them.

Gunther Cornelissen (Utrecht University, Netherlands): *Counting in algebraic groups in positive characteristic.*

Our starting point are the well-known formulae for the number of places of a given degree in a global function field and for the order of groups of Lie type over finite fields. We consider these as results about the action of Frobenius on jacobians and on the additive group, and investigate what happens for general endomorphisms of general algebraic groups. New p-adic phenomena appear, zeta functions are no longer rational in general, and the analogue of the Prime Number Theorem can fail. (Joint work with Jakub Byszewski and Marc Houben.)

Madeline Locus Dawsey (Emory University, USA): *Higher Width Moonshine.*

Weak moonshine for a finite group G is the phenomenon where an infinite dimensional graded G-module

$$V_G = \bigoplus_{n \gg -\infty} V_G(n)$$

has the property that its trace functions, known as McKay-Thompson series, are modular functions. Recent work of Dehority, Gonzalez, Vafa, and Van Peski established that weak moonshine holds for every finite group. Since weak moonshine only relies on character tables, which are not isomorphism class invariants, non-isomorphic groups can have the same McKay-Thompson series. We address this problem by extending weak moonshine to arbitrary width $s \in \mathbb{Z}^+$. Namely, for each $1 \le r \le s$ and each irreducible character χ_i, we employ Frobenius' r-character extension $\chi_i^{(r)} : G^{(r)} \to \mathbb{C}$ to define McKay-Thompson series of $V_G^{(r)} := V_G \times \cdots \times V_G$ (r copies) for each r-tuple in $G^{(r)} := G \times \cdots \times G$ (r copies). These series are modular functions. We find that *complete* width 3 weak moonshine always determines a group up to isomorphism. Furthermore, we establish orthogonality relations for the Frobenius r-characters, which dictate the compatibility of the extension of weak moonshine for V_G to width s weak moonshine.

Luca Demangos (University of Stellenbosch, South Africa): *Quantum Drinfeld modules II: quantum exponential and ray class field generation - Joint work with T. M. Gendron.*

We provide a more detailed overview of the second part of our joint work with T. M. Gendron on Hilbert's 12th problem for real quadratic global function fields, concerning the algebraic aspect of our construction. This involves the formulation of a quantum exponential and the corresponding quantum Hayes module associated to our initial real quadratic global function field K. This construction is used in order to propose a notion of torsion point for such a corresponding object, which we use to generate the relative abelian closure of K as a field extension of the relative Hilbert Class Field of K.

Sumita Garai (Penn State University, USA): *Endomorphism Rings of Finite Drinfeld Modules.*

The theory of Drinfeld modules runs parallel to the theory of Elliptic curves, and our result was motivated by a similar result for Elliptic curves. Let $A = \mathbb{F}_q[T]$ be the polynomial ring over $\mathbb{F}_q$, and F be the field of fractions of A. Let ϕ be a Drinfeld A-module of rank r over F. For all but finitely many primes $\mathfrak{p} \lhd A$, one can reduce ϕ modulo $\mathfrak{p}$ to obtain a Drinfeld A-module $\phi \otimes \mathbb{F}_\mathfrak{p}$ of rank r over $\mathbb{F}_\mathfrak{p} = A/\mathfrak{p}$. It is known that the endomorphism ring $\mathcal{E}_{\mathfrak{p}} = \mathrm{End}_{\mathbb{F}_{\mathfrak{p}}}(\phi \otimes \mathbb{F}_{\mathfrak{p}})$ is an order in an imaginary field extension K of F of degree r. Let $\mathcal{O}_{\mathfrak{p}}$ be the integral closure of A in K, and let $\pi_{\mathfrak{p}} \in \mathcal{E}_{\mathfrak{p}}$ be the Frobenius endomorphism of $\phi \otimes \mathbb{F}_{\mathfrak{p}}$. Then we have the inclusion of orders $A[\pi_{\mathfrak{p}}] \subset \mathcal{E}_{\mathfrak{p}} \subset \mathcal{O}_{\mathfrak{p}}$ in K. In a joint work with my advisor, Mihran Papikian, we showed that if ϕ is a Drinfeld Module without complex multiplication, then for arbitrary non-zero ideals $\mathfrak{n}, \mathfrak{m}$ of A, there are infinitely many $\mathfrak{p}$ such that $\mathfrak{n}$ divides the index $\chi(\mathcal{E}_{\mathfrak{p}}/A[\pi_{\mathfrak{p}}])$ and $\mathfrak{m}$ divides the index $\chi(\mathcal{O}_{\mathfrak{p}}/\mathcal{E}_{\mathfrak{p}})$. We also give an algorithm to compute $\mathcal{E}_{\mathfrak{p}}$ in the rank-2 case.

Ernst-Ulrich Gekeler (Saarland University, Germany): *The Eisenstein compactification of a Drinfeld moduli scheme of higher rank and its relation with modular forms.*

Let F be a finite field, $A = F[T]$ its polynomial ring, N a non-constant element of A, and $M^r(N)/C_\infty$ the moduli scheme of Drinfeld A-modules of rank $r >= 2$ with structure of level N, regarded as a variety over the "complex numbers" C_∞. Using the algebra $Eis(N)$ generated over C_∞ by the Eisenstein series of level N, we construct a (not necessarily normal) compactification of $M^r(N)/C_\infty$ and use it to describe the algebra $Mod(N)$ of modular forms of rank r and level N. $Mod(N)$ is the normalization of $Eis(N)$ in their common field of fractions, and consists of those weak modular forms of level N which, along with their transforms under $GL(r, A)$, are bounded on the natural fundamental domain for $GL(r, A)$ on the Drinfeld space of rank r.

Tim Gendron (UNAM, Mexico): *Quantum Modular Invariant, Hilbert Class Fields and Quasicrystalline Drinfeld Modules.*

The quantum modular invariant

$$j^{\mathrm{qt}} : \mathbb{R} \multimap \mathbb{R} \cup \{\infty\}$$

is a multi-valued, discontinuous and $\mathrm{GL}_2(\mathbb{Z})$-invariant function, defined using diophantine approximations. If θ is a fundamental quadratic unit, $j^{\mathrm{qt}}(\theta)$ is a Cantor set, and we conjecture that the Hilbert class field H_K of $K = \mathbb{Q}(\theta)$ is generated by a weighted product of the elements of $j^{\mathrm{qt}}(\theta)$. The analog of this conjecture in positive characteristic is now a theorem:

Let $\mathbf{K}/\mathbb{F}_q(T)$ be real and quadratic, generated by f a fundamental unit. Then $\mathrm{j}^{\mathrm{qt}}(f)$ is finite and

$$\mathbf{H}_{\mathcal{O}_{\mathbf{K}}} = \mathbf{K}\bigg(\prod_{j_i \in \mathrm{j}^{\mathrm{qt}}(f)} j_i\bigg),$$

where $\mathbf{H}_{\mathcal{O}_{\mathbf{K}}}$ is the Hilbert class field associated to $\mathcal{O}_{\mathbf{K}}$.

The key step in the proof is the identification of the elements $j_i \in j^{\mathrm{qt}}(f)$ with modular invariants $j(\mathfrak{a}_i)$ of ideals $\mathfrak{a}_i$ in the ring $\mathbf{A}_{\infty_1}$ of functions regular outside of a point $\infty_1 \in \Sigma_{\mathbf{K}}$ = the curve associated to $\mathbf{K}$. In 2017, Richard Pink (private communication) discovered the analog of this latter property in characteristic zero: if $\theta \in \mathbb{R}$ is a fundamental quadratic unit, each element of $j^{\mathrm{qt}}(\theta)$ is of the form $j(\mathfrak{a})$ where $\mathfrak{a} \subset \mathcal{O}_K$ is a *quasicrystal*. The quasicrystals in Pink's Theorem are examples of quasicrystalline ideals, with which one may define the characteristic zero analog of Drinfeld module: a 1-dimensional solenoid equipped with an exponential map with values in $\mathbb{C}$. We finish by discussing the prospects of developing Hayes theory in characteristic zero using these ideas. This is work in collaboration with Carlos Castaño Bernard, Luca Demangos, Éric Leichtnam and Pierre Lochak.

Oguz Gezmis (Texas A & M University, USA): *Pellarin L-Values for Drinfeld Modules Over Tate Algebras.*

In 2012, Pellarin defined an L-series which can be seen as a deformation of Goss's L-series. In order to give new identities for Pellarin L-Series, Anglès, Pellarin and Tavares Ribeiro introduced Drinfeld modules over Tate algebras and also generalized the theory of Taelman for Drinfeld modules of rank 1 over Tate algebras. Later on, Anglès and Tavares Ribeiro expanded the theory for the special type of Drinfeld modules of arbitrary rank over Tate algebras. In this talk, we investigate Taelman L-values corresponding to Drinfeld modules over Tate algebras of arbitrary rank. Using our results, we also introduce an L-series which can be seen as a generalization of Pellarin L-series.

Nathan Green (UCSD, USA): *Hyperderivatives of t-motivic multiple zeta values.*

For each function field multiple zeta value (defined by Thakur), we construct a t-module with an attached logarithmic vector such that a specific coordinate of the logarithmic vector is a rational multiple of that multiple zeta value. We then show that the other coordinates of this logarithmic vector contain hyperderivatives of a deformation of these multiple zeta values, which we call t-motivic multiple zeta values. This allows us to give a logarithmic expression for monomials of multiple zeta values. Joint work with Chieh-Yu Chang and Yoshinori Mishiba.

Richard Griffon (University of Basel, Switzerland): *Analogue of the Brauer-Siegel theorem for some families of elliptic curves.*

L-functions of elliptic curves over global fields conjecturally encode a lot of information about their arithmetic. In general though, even for elliptic curves E over $F_q(t)$, not much is known about their L-functions $L(E,s)$. For example, consider the first non-zero coefficient $L^*(E,1)$ in the Taylor expansion of $L(E,s)$ around the point $s=1$ (the so-called 'special value): the size of $L^*(E,1)$, when compared to the degree of the conductor of E, remains mysterious. One expects that $L^*(E,1)$ is generically as big as it possibly can, but this has only been proved in a very limited number of cases. In this talk, I would like to report on recent works about this question: I studied some special sequences of elliptic curves and proved that the above expectation is true (unconditionally). Via the Birch and Swinnerton-Dyer conjecture, one can translate these bounds on $L^*(E,1)$ into asymptotic estimates on certain arithmetic invariants of the elliptic curves E under consideration. These provide examples of family of elliptic curves for which an analogue of the classical Brauer-Siegel theorem holds.

Simon Häberli (ETH, Zurich, Switzerland): *Satake compactification of analytic Drinfeld modular varieties.*

Drinfeld modular forms have been defined and studied in terms of Fourier expansions by Gekeler and by Goss mainly in the case of rank 2 and by Basson, Breuer and Pink in general. It is a classical approach to interpret them as the global sections of ample invertible sheaves on a compact space. This yields finiteness results for rings and vector spaces of modular forms. In this talk, we shall explain the construction of a normal projective compactification of any irreducible component of any analytic Drinfeld modular variety. We shall describe explicit sheaves on the compactification whose global sections correspond bijectively to modular forms. An important role is played by Eisenstein series which we shall write directly as global sections. The compactification is analogous to the ones of Shimura varieties by Satake, Baily and Borel. Its underlying topological space coincides, in the case of the polynomial ring, with Kapranov's compactification. This is work advised by Richard Pink.

Urs Hartl (University of Münster, Germany): *Zeta-functions and Groessencharacters of A-Motives with complex multiplication.*

We will study uniformizable A-Motives with complex multiplication over finite field extensions of Quot(A) and express their Goss Zeta-function as a product of L-series attached to Groessencharacters. As a tool we construct the Serre-Tate reciprocity homomorphisms for uniformizable CM A-Motives.

Shin Hattori (Tokyo City University, Japan): *Dimension Variation of Gouvêa-Mazur type for Drinfeld cuspforms of level* $\Gamma_1(t)$*.*

Let p be a rational prime and $q > 1$ a p-power. Let U be the operator on the space of Drinfeld cuspforms of level $\Gamma_1(t)$ and weight k defined by $Uf(z) = 1/t \sum_{b \in \mathbf{F}_q} f((z+b)/t)$. The t-adic valuation of its eigenvalue is called slope. We denote by $d(k,a)$ the dimension of the generalized U-eigenspace with eigenvalues of slope a.

Recently, it has been observed that slopes have some patterns which are comparable to the case of p-adic slopes of elliptic modular forms. In the elliptic case, such patterns often reflect/indicate the existence of p-adic analytic families of eigenforms including eigencurves, while in the Drinfeld case a parallel construction breaks down and we do not have a good theory of t-adic analytic families. In this talk, I will explain that an analogue of the Gouvêa-Mazur conjecture nonetheless holds for our case. Namely, for any non-negative integer m and non-negative rational number $a \leq m$, we have $d(k,a) = d(k',a)$ if $k, k' > a+1$ and $k \equiv k' \bmod p^m$.

Masanobu Kaneko (Kyushu University, Japan): *On the "value" of the elliptic modular function at real quadratics.*

About ten years ago, I have made some observations on the "values" (or cycle integrals) of the elliptic modular function at real quadratics and posed several conjectures. Recently, Imamoglu, Bengoechea, and Paepcke solved some of the conjectures. I will review these conjectures and their work, and give some generalizations of their results.

Andreas Maurischat (University of Heidelberg, Germany): *On algebraic independence of period coordinates of t-modules.*

We will explain a construction to obtain the period coordinates of uniformizable abelian t-modules as the special values at $t = \theta$ of a rigid analytic trivialization of a related dual t-motive. This allows to use the ABP-criterion for showing transcendence results for period coordinates via proving corresponding results for the entries of the rigid analytic trivialization. Actually, the dual t-motive will not show up in the talk, but only a matrix Ψ representing the rigid analytic trivialization. In the talk, we will also give such a result for the n-th tensor power of the Carlitz module.

Changningphaabi Namoijam (Texas A & M University, USA): *Hyperderivatives of periods and quasi-periods of t-modules.*

Brownawell and Denis constructed, as extensions of Drinfeld modules by additive groups, Divided Derivatives of a Drinfeld module whose periods can be expressed in terms of hyperderivatives of the periods and quasi-periods of the given Drinfeld module. In this talk, we discuss how to obtain hyperderivatives of periods and quasi-periods of an abelian Anderson t-module as periods and quasi-periods of the t-module given by the minimal quasi-periodic extension of Maurischats prolongation t-module of the given t-module. We also determine how periods, quasi periods, logarithms and quasi-logarithms of an abelian Anderson t-module appear as evaluations of solutions of Frobenius difference equations. This is joint work with Matthew A. Papanikolas.

Ken Ono (Emory University, USA): *Polyas Program for the Riemann Hypothesis and Related Problems.*

In 1927 Polya proved that the Riemann Hypothesis is equivalent to the hyperbolicity of Jensen polynomials for Riemanns Xi-function. This hyperbolicity has only been proved for degrees d=1, 2, 3. We prove the hyperbolicity of all (but possibly finitely many) the Jensen polynomials of every degree d. We obtain a general theorem which models such polynomials by Hermite polynomials, which in the case of the zeta-function can be thought of as a "degree aspect" GUE distribution. The general theorem also allows us to prove a conjecture of Chen, Jia, and Wang on the partition function, as well as the coefficients of generic weakly holomorphic modular forms. This is joint work with Michael Griffin, Larry Rolen, and Don Zagier.

Matthew Papanikolas (Texas A & M University, USA): *Equidistribution of Gross points over rational function fields.*

Gross points over function fields have been investigated by Papikian, Wei, and Yu, as special points on certain Shimura curves that are disjoint unions of genus 0 curves. They are connected to isomorphism classes of supersingular Drinfeld modules of rank two with endomorphisms by a prescribed quaternion algebra, as well as to isomorphism classes of rank two Drinfeld modules with complex multiplication. We will discuss a sparse equidistribution result for Gross points over the rational function field, with applications to the surjectivity of the reduction map from CM Drinfeld modules to the supersingular locus. Joint work with A. El-Guindy, R. Masri, and G. Zeng.

Giovanni Rosso (Concordia University, Canada & University of Cambridge, UK): *Drinfeld Modular Varieties for GL(r) and Families of Modular Forms.*

Classical modular curves associated with GL(2) are moduli spaces of elliptic curves with additional structure. While there are no Shimura varieties associated with the general linear group GL(r) for $r > 2$, the situation is sharply different over function fields. The Drinfeld modular variety for GL(r) is the moduli space of Drinfeld modules of rank r (with auxiliary level structure). It is a smooth, affine scheme of relative dimension $r - 1$. I will recall how various analogues of well-established theories in the classical context extend to Drinfeld modular varieties and their modular forms: Hasse invariant and ordinary locus, Igusa tower, Katz's theory of p-adic modular forms. I will construct Hida families of modular forms and explain what can be done for finite slope forms. Joint work with M.-H. Nicole.

Ari Shnidman (CCR Princeton, USA): *Intersections of Heegner-Drinfeld cycles.*

I'll describe a function field and "higher order" analogue of the Gross-Kohnen-Zagier formula. Our formula relates the higher order central derivatives of automorphic L-functions to the intersection pairing of two (possibly different) Heegner-Drinfeld cycles in the moduli stack of shtukas for PGL_2. Intervening is a certain toric period integral. The proof follows Jacquet's relative trace-formula approach. Along the way, I will outline the basics of this approach and its geometrization, due to Yun and Zhang. This is joint work with Ben Howard.

Douglas Ulmer (University of Arizona, USA): *On elliptic divisibility sequences over function fields.*

Let K be the function field of a curve C over a field k. If E is an elliptic curve over K and P is a K-rational point of E of infinite order, then there is an associated "elliptic divisibility sequence" $D_n (n = 1, 2, ...)$ which is a sequence of divisors on C encoding the intersections between nP and the zero section of E. We prove that if k has characteristic zero, then for any P as above, there is a finite list of integers $n_1, ..., n_k$ such that D_n is reduced if and only if n is not divisible by any of the n_i. Our proof is rather analytic, and we invite the audience to look for a more algebraic proof that might extend to positive characteristic. Time permitting, we will introduce a mysterious non-algebraic distribution (or connection) and a closely related algebraic distribution connected to classical modular forms which are lurking in the background.

Maria Valentino (Università di Parma, Italy): *Slopes of Drinfeld cusp forms.*

Let $S_{k,m}(GL_2(\mathbb{F}_q[t]))$ be the space of Drinfeld cusp forms of weight $k \in \mathbb{Z}$ and type $m \in \mathbb{Z}/(q-1)\mathbb{Z}$, where $\mathbb{F}_q$ is the finite field with q elements. Let $\Gamma_0(t)$ be the Hecke congruence subgroup and $S_{k,m}(\Gamma_0(t))$ the space of Drinfeld cusp forms of level t. Finally, let U_t be the Atkin operator acting on $S_{k,m}(\Gamma_0(t))$.
We study the degeneracy maps $S_{k,m}(GL_2(\mathbb{F}_q[t])) \to S_{k,m}(\Gamma_0(t))$ and trace maps (the other way around) and we use them to define oldforms and newforms.
Starting from a computational search for small q we observed that as the weight k varies the distribution of slopes, i.e. the t-adic valuation of U_t eigenvalues, resembles that of classical cusp forms is several aspects. In this talk we shall describe various results and conjectures on the structure of $S_{k,m}(\Gamma_0(t))$. Moreover, we will look at patterns in slopes and state conjectures on what we think is going on trying to explain the arithmetic/geometric aspects behind these schemes. Furthermore, we will prove, usually in some special cases, these conjectures.
This is based on a joint work with A. Bandini.

Ian Wagner (Emory University, USA): *Harmonic Hecke eigenlines and Mazur's problem.*

We construct two families of harmonic Maass Hecke eigenforms. Using these families, we construct p-adic harmonic Maass forms in the sense of Serre. The p-adic properties of these forms answer a question of Mazur about the existence of an "eigencurve-type" object in the world of harmonic Maass forms.

Fu-Tsun Wei (National Central University, Taiwan): *A Sturm-type bound for Drinfeld-type automorphic forms over function fields.*

In the function field context, Drinfeld-type automorphic forms can be viewed as analogue of classical weight 2 modular forms. The structure of the Hecke algebra associated to these automorphic forms is related to various topics in function field arithmetic. In this talk, I shall present a Sturm-type bound for the generators of the Hecke algebra in question. One consequence is to determine effectively whether two elliptic curves over function fields are isogenous. This is a joint work with Cécile Armana.

Chia-Fu Yu (Academia Sinica, Taiwan): *Arithmetic Satake compactifications and algebraic Drinfeld modular forms.*

Drinfeld modular schemes were introduced by Drinfeld in order to prove the Langlands reciprocity of the function field analogue. Compactifying these moduli spaces is one of key steps for realizing the Langlands correpsondence in their l-adic cohomologies. Drinfeld constructed the compactification for rank 2 moduli spaces. Higher rank moduli spaces were constructed by Kapranov, Gekeler and Pink by different methods. In this talk we shall discuss the arithmetic Satake compacfication of Drinfeld moduli schemes of any rank following Pink's approach. Applications to algebraic Drinfeld modular forms are addressed. This is joint work with Urs Hartl.

Jing Yu (National Taiwan University, Taiwan): *On Arithmetic Properties of Drinfeld Modular Forms.*

We will report on the arithmetic properties of Drinfeld modular forms, e.g. Drinfeld-Eisenstein Series, Drinfeld Discriminant forms, asking what could be expected in the higher rank case. This concerns progress on values of these modular forms taken at various CM points on the Drinfeld period domain, the transcendence, and algebraic independence of such CM values (due to Chang in rank 2 case) in relating to values of Goss zeta function at positive integers divisible by $q-1$. Another aspect of interests to us is a development of Kronecker type formulas for GL_n (by Kondo, Wei) which involves ∞-adic absolute value of the Discriminant forms.

Guchao Zeng (Texas A & M University, Qatar): *v-adic limits of Bernoulli-Carlitz numbers.*

The Bernoulli-Carlitz numbers BC_m for the rational function field K over a finite field of order q do not behave the same as the classical Bernoulli numbers. Serre has proved that Bernoulli numbers B_m have p-adic limits and the limits are closely related to the p-adic limits of m. However, it is not the case for Bernoulli-Carlitz numbers. We show that BC_m has v-adic limits (v is a finite place of K of degree d) for m of the form $aq^{dj}+b$, where a and b are positive integers. Moreover, the limit is in a constant field extension of K and invariant under the permutation of distance d. Joint work with M. Papanikolas.