BRANCHING ANALYSIS OF A COUNTABLE FAMILY OF GLOBAL SIMILARITY SOLUTIONS OF A FOURTH-ORDER THIN FILM EQUATION

P. ÁLVAREZ-CAUDEVILLA AND VICTOR A. GALAKTIONOV

Abstract. The main goal of the paper is to justify that source-type and other global-in-time similarity solutions of the Cauchy problem for the fourth-order thin film equation

$$u_t = -\nabla \cdot (|u|^n \nabla \Delta u) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad \text{where} \quad n > 0, \;\; N \geq 1, \tag{0.1}$$

can be obtained by a continuous deformation (a homotopy path) as $n \to 0^+$ by reducing to similarity solutions (given by eigenfunctions of a rescaled linear operator $\mathbf{B}$) of the classic *bi-harmonic equation*

$$u_t = -\Delta^2 u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \;\; \text{where } \mathbf{B} = -\Delta^2 + \tfrac{1}{4}\, y \cdot \nabla + \tfrac{N}{4}\, I. \tag{0.2}$$

This approach leads to a countable family of various global similarity patterns of (0.1) and describes their oscillatory sign-changing behaviour by using the known asymptotic properties of the fundamental solution of (0.2). The branching from $n = 0^+$ for (0.1) requires Hermitian spectral theory for a pair $\{\mathbf{B}, \mathbf{B}^*\}$ of non-self adjoint operators and leads to a number of difficult mathematical problems. These include, as a key part, the problem of multiplicity of solutions, which is under particular scrutiny.

1. Introduction: TFEs, connections with classic PDE theory, layout

1.1. **Main models, their applications, and preliminaries.** We study the global-in-time behaviour of solutions of the fourth-order quasilinear evolution equation of parabolic type, called the *thin film equation* (the TFE–4, in short)

$$u_t = -\nabla \cdot (|u|^n \nabla \Delta u) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+\,, \tag{1.1}$$

where $\nabla = \text{grad}_x$, $\Delta = \nabla \cdot \nabla$ stands for the Laplace operator in $\mathbb{R}^N$, and $n > 0$ is a real parameter. Fourth- and sixth-order TFEs (the TFE–6) having a similar form,

$$u_t = \nabla \cdot (|u|^n \nabla \Delta^2 u)\,, \tag{1.2}$$

as well as more complicated doubly nonlinear degenerate parabolic models (see typical examples in [29]), have various applications in thin film, lubrication theory, and in several

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other hydrodynamic-type problems. We refer e.g., to [13, 15, 25, 26] for most recent surveys and for extended lists of references concerning physical derivations of various models, key mathematical results, and further applications. Since the 1980s, such equations also play a quite special role in nonlinear PDE theory to be discussed in greater detail below.

The TFE–4 (1.1) is written for solutions of changing sign, which can occur in the *Cauchy problem* (the CP) and also in some *free–boundary problems* (FBPs); see proper settings shortly. It is worth mentioning that *nonnegative solutions* with compact support of various FBPs are mostly physically relevant, and that the pioneering mathematical approaches by Bernis and Friedman in 1990 [5] were developed mainly for such solutions.

However, *solutions of changing sign* have been already under scrutiny for a few years (see [9, 14, 16]), which in particular can have some biological motivations [30], to say nothing of general PDE theory. It turned out that these classes of the so-called "oscillatory solutions of changing sign" of (1.1) were rather difficult to tackle rigorously by standard and classic methods. Moreover, even their self-similar (i.e., ODE) representatives can lead to several surprises in trying to describe sign-changing features close to interfaces; see [14] for a collection of such hard properties. It turned out also that, for better understanding of such singular oscillatory properties of solutions of the CP for (1.1), it is fruitful to consider the (homotopic) limit $n \to 0^+$, owing to Hermitian spectral theory developed in [12] for a pair $\{\mathbf{B}, \mathbf{B}^*\}$ of linear rescaled operators for $n = 0$, i.e., for the *bi-harmonic equation*

$$(1.3) \quad u_t = -\Delta^2 u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad \mathbf{B} = -\Delta^2 + \tfrac{1}{4}\, y \cdot \nabla + \tfrac{N}{4}\, I, \quad \mathbf{B}^* = -\Delta^2 - \tfrac{1}{4}\, y \cdot \nabla,$$

which will always be key for our further analysis.

In the present paper, using this continuity/homotopy deformation approach "$n \to 0^+$", we shall focus our analysis to the Cauchy problem for (1.1) for exponents $n > 0$, which are assumed to be sufficiently small. Some key and necessary references will be presented later on. We study the large time behaviour of the solutions of (1.1). To this end, we will use some natural connections with a similar analysis for more complicated models such as the *limit unstable fourth-order thin film equation* (the unstable TFE–4):

$$(1.4) \qquad u_t = -\nabla \cdot (|u|^n \nabla \Delta u) - \Delta(|u|^{p-1} u)\,,$$

with the unstable homogeneous second-order diffusion term, where $p > 1$ is a fixed exponent; see [13] for physical motivations, references, and other basics. Here, (1.4) represents a fourth-order nonlinear parabolic equation with the backward (unstable) diffusion term in the second-order operator. Blow-up and global self-similar solutions of (1.4) have been extensively studied in [13, 14] for the unstable TFE–4 (1.4) and in [15, 16] for the *unstable TFE–6*,

$$(1.5) \qquad u_t = \nabla \cdot (|u|^n \nabla \Delta^2 u) - \Delta(|u|^{p-1} u)\,,$$

where further references and other related higher-order TFEs can be found.

From the application point of view, it is well known (see references to surveys above) that (1.1) and (1.4) arises in numerous areas. In particular, those equations model the dynamics of a thin film of viscous fluid, as the spreading of a liquid film along a surface, where u stands the height of the film (then clearly $u \geq 0$ that naturally leads to a FBP setting). In particular, when $n = 3$ we are dealing with a problem in the context of lubrication theory for thin viscous films that are driven by surface tension and when $n = 1$ with Hele–Shaw flows. It is also important to note that, in (1.4), the fourth-order term reflects surface tension effects and the second-order term can reflect gravity, van der Waals interactions, thermocapillary effects, or geometry of the solid substrate.

Finally, in order to summarize let us mention again that higher-order semilinear and quasilinear parabolic equations occur in applications to thin film theory, nonlinear diffusion, lubrication theory, flame and wave propagation (the Kuramoto–Sivashinsky equation and the extended Fisher–Kolmogorov equation), phase transition at critical Lifshitz points and bi-stable systems (see Peletier–Troy [39] for further details, models, and results). Moreover, in the special situation when $n = 0$ we should notice that (1.4) is the well known *unstable Cahn–Hilliard equation* (the CHE)

$$u_t = -\Delta^2 u - \Delta(|u|^{p-1}u) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+ \, ; \tag{1.6}$$

see main references and full details in [17]. The unstable semilinear model (1.6) and similar stable ones are much better known and are connected with several new applications that have increased the interest of the study of their quasilinear TFE extensions (1.4) and consequently of (1.1). Note that, without any doubts, the semilinear CHE (1.6) in the CP setting, admits *oscillatory* solutions of changing sign, though with no finite (i.e., *infinite*) interfaces. As our main goal, we plan to extend those properties of the CHE (1.6) to the TFEs for small $n > 0$, where oscillations begin to concentrate at *finite* interfaces.

1.2. **A digression to reaction-diffusion theory.** Furthermore, in the CP setting for (1.6), one can write (1.6) in the form

$$\mathcal{A} u_t = \Delta u + u^p, \quad \text{where} \quad \mathcal{A} := (-\Delta)^{-1} \tag{1.7}$$

is a standard positive operator, so that (1.7) is a pseudo-parabolic second-order equation. Such equations had been widely studied since the 1970's, and nowadays are well-understood with both the existence and uniqueness results of local and global classical or blow-up solutions obtained. first blow-up results for such pseudo-parabolic PDEs were due to Levine in 1973, [33]. As was noted in [17], there are some similarities between (1.7) and the classical semilinear heat equation from combustion theory (the solid fuel model)

$$u_t = \Delta u + u^p \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad p > 1 \, , \tag{1.8}$$

with $N \geq 1$, especially for the blow-up singularity formation phenomena. Mathematical literature devoted to the study of (1.8) include a huge number of papers published since Fujita's classic papers in the 1960s, and this remarkable history has been already explained in a dozen of well-known monographs; we refer to [1, 18, 24, 34, 38, 40, 41, 42], where further extensions and references can be found. In particular, concerning blow-up patterns, a complete description of all possible types of blow-up have been achieved for some ranges of the parameters p and N, especially in the subcritical Sobolev range

$$p < p_{\text{Sobolev}} = \tfrac{N+2}{(N-2)_+}.$$

Note that, for $p \geq p_{\text{Sobolev}}$, such a classification of blow-up patterns is far away from being complete, with a number difficult open problems posed.

Nevertheless, using a standard *pseudo-parabolic* form (1.7) of the C–H flows could rise a hope to apply a huge experience achieved earlier for classic reaction-diffusion models (1.8), though this is not expected to be that straightforward.

1.3. **A digression to porous medium equation: homotopy to the heat equation.** Returning to the TFEs, note that the unstable nonlinear operator in (1.4) gives us the notorious classic *porous medium equation* (the PME–2, but posed backwards in time),

$$u_t = \Delta(|u|^{n-1}u) \quad \text{(for convenience, } p \text{ is replaced by } n \text{ as in (1.1))}, \tag{1.9}$$

which derives its name from the role in the description of flows in porous media. Parabolic PDE models in filtration theory of liquids and gases in porous media were derived by Leibenzon in the 1920s and 1930s, as well as by Richard's (1931), and Muskat (1937). In fact, modern filtration theory goes back to the beginning of the twentieth century initiated in the works by N.Ye. Zhukovskii, who is better known for his fundamental research in aerodynamics, hydrodynamics, and ODE theory (on his pioneering non-oscillation test in 1892, see [18, p. 19]). His contribution to "theory of ground waters" is explained in P.Ya. Kochina's paper [31]. For an extended list of references on this subject and more filtration history, see [19].

It is well understood that, for any $n > 1$, (1.9) has a family of exact self-similar compactly supported source type solutions (the ZKB ones from 1950s), which describe the large time behaviour of compactly supported solutions with conservation of mass in the case on a non-zero mass, i.e.,

$$\textstyle\int u(x,t)\,\mathrm{d}x \neq 0.$$

On the other hand, (1.9) also admits a countable (at least) family of other similarity solutions; see [21] for key references and most recent results.

The PME–2 (1.9) can be interpreted as a nonlinear degenerate version of the classic *heat equation* for $n = 0$,

$$u_t = \Delta u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+.$$

Note that passing to the limit $n \to 0^+$ in (1.9) for nonnegative solutions used to be a difficult mathematical problem in the 1970s-80s, which exhibited typical (but clearly simpler than in the TFE case) features of a "homotopy" transformation of PDEs. This study in 1D was initiated by Kalashnikov in 1978 [28]. Further detailed results in $\mathbb{R}^N$ were obtained in [2]; see also [10]. More recent involved estimates were obtained in [36, 37] for the 1D PME–2 (1.9) establishing the rate of convergence of solutions as $n \to 0^{\pm}$, such as $O(n)$ as $n \to 0^-$ (i.e, from $n < 0$, the fast diffusion range, where solutions are smoother) in $L^1(\mathbb{R})$ [36], and $O(n^2)$ as $n \to 0^+$ in $L^2(\mathbb{R} \times (0,T))$ [37]. However, the most of such convergence results are obtained for *nonnegative* solutions of (1.9). For solutions of changing sign, there are some open problems; see [21] for references and further details.

Note that, as customary, any kind of detailed asymptotic analysis for higher-order equations is much more difficult than for second-order counterparts in view of the lack of

Maximum Principle, comparison, and order preserving semigroups and potential properties of the operators involved. Thus, practically all the existing methods for the PME–2 (1.9) are not applicable to the TFE–4 (1.1) or (1.4).

Thus, in the twenty-first century, higher-order TFEs such as (1.1) and (1.2), though looking like a natural counterpart/extension of the PME–2 (1.9), corresponding mathematical TFE theory is more complicated with several problems, remaining open still.

1.4. **Main approaches, results, and layout.** It is worth mentioning that, unlike the FBPs, studied in hundreds of papers since 1980s (see [25] and [14] for key references and alternative versions of uniqueness approaches), thin film theory for the Cauchy problem for (1.1) or (1.4) led recently to a number of difficult open problems and is not still fully developed; see the above references as a guide to main difficulties and ideas. In fact, the concept of proper solutions is still rather obscure for the Cauchy problem, since any classic or standard notions of weak-mild-generalized-... solutions fail in the CP setting.

In this work, we perform a more systematic than before analysis of the behaviour of the similarity solutions through a so-called *homotopic approach* (branching from $n = 0$) via branching theory, using the Lyapunov–Schmidt methods for obtaining relevant results and properties for the solutions of the self-similar equation associated with (1.1) and, hence, for the proper solutions of (1.1). Overall, loosely speaking, this approach is characterized as follows: good proper (similarity or not) solutions of the Cauchy problem for the TFE (1.1) are those that can be continuously deformed (via a homotopic path) as $n \to 0^+$ to the corresponding solutions of the bi-harmonic equation (1.3), which will play a crucial role in the subsequent analysis. This homotopic-like approach is based upon the spectral properties known for the linear counterpart (1.3) of the TFE (1.1). Moreover, owing to the oscillatory character of the solutions of the bi-harmonic equation (1.3) being a "limit case" of the TFE (1.1), close to the interfaces, this homotopy study exhibits a typical difficulty concerning the desired structure of the transversal zeros of solutions, at least for small $n > 0$. Proving such a transversality zero property is a difficult open problem, though qualitatively, this was rather well understood, [13].

Indeed, we ascertain through an analytic "homotopy", which is understood as just the existence of a continuity as $n \to 0^+$, over the CP performed in the last section of this paper that the solutions of (1.1) are homotopic to the solutions of the bi-harmonic equation (1.3) in a weak sense. However, we must admit that this does not solve the problem of uniqueness of solutions of the CP (see details in Section 5), since the final identification of the solutions obtained via analytic ε-regularization and passing to the limit $\varepsilon \to 0^+$ remains not fully understood. Overall, it seems that ε-regularizations of solutions of the TFEs via families of uniformly parabolic analytic flows, which was a powerful and successful tool for second-order degenerate parabolic PDEs (such as the PME–2 (1.9)), for TFEs again leads to difficult *boundary layer*-type problems that remain open in a sufficient generality.

Some parts of the study of the thin film equation (1.4) can be performed in similar lines, though a full homotopy approach would include the passage to the limit $p \to 1^+$, leading to the limit linear equation (not treated here)

$$u_t = -\Delta^2 u - \Delta u.$$

Thus, the layout of the paper is as follows:
(I) Study of a countable family of global self-similar solutions of (1.1) via their branching from eigenspaces at $n = 0^+$, Sections 2–4, and
(II) Some general aspects of the CP for (1.1) by another homotopy approach, Section 5.

2. Problem setting and self-similar solutions

2.1. **The FBP and CP.** For both the FBP and the CP, the solutions are assumed to satisfy standard free-boundary conditions:

$$\begin{cases} u = 0, & \text{zero-height,} \\ \nabla u = 0, & \text{zero contact angle,} \\ -\mathbf{n} \cdot \nabla(|u|^n \Delta u) = 0, & \text{conservation of mass (zero-flux)} \end{cases} \tag{2.1}$$

at the singularity surface (interface) $\Gamma_0[u]$, which is the lateral boundary of

$$\text{supp}\, u \subset \mathbb{R}^N \times \mathbb{R}_+, \quad N \ge 1\,,$$

where $\mathbf{n}$ stands for the unit outward normal to $\Gamma_0[u]$. Note that, for sufficiently smooth interfaces, the condition on the flux can be read as

$$\lim_{\text{dist}(x,\Gamma_0[u]) \downarrow 0} -\mathbf{n} \cdot \nabla(|u|^n \Delta u) = 0.$$

It is key that, for the CP, the solutions are assumed to be "smoother" at the interface than those for the FBP, i.e., (2.1) are not sufficient to define their regularity. These *maximal regularity* issues for the CP, leading to oscillatory solutions, are under scrutiny in [14].

Next, denote by

$$M(t) := \textstyle\int u(x,t)\, \mathrm{d}x$$

the mass of the solution, where integration is performed over smooth support ($\mathbb{R}^N$ is allowed for the CP only). Then, differentiating $M(t)$ with respect to t and applying the Divergence Theorem (under natural regularity assumptions on solutions and free boundary), we have that

$$J(t) := \tfrac{\mathrm{d}M}{\mathrm{d}t} = -\int_{\Gamma_0 \cap \{t\}} \mathbf{n} \cdot \nabla(|u|^n \Delta u)\,.$$

The mass is conserved if $J(t) \equiv 0$, which is assured by the flux condition in (2.1).

The problem is completed with bounded, smooth, integrable, compactly supported initial data

$$u(x,0) = u_0(x) \quad \text{in} \quad \Gamma_0[u] \cap \{t = 0\}. \tag{2.2}$$

In the CP for (1.1) in $\mathbb{R}^N \times \mathbb{R}_+$, one needs to pose bounded compactly supported initial data (2.2) prescribed in $\mathbb{R}^N$. Then, under the same zero flux condition at finite interfaces (to be established separately), the mass is preserved.

2.2. **Global similarity solutions: towards a nonlinear eigenvalue problem.** We now begin to specify the self-similar solutions of the equation (1.1), which are admitted due to its natural scaling-invariant nature. In the case of the mass being conserved, we have global in time source-type solutions.

Using the following scaling in (1.1)

$$x := \mu \bar{x}, \quad t := \lambda \bar{t}, \quad u := \nu \bar{u}, \quad \text{with}$$

$$\frac{\partial u}{\partial t} = \frac{\nu}{\lambda}\frac{\partial \bar{u}}{\partial t}, \quad \frac{\partial u}{\partial x_i} = \frac{\nu}{\mu}\frac{\partial \bar{u}}{\partial x_i}, \quad \frac{\partial^2 u}{\partial x_i^2} = \frac{\nu}{\mu^2}\frac{\partial^2 \bar{u}}{\partial x_i^2}, \tag{2.3}$$

and substituting those expressions in (1.1) yields

$$\frac{\nu}{\lambda}\frac{\partial \bar{u}}{\partial t} = -\frac{\nu^{n+1}}{\mu^4}\nabla \cdot \left(|\bar{u}|^n \nabla \Delta \bar{u}\right).$$

To keep this equation invariant, the following must be fulfilled:

$$\frac{\nu}{\lambda} = \frac{\nu^{n+1}}{\mu^4}, \quad \text{so that} \tag{2.4}$$

$$\mu := \lambda^{\beta} \Longrightarrow \nu := \lambda^{\frac{4\beta-1}{n}} \quad \text{and} \quad u(x,t) := \lambda^{\frac{4\beta-1}{n}} \bar{u}(\bar{x},\bar{t}) = \lambda^{\frac{4\beta-1}{n}} \bar{u}(\tfrac{x}{\mu}, \tfrac{t}{\lambda}).$$

Consequently,

$$u(x,t) := t^{\frac{4\beta-1}{n}} f(\tfrac{x}{t^{\beta}}),$$

where $t = \lambda$ and $f(\frac{x}{t^{\beta}}) = \bar{u}(\frac{x}{t^{\beta}}, 1)$. Owing to (2.4), we obtain

$$n\alpha + 4\beta = 1,$$

which links the parameters α and β. Hence, substituting

$$u(x,t) := t^{-\alpha} f(y), \quad \text{with} \quad y := \tfrac{x}{t^{\beta}}, \quad \beta = \tfrac{1-n\alpha}{4}, \tag{2.5}$$

into (1.1) and rearranging terms, we find that the function f solves a quasilinear elliptic equation of the form

$$\nabla \cdot (|f|^n \nabla \Delta f) = \alpha f + \beta y \nabla \cdot f. \tag{2.6}$$

Finally, thanks to the above relation between α and β, we find a *nonlinear eigenvalue problem* of the form

$$\boxed{-\nabla \cdot (|f|^n \nabla \Delta f) + \tfrac{1-\alpha n}{4}\, y \nabla \cdot f + \alpha f = 0, \quad f \in C_0(\mathbb{R}^N)}\,, \tag{2.7}$$

where we add to the equation (2.6) a natural assumption that f must be compactly supported (and, of course, sufficiently smooth at the interface, which is an accompanying question to be discussed as well).

Thus, for such degenerate elliptic equations, the functional setting in (2.7) assumes that we are looking for (weak) *compactly supported* solutions $f(y)$ as certain "nonlinear eigenfunctions" that hopefully occur for special values of nonlinear eigenvalues $\{\alpha_k\}_{k \geq 0}$. Our goal is to justify that, labelling the eigenfunctions via a multiindex σ,

$$\boxed{\text{(2.7) possesses a countable set of eigenfunction/value pairs } \{f_\sigma,\, \alpha_k\}_{|\sigma|=k\geq 0}.} \tag{2.8}$$

Concerning the well-known properties of finite propagation for TFEs, we refer to papers [13]–[16], where a large amount of earlier references is available; see also [22] for more recent results and references in this elliptic area. However, there are still a little of entirely rigorous results, especially those that are attributed to the Cauchy problem for TFEs. In the linear case $n=0$, the condition $f \in C_0(\mathbb{R}^N)$, is naturally replaced by the requirement that the eigenfunctions $\psi_\beta(y)$ exhibit typical exponential decay at infinity, a property that is reinforced by introducing appropriate weighted L^2-spaces. Actually, using the homotopy limit $n \to 0^+$, we will be obliged for small $n>0$, instead of C_0-setting in (2.7), we will use the following weighted L^2-space:

$$f \in L^2_\rho(\mathbb{R}^N), \quad \text{where} \quad \rho(y) = \mathrm{e}^{a|y|^{4/3}}, \quad a>0 \text{ small}. \tag{2.9}$$

Note that, in the case of the Cauchy problem with conservation of mass making use of the self-similar solutions (2.5), we have that

$$M(t) := \int_{\mathbb{R}^N} u(x,t)\,\mathrm{d}x = t^{-\alpha} \int_{\mathbb{R}^N} f\big(\tfrac{x}{t^\beta}\big)\,\mathrm{d}x = t^{-\alpha+\beta N} \int_{\mathbb{R}^N} f(y)\,\mathrm{d}y,$$

where the actual integration is performed over the support supp f of the nonlinear eigenfunction. Then, as is well known, if $\int f \neq 0$, the exponents are calculated giving the first explicit nonlinear eigenvalue:

$$-\alpha+\beta N = 0 \quad \Longrightarrow \quad \alpha_0(n) = \tfrac{N}{4+Nn} \quad \text{and} \quad \beta_0(n) = \tfrac{1}{4+Nn}. \tag{2.10}$$

3. Hermitian spectral theory of the linear rescaled operators

In this section, we establish the spectrum $\sigma(\mathbf{B})$ of the linear operator $\mathbf{B}$ obtained from the rescaling of the linear counterpart of (1.1), the bi-harmonic equation (1.3), which will be essentially used in what follows.

3.1. How the operator B appears: a linear eigenvalue problem.

Let $u(x,t)$ be the unique solution of the CP for the linear parabolic bi-harmonic equation (1.3) with the initial data (the space as in (2.9) to be more properly introduced shortly)

$$u_0 \in L^2_\rho(\mathbb{R}^N),$$

given by the convolution Poisson-type integral

$$u(x,t) = b(t) \, * \, u_0 \equiv t^{-\frac{N}{4}} \int_{\mathbb{R}^N} F((x-z)t^{-\frac{1}{4}}) u_0(z)\,\mathrm{d}z. \tag{3.1}$$

Here, by scaling invariance of the problem, in a similar way as was done in the previous section for (1.1), the unique the fundamental solution of the operator $\frac{\partial}{\partial t} + \Delta^2$ has the self-similar structure

$$b(x,t) = t^{-\frac{N}{4}} F(y), \quad y := \tfrac{x}{t^{1/4}} \quad (x \in \mathbb{R}^N). \tag{3.2}$$

Substituting $b(x,t)$ into (1.3) yields that the rescaled fundamental kernel F in (3.2) solves the linear elliptic problem

$$\mathbf{B}F \equiv -\Delta_y^2 F + \tfrac{1}{4}\, y \cdot \nabla_y F + \tfrac{N}{4}\, F = 0 \quad \text{in} \quad \mathbb{R}^N, \quad \int_{\mathbb{R}^N} F(y)\,\mathrm{d}y = 1. \tag{3.3}$$

$\mathbf{B}$ is a non-symmetric linear operator, which is bounded from $H^4_\rho(\mathbb{R}^N)$ to $L^2_\rho(\mathbb{R}^N)$ with the exponential weight as in (2.9). Here, $a \in (0, 2d)$ is any positive constant, depending on the parameter $d > 0$, which characterises the exponential decay of the kernel $F(y)$:

$$|F(y)| \le D\, \mathrm{e}^{-d|y|^{4/3}} \quad \text{in} \quad \mathbb{R}^N \quad \big(D > 0, \;\; d = 3 \cdot 2^{-\frac{11}{3}}\big). \tag{3.4}$$

Later on, by F we denote the oscillatory rescaled kernel as the only solution of (3.3), which has exponential decay, oscillates as $|y| \to \infty$, and satisfies the standard pointwise estimate (3.4).

Thus, we need to solve the corresponding *linear eigenvalue problem*:

$$\boxed{\mathbf{B}\psi = \lambda\psi \quad \text{in} \quad \mathbb{R}^N, \quad \psi \in L^2_\rho(\mathbb{R}^N).} \tag{3.5}$$

One can see that the nonlinear one (2.7) formally reduces to (3.5) at $n = 0$ with the following shifting of the corresponding eigenvalues:

$$\lambda = -\alpha + \tfrac{N}{4}.$$

In fact, this is the main reason to calling (2.7) a nonlinear eigenvalue problem, and, crucially, the discreteness of the real spectrum of the linear one (3.5) will be shown to be inherited by the nonlinear problem, but a long way is needed to justify such an issue.

3.2. **Functional setting and semigroup expansion.** Thus, we solve (3.5) and calculate the spectrum of $\sigma(\mathbf{B})$ in the weighted space $L^2_\rho(\mathbb{R}^N)$. We then need the following Hilbert space:

$$H^4_\rho(\mathbb{R}^N) \subset L^2_\rho(\mathbb{R}^N) \subset L^2(\mathbb{R}^N).$$

The Hilbert space $H^4_\rho(\mathbb{R}^N)$ has the following inner product:

$$\big\langle v, w \big\rangle_\rho := \int_{\mathbb{R}^N} \rho(y) \sum_{k=0}^{4} D^k v(y) \overline{D^k w(y)}\, \mathrm{d}y,$$

where $D^k v$ stands for the vector $\{D^\beta v\,,\, |\beta| = k\}$, and the norm

$$\|v\|^2_\rho := \int_{\mathbb{R}^N} \rho(y) \sum_{k=0}^{4} |D^k v(y)|^2\, \mathrm{d}y.$$

Next, introducing the rescaled variables

$$u(x,t) = t^{-\frac{N}{4}} w(y,\tau), \quad y := \tfrac{x}{t^{1/4}}, \quad \tau = \ln t \,:\, \mathbb{R}_+ \to \mathbb{R}, \tag{3.6}$$

we find that the rescaled solution $w(y,\tau)$ satisfies the evolution equation

$$w_\tau = \mathbf{B} w\,, \tag{3.7}$$

since, substituting the representation of $u(x,t)$ (3.6) into (1.3) yields

$$-\Delta^2_y w + \tfrac{1}{4}\, y \cdot \nabla_y w + \tfrac{N}{4}\, w = t\, \tfrac{\partial w}{\partial t} \tfrac{\partial \tau}{\partial t}.$$

Thus, to keep this invariant, the following should be satisfied:

$$t\, \tfrac{\partial \tau}{\partial t} = 1 \implies \tau = \ln t, \quad \text{i.e., as defined in (3.6).}$$

Hence, $w(y,\tau)$ is the solution of the Cauchy problem for the equation (3.7) and with the following initial condition at $\tau=0$, i.e., at $t=1$:

$$w_0(y)=u(y,1)\equiv b(1)\,*\,u_0=F\,*\,u_0\,. \tag{3.8}$$

Thus, the linear operator $\frac{\partial}{\partial\tau}-\mathbf{B}$ is also a rescaled version of the standard parabolic one $\frac{\partial}{\partial t}+\Delta^2$. Therefore, the corresponding semigroup $\mathrm{e}^{\mathbf{B}\tau}$ admits an explicit integral representation. This helps to establish some properties of the operator $\mathbf{B}$ and describes other evolution features of the linear flow. From (3.1) we find the following explicit representation of the semigroup:

$$w(y,\tau)=\int_{\mathbb{R}^N} F\big(y-z\mathrm{e}^{-\frac{\tau}{4}}\big)\,u_0(z)\,\mathrm{d}z\equiv \mathrm{e}^{\mathbf{B}\tau}w_0,\quad \text{where}\quad x=t^{\frac14}y,\quad \tau=\ln t. \tag{3.9}$$

Subsequently, consider Taylor's power series of the analytic kernel[1]

$$F\big(y-z\mathrm{e}^{-\frac{\tau}{4}}\big)=\sum_{(\beta)}\mathrm{e}^{-\frac{|\beta|\tau}{4}}\frac{(-1)^{|\beta|}}{\beta!}D^\beta F(y)z^\beta\equiv\sum_{(\beta)}\mathrm{e}^{-\frac{|\beta|\tau}{4}}\frac{1}{\sqrt{\beta!}}\psi_\beta(y)z^\beta, \tag{3.10}$$

for any $y\in\mathbb{R}^N$, where

$$z^\beta:=z_1^{\beta_1}\cdots z_N^{\beta_N}$$

and ψ_β are the normalized eigenfunctions for the operator $\mathbf{B}$. The series in (3.10) converges uniformly on compact subsets in $z\in\mathbb{R}^N$. Indeed, denoting $|\beta|=l$ and estimating the coefficients

$$\Big|\sum_{|\beta|=l}\frac{(-1)^l}{\beta!}D^\beta F(y)z_1^{\beta_1}\cdots z_N^{\beta_N}\Big|\le b_l|z|^l,$$

by Stirling's formula we have that, for $l\gg1$,

$$b_l=\tfrac{N^l}{l!}\sup_{y\in\mathbb{R}^N,|\beta|=l}|D^\beta F(y)|\approx\tfrac{N^l}{l!}l^{-l/4}\mathrm{e}^{l/4}\approx l^{-3l/4}c^l=\mathrm{e}^{-l\ln 3l/4+l\ln c}. \tag{3.11}$$

Note that, the series

$$\sum b_l|z|^l$$

has the radius of convergence $R=\infty$.

Thus, we obtain the following representation of the solution:

$$w(y,\tau)=\sum_{(\beta)}\mathrm{e}^{-\frac{|\beta|}{4}\tau}M_\beta(u_0)\psi_\beta(y),\quad\text{where}\quad \lambda_\beta=:-\tfrac{|\beta|}{4} \tag{3.12}$$

and $\{\psi_\beta\}$ are the eigenvalues and eigenfunctions of the operator $\mathbf{B}$, respectively, and

$$M_\beta(u_0):=\frac{1}{\sqrt{\beta!}}\int_{\mathbb{R}^N}z_1^{\beta_1}\cdots z_N^{\beta_N}u_0(z)\,\mathrm{d}z$$

are the corresponding momenta of the initial datum w_0 defined by (3.8).

[1] We hope that returning here to the multiindex β instead of σ in (2.8) will not lead to a confusion with the exponent β in self-similar scaling (2.5).

3.3. **Main spectral properties of the pair $\{\mathbf{B}, \mathbf{B}^*\}$.** Thus, the following holds [12]:

Theorem 3.1. (i) *The spectrum of* $\mathbf{B}$ *comprises real eigenvalues only with the form*

$$\sigma(\mathbf{B}) := \big\{\lambda_\beta = -\tfrac{|\beta|}{4}\,,\ |\beta| = 0, 1, 2, ...\big\}. \tag{3.13}$$

Eigenvalues λ_β *have finite multiplicity with eigenfunctions,*

$$\psi_\beta(y) := \tfrac{(-1)^{|\beta|}}{\sqrt{\beta!}} D^\beta F(y) \equiv \tfrac{(-1)^{|\beta|}}{\sqrt{\beta!}} \big(\tfrac{\partial}{\partial y_1}\big)^{\beta_1} \cdots \big(\tfrac{\partial}{\partial y_N}\big)^{\beta_N} F(y). \tag{3.14}$$

(ii) *The subset of eigenfunctions* $\Phi = \{\psi_\beta\}$ *is complete in* $L^2(\mathbb{R}^N)$ *and in* $L^2_\rho(\mathbb{R}^N)$.
(iii) *For any* $\lambda \notin \sigma(\mathbf{B})$*, the resolvent* $(\mathbf{B} - \lambda I)^{-1}$ *is a compact operator in* $L^2_\rho(\mathbb{R}^N)$.

Subsequently, it was also shown in [12] that the adjoint (in the dual metric of $L^2(\mathbb{R}^N)$) operator of $\mathbf{B}$ given by

$$\mathbf{B}^* := -\Delta^2 - \tfrac{1}{4}\, y \cdot \nabla, \tag{3.15}$$

in the weighted space $L^2_{\rho^*}(\mathbb{R}^N)$, with the exponentially decaying weight function

$$\rho^*(y) \equiv \tfrac{1}{\rho(y)} = \mathrm{e}^{-a|y|^\alpha} > 0, \tag{3.16}$$

is a bounded linear operator,

$$\mathbf{B}^* : H^4_{\rho^*}(\mathbb{R}^N) \to L^2_{\rho^*}(\mathbb{R}^N),\ \text{so}\ \big\langle \mathbf{B}v, w\big\rangle = \big\langle v, \mathbf{B}^* w\big\rangle,\ v \in H^4_\rho(\mathbb{R}^N),\ w \in H^4_{\rho^*}(\mathbb{R}^N).$$

Moreover, the following theorem establishes the spectral properties of the adjoint operator which will be very similar to those shown in Theorem 3.1 for the operator $\mathbf{B}$.

Theorem 3.2. (i) *The spectrum of* $\mathbf{B}^*$ *consists of eigenvalues of finite multiplicity,*

$$\sigma(\mathbf{B}^*) = \sigma(\mathbf{B}) := \big\{\lambda_\beta = -\tfrac{|\beta|}{4}\,,\ |\beta| = 0, 1, 2, ...\big\}, \tag{3.17}$$

and the eigenfunctions $\psi^*_\beta(y)$ *are polynomials of order* $|\beta|$.
(ii) *The subset of eigenfunctions* $\Phi^* = \{\psi^*_\beta\}$ *is complete in* $L^2_{\rho^*}(\mathbb{R}^N)$.
(iii) *For any* $\lambda \notin \sigma(\mathbf{B}^*)$*, the resolvent* $(\mathbf{B}^* - \lambda I)^{-1}$ *is a compact operator in* $L^2_{\rho^*}(\mathbb{R}^N)$.

It should be pointed out that, since $\psi_0 = F$ and $\psi^*_0 \equiv 1$, we have

$$\langle \psi_0, \psi^*_0 \rangle = \int_{\mathbb{R}^N} \psi_0 \,\mathrm{d}y = \int_{\mathbb{R}^N} F(y)\,\mathrm{d}y = 1.$$

However, thanks to (3.14), we have that

$$\int_{\mathbb{R}^N} \psi_\beta \equiv \langle \psi_\beta, \psi^*_0 \rangle = 0 \quad \text{for any} \quad |\beta| \neq 0.$$

This expresses the orthogonality property to the adjoint eigenfunctions in terms of the dual inner product.

Note that [12], for the eigenfunctions $\{\psi_\beta\}$ of $\mathbf{B}$ denoted by (3.14), the corresponding adjoint eigenfunctions are *generalized Hermite polynomials* given by

$$\psi^*_\beta(y) := \tfrac{1}{\sqrt{\beta!}} \Big[y^\beta + \sum_{j=1}^{[|\beta|/4]} \tfrac{1}{j!}\, \Delta^{2j} y^\beta \Big]. \tag{3.18}$$

Hence, the orthonormality condition holds

$$\langle \psi_\beta, \psi_\gamma \rangle = \delta_{\beta\gamma} \quad \text{for any} \quad \beta,\ \gamma, \tag{3.19}$$

where $\langle \cdot, \cdot \rangle$ is the duality product in $L^2(\mathbb{R}^N)$ and $\delta_{\beta\gamma}$ is Kronecker's delta. Also, operators $\mathbf{B}$ and $\mathbf{B}^*$ have zero Morse index (no eigenvalues with positive real parts are available).

Key spectral results can be extended [12] to $2m$th-order linear poly-harmonic flows

$$u_t = -(-\Delta)^m u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \tag{3.20}$$

where the elliptic equation for the rescaled kernel $F(y)$ takes the form

$$\mathbf{B}F \equiv -(-\Delta_y)^m F + \tfrac{1}{2m}\, y \cdot \nabla_y F + \tfrac{N}{2m}\, F = 0 \quad \text{in} \quad \mathbb{R}^N, \quad \int\limits_{\mathbb{R}^N} F(y)\, {\rm d}y = 1. \tag{3.21}$$

In particular, for $m = 1$, we find the *Hermite operator* and the *Gaussian kernel* (see [8] for further information)

$$\mathbf{B}F \equiv \Delta F + \tfrac{1}{2}\, y \cdot \nabla F + \tfrac{N}{2}\, F = 0 \quad \Longrightarrow \quad F(y) = \tfrac{1}{(4\pi)^{N/2}}\, {\rm e}^{-\frac{|y|^2}{4}},$$

whose name is connected with fundamental works of Charles Hermite on orthogonal polynomials $\{H_\beta\}$ about 1870. These classic Hermite polynomials are obtained by differentiating the Gaussian: up to normalization constants,

$$D^\beta {\rm e}^{-\frac{|y|^2}{4}} = H_\beta(y)\, {\rm e}^{-\frac{|y|^2}{4}} \quad \text{for any} \ \ \beta. \tag{3.22}$$

Note that, for $N = 1$, such operators and polynomial eigenfunctions in 1D were studied earlier by Jacques C.F. Sturm in 1836; on this history and Sturm's main original calculations, see [18, Ch. 1].

The generating formula (3.22) for (generalized) Hermite polynomials is not available if $m \ge 2$, so that (3.18) are obtained via a different procedure, [12].

4. Similarity profiles for the Cauchy problem via n-branching

4.1. **Derivation of the branching equation.** In general, construction of oscillatory similarity solutions of the Cauchy problem for the TFE–4 (1.1) is a difficult nonlinear problem, which is harder than for the corresponding FBP one. On the other hand, for $n = 0$, such similarity profiles exist and are given by eigenfunctions $\{\psi_\beta\}$. In particular, the first mass-preserving profile is just the rescaled kernel $F(y)$, so it is unique, as was shown in Section 3. Hence, somehow, there can be expected a possibility to visualize such an oscillatory first "nonlinear eigenfunction" $f(y)$ of changing sign, which satisfies the *nonlinear eigenvalue problem* (2.7), at least, for sufficiently small $n > 0$. This assumes using the n-branching approach that "connects" f with the rescaled fundamental profile F satisfying the corresponding linear equation (3.3), with all the necessary properties of F presented in Section 3.

Thus, we plan to describe the behaviour of the similarity profiles $\{f_\beta\}$, as nonlinear eigenfunctions of (2.7) for the TFE performing a "homotopic" approach when $n \downarrow 0$. Homotopic approaches are well-known in the theory of vector fields, degree, and nonlinear operator theory (see [11, 32] for details). However, we shall be less precise in order to apply that approach, and here a "homotopic path" just declares existence of a continuous connection (a curve) of solutions $f \in C_0$ that ends up at $n = 0^+$ at the linear eigenfunction $\psi_0(y) = F(y)$ or further eigenfunctions $\psi_\beta(y) \sim D^\beta F(y)$, as (3.14) claims.

Using classical branching theory in the case of finite regularity of nonlinear operators involved, we formally show that the necessary orthogonality condition holds deriving the corresponding *Lyapunov–Schmidt branching equation*. We will try to be as much rigorous as possible in supporting of delivering the nonlinear eigenvalues $\{\alpha_k\}$. Further extensions of solutions for non-small $n > 0$ require a novel essentially non-local technique of such nonlinear analysis, which remains an open problem.

Those critical eigenvalues $\{\alpha_k\}$ are obtained according to non-self-adjoint spectral theory from Section 3. We then use the explicit expressions for the eigenvalues and eigenfunctions of the linear eigenvalue problem (3.5) given in Theorem 3.1, where we also need the main conclusions of the "adjoint" Theorem 3.2. Then, taking the corresponding linear equation from (2.7) with $n = 0$, we find, at least, formally, that

$$n = 0: \quad \mathcal{L}(\alpha)f := -\Delta^2 f + \tfrac{1}{4}\, y \cdot \nabla f + \alpha f = 0.$$

From that equation combined with the eigenvalues expressions obtained in the previous section, we ascertain the following critical values for the parameter $\alpha_k = \alpha_k(n)$,

$$n = 0: \quad \alpha_k(0) := -\lambda_k + \tfrac{N}{4} \equiv \tfrac{k+N}{4} \quad \text{for any} \quad k = 0, 1, 2, \ldots, \tag{4.1}$$

where λ_k are the eigenvalues defined in Theorem 3.1, so that

$$\alpha_0(0) = \tfrac{N}{4},\ \alpha_1(0) = \tfrac{N+1}{4},\ \alpha_2(0) = \tfrac{N+2}{4}, \ldots, \alpha_k(0) = \tfrac{k+N}{4} \ldots .$$

In particular, when $k = 0$, we have that $\alpha_0(0) = \frac{N}{4}$ and the eigenfunction satisfies

$$\mathbf{B}F = 0, \quad \text{so that} \quad \ker \mathcal{L}(\alpha_0) = \operatorname{span}\{\psi_0\} \quad (\psi_0 = F),$$

and, hence, since $\lambda_0 = 0$ is a simple eigenvalue for the operator $\mathcal{L}(\alpha_0) = \mathbf{B}$, its algebraic multiplicity is 1. In general, we find that

$$\ker\big(\mathbf{B} + \tfrac{k}{4}\, I\big) = \operatorname{span}\{\psi_\beta,\ |\beta| = k\}, \quad \text{for any} \quad k = 0, 1, 2, 3, \cdots, \tag{4.2}$$

where the operator $\mathbf{B} + \frac{k}{4}\, I$ is Fredholm of index zero since it is a compact perturbation of the identity of linear type with respect to k. In other words, $R[\mathcal{L}(\alpha_k)]$ is a closed subspace of $L^2_\rho(\mathbb{R}^N)$ and, for each α_k,

$$\dim \ker(\mathcal{L}(\alpha_k)) < \infty \quad \text{and} \quad \operatorname{codim} R[\mathcal{L}(\alpha_k)] < \infty.$$

Then, for small $n > 0$ in (2.7), we can use the asymptotic expansions

$$\alpha_k(n) := \alpha_k + \mu_{1,k} n + o(n), \quad \text{and} \tag{4.3}$$

$$|f|^n \equiv \mathrm{e}^{n \ln |f|} := 1 + n \ln |f| + o(n). \tag{4.4}$$

As customary in bifurcation-branching theory [32, 43], existence of an expansion such as (4.3) will allow one to get further expansion coefficients in

$$\alpha_k(n) := \alpha_k + \mu_{1,k} n + \mu_{2,k} n^2 + \mu_{3,k} n^3 + \ldots, \tag{4.5}$$

as the regularity of nonlinearities allows and suggests, though the convergence of such an analytic series can be questionable and is not under scrutiny here.

Another principle question is that, for oscillatory sign changing profiles $f(y)$, the last expansion (4.4) cannot be understood in the pointwise sense, but can be naturally expected to be valid in other metrics such as weighted L^2 or Sobolev spaces as in Section 3 that used to be appropriate for the functional setting of the equivalent integral equation and for that with $n = 0$. Since (4.4) is obviously pointwise violated at the nodal set $\{f = 0\}$ of $f(y)$, this imposes some restrictions on the behaviour of corresponding eigenfunctions $\psi_\beta(y)$ ($n = 0$) close to their zero sets. Using well-known asymptotic and other related properties of the *radial* analytic rescaled kernel $F(y)$ of the fundamental solutions (3.2), the generating formula of eigenfunctions (3.14) confirms that the nodal set of analytic eigenfunctions $\{\psi_\beta = 0\}$ consists of isolated zero surfaces, which are "transversal", at least in the a.e. sense, with the only accumulation point at $y = \infty$. Overall, under such conditions, this indicates that

(4.6) expansion (4.4) contains not more than "logarithmic" singularities a.e.,

which well suited the integral compact operators involved into a branching analysis, though we are far away to claim this as any rigorous issue. Moreover, when $n > 0$ is not small enough, such an analogy and statements like (4.6) become not that clear, and global extensions of continuous n-branches induced by some compact integral operators, i.e., nonexistence of turning (saddle-node) points in n, require, as usual, some unknown monotonicity-like results.

Now, we assume the expansion (4.4) away from possible zero surfaces of $f(y)$, which, by transversality, can be localized in arbitrarily small neighbourhoods. Indeed, it is clear that when $|f| > \delta > 0$, for any $\delta > 0$, there is no problem in approximating $|f|^n$ by (4.3), i.e., $|f|^n = 1 + O(n)$ as $n \to 0^+$. However, when $|f| \le \delta$, for any $\delta \ge 0$ sufficiently small, the proof of such an approximation in weak topology (as suffices for dealing with equivalent integral equations) is far from clear unless the zeros of the f's are also transversal a.e. with a standard accumulating property at the only interface zero surface. The latter issues have been studied and described in [14] in the radial setting. Hence, we can suppose that such nonlinear eigenfunctions $f(y)$ are oscillatory and infinitely sign changing close to the interface surface. Therefore, if we assume that their zero surface are transversal a.e. with a known geometric-like accumulation at the interface, we find that, for any n close to zero and any $\delta = \delta(n) > 0$ sufficiently small,

$$n|\ln|f|| \gg 1, \quad \text{if} \quad |f| \le \delta(n),$$

and, hence, on such subsets, $f(y)$ must be exponentially small:

$$|\ln|f|| \gg \tfrac{1}{n} \implies \ln|f| \ll -\tfrac{1}{n} \implies |f| \ll \mathrm{e}^{-\frac{1}{n}}.$$

Recall that this happens in also exponentially small neighbourhoods of the transversal zero surfaces.

Overall, using the periodic structure of the oscillatory component at the interface [14] (we must admit that such delicate properties of oscillatory structures of solutions are known for the 1D and radial cases only, though we expect that these phenomena are generic), we can control the singular coefficients in (4.3), and, in particular, to see that

$$\ln|f| \in L^1_{\rm loc}(\mathbb{R}^N). \tag{4.7}$$

However, for most general geometric configurations of nonlinear eigenfunctions $f(y)$, we do not have a proper proof of (4.7) or similar estimates, so our further analysis is still essentially formal. It is worth recalling again that our computations below are to be understood as those dealing with the equivalent integral equations and operators, so, in particular, we can use the powerful facts on compactness of the resolvent $(\mathbf{B}-\lambda I)^{-1}$ and of the adjoint one $(\mathbf{B}^*-\lambda I)^{-1}$ in the corresponding weighted L^2-spaces. Note that, in such an equivalent integral representation, the singular term in (4.4) satisfying (4.7) makes no principal difficulty, so the expansion (4.4) makes rather usual sense for applying standard nonlinear operator theory.

Thus, under natural assumptions, substituting (4.3) into (2.7), for any $k=0,1,2,3,\cdots$, we find that, omitting $o(n)$ terms when necessary,

$$-\nabla\cdot[(1+n\ln|f|)\nabla\Delta f]+\tfrac{1-\alpha_k n-\mu_{1,k}n^2}{4}\,y\cdot\nabla f+(\alpha_k+\mu_{1,k}n)f=0\,,$$

and, rearranging terms,

$$-\Delta^2 f-n\nabla\cdot(\ln|f|\nabla\Delta f)+\tfrac14\,y\cdot\nabla f-\tfrac{\alpha_k n+\mu_{1,k}n^2}{4}\,y\cdot\nabla f+\alpha_k f+\mu_{1,k}nf=0\,.$$

Hence, we finally have

$$\big(\mathbf{B}+\tfrac{k}{4}\,I\big)f+n\big[-\nabla\cdot(\ln|f|\nabla\Delta f)-\tfrac{\alpha_k}{4}\,y\cdot\nabla f+\mu_{1,k}f\big]+o(n)=0\,, \tag{4.8}$$

which can be written in the following form:

$$\big(\mathbf{B}+\tfrac{k}{4}\,I\big)f+n\mathcal{N}_k(f)+o(n)=0\,, \tag{4.9}$$

with the operator

$$\mathcal{N}_k(f):=-\nabla\cdot(\ln|f|\nabla\Delta f)-\tfrac{\alpha_k}{4}\,y\cdot\nabla f+\mu_{1,k}f\,. \tag{4.10}$$

Subsequently, as was shown in Section 3, we have that

$$\ker\big(\mathbf{B}+\tfrac{k}{4}\,I\big)={\rm span}\,\{\psi_\beta\}_{|\beta|=k}\quad\text{for any}\quad k=0,1,2,3,\cdots, \tag{4.11}$$

where the operator $\mathbf{B}+\frac{k}{4}\,I$ is Fredholm of index zero and

$$\dim\ker\big(\mathbf{B}+\tfrac{k}{4}\,I\big)=M_k\ge 1\quad\text{for any}\quad k=0,1,2,3,\cdots,$$

where M_k stands for the length of the vector $\{D^\beta v,\,|\beta|=k\}$, so that $M_k>1$ for $k\ge 1$.

<u>Simple eigenvalue for $k=0$</u>. Since 0 is a simple eigenvalue of $\mathbf{B}$ when $k=0$, i.e.,

$$\ker\,\mathbf{B}\oplus R[\mathbf{B}]=L^2_\rho(\mathbb{R}^N),$$

the study of the case $k=0$ seems to be simpler than for other different k's because the dimension of the eigenspace is $M_0=1$. Thus, we shall describe the behaviour of solutions for small $n>0$ and apply the classical Lyapunov–Schmidt method to (4.9) (assuming as usual some extra necessary regularity hypothesis to be clarified later on), in order to accomplish the branching approach as $n \downarrow 0$, in two steps, when $k=0$ and k is different from 0.

Thus, owing to Section 3, we have already known that 0 is a simple eigenvalue of $\mathbf{B}$, i.e., $\ker \mathbf{B} = \text{span}\,\{\psi_0\}$ is one-dimensional. Hence, denoting by Y_0 the complementary invariant subspace, orthogonal to ψ_0^*, we set

$$f = \psi_0 + V_0,$$

where $V_0 \in Y_0$. According to the already well known spectral properties of the operator $\mathbf{B}$, we define P_0 and P_1 such that $P_0 + P_1 = I$, to be the projections onto $\ker \mathbf{B}$ and Y_0 respectively. Finally, setting

$$V_0 := n\Phi_{1,0} + o(n), \tag{4.12}$$

substituting the expression for f into (4.9) and passing to the limit as $n \to 0^+$ leads to a linear inhomogeneous equation for $\Phi_{1,0}$,

$$\mathbf{B}\Phi_{1,0} = -\mathcal{N}_0(\psi_0), \tag{4.13}$$

since $\mathbf{B}\psi_0 = 0$. Moreover, by Fredholm theory, $V_0 \in Y_0$ exists if and only if the right-hand side is orthogonal to the one dimensional kernel of the adjoint operator $\mathbf{B}^*$ with $\psi_0^* = 1$, because of (3.18). Hence, in the topology of the dual space L^2, this requires the standard orthogonality condition:

$$\big\langle \mathcal{N}_0(\psi_0), 1 \big\rangle = 0. \tag{4.14}$$

Then, (4.13) has a unique solution $\Phi_{1,0} \in Y_0$ determining by (4.12) a bifurcation branch for small $n > 0$. In fact, the algebraic equation (4.14) yields the following explicit expression for the coefficient $\mu_{1,0}$ of the expansion (4.3) of the first eigenvalue $\alpha_0(n)$:

$$\mu_{1,0} := \tfrac{\langle \nabla\cdot(\ln|\psi_0|\nabla\Delta\psi_0) + \frac{N}{16}\, y\cdot\nabla\psi_0, \psi_0^* \rangle}{\langle \psi_0, \psi_0^* \rangle} = \langle \nabla\cdot(\ln|\psi_0|\nabla\Delta\psi_0) + \tfrac{N}{16}\, y\cdot\nabla\psi_0, \psi_0^* \rangle. \tag{4.15}$$

<u>Multiple eigenvalues for $k \geq 1$</u>. For any $k \geq 1$, we know that

$$\dim\ker\big(\mathbf{B} + \tfrac{k}{4}\, I\big) = M_k > 1.$$

Hence, we have to use the full eigenspace expansion

$$f = \sum_{|\beta|=k} c_\beta \hat{\psi}_\beta + V_k, \tag{4.16}$$

for every $k \geq 1$. Currently, for convenience, we denote $\{\hat{\psi}_\beta\}_{|\beta|=k} = \{\hat{\psi}_1, ..., \hat{\psi}_{M_k}\}$ the natural basis of the M_k-dimensional eigenspace $\ker\big(\mathbf{B} + \frac{k}{4}\, I\big)$ and set $\psi_k = \sum_{|\beta|=k} c_\beta \hat{\psi}_\beta$. Moreover, $V_k \in Y_k$ and $V_k = \sum_{|\beta|>k} c_\beta \psi_\beta$, where Y_k is the complementary invariant subspace of $\ker\big(\mathbf{B} + \frac{k}{4}\, I\big)$. Furthermore, in the same way, as we did for the case $k=0$, we define the $P_{0,k}$ and $P_{1,k}$, for every $k \geq 1$, to be the projections of $\ker\big(\mathbf{B} + \frac{k}{4}\, I\big)$ and Y_k respectively. We also expand V_k as

$$V_k := n\Phi_{1,k} + o(n). \tag{4.17}$$

Subsequently, substituting (4.16) into (4.9) and passing to the limit as $n \downarrow 0^+$, we obtain the following equation:

$$\big(\mathbf{B} + \tfrac{k}{4}\, I\big)\Phi_{1,k} = -\mathcal{N}_k\big(\textstyle\sum_{|\beta|=k} c_\beta \psi_\beta\big), \tag{4.18}$$

under the natural "normalizing" constraint

$$\sum_{|\beta|=k} c_\beta = 1 \quad (c_\beta \ge 0). \tag{4.19}$$

Therefore, applying the Fredholm alternative, $V_k \in Y_k$ exists if and only if the term on the right-hand side of (4.18) is orthogonal to ker $\big(\mathbf{B} + \frac{k}{4}\, I\big)$. Multiplying the right-hand side of (4.18) by ψ_β^*, for every $|\beta| = k$, in the topology of the dual space L^2, we obtain an algebraic system of $M_k + 1$ equations and the same number of unknowns, $\{c_\beta,\, |\beta| = k\}$ and $\mu_{1,k}$:

$$\big\langle \mathcal{N}_k(\textstyle\sum_{|\beta|=k} c_\beta \psi_\beta), \psi_\beta^* \big\rangle = 0 \quad \text{for all} \quad |\beta| = k, \tag{4.20}$$

which is indeed the Lyapunov–Schmidt branching equation [43]. In general, such algebraic system are assumed to allow us to obtain the branching parameters and hence establish the number of different solutions induced on the given M_k-dimensional eigenspace as the kernel of the operator involved.

However, we must admit and urge that the algebraic system (4.20) is a truly difficult issue. One of the main features of it is as follows:

$$\boxed{\text{(4.20) is not variational.}} \tag{4.21}$$

In other words, one cannot use for (4.20) the classic category-genus theory of calculus of variation [4, 32], to claim that the category of the kernel (equal to M_k) is the least number of different critical points and hence of different solutions.

To see (4.21), it suffices to note that, due to (3.14) and (3.18), the generalized Hermite polynomials ψ_β^* have nothing common in the algebraic sense with the eigenfunctions ψ_β in the L^2-scalar products in (4.20).

4.2. **A digression to Hermite classic self-adjoint theory.** It is worth mentioning that, for classic second-order Hermite operator

$$\mathbf{B} = \Delta + \tfrac{1}{2}\, y \cdot \nabla + \tfrac{N}{2}\, I \quad \big(\text{then, in the } L^2\text{-metric,} \quad \mathbf{B}^* = \Delta - \tfrac{1}{2}\, y \cdot \nabla\big), \tag{4.22}$$

(4.21) is not the case. Indeed, by classic self-adjoint theory [8, p. 48], these eigenfunctions are related to each other by

$$\psi_\beta(y) = D^\beta F(y) \equiv H_\beta(y)\, F(y), \quad \text{where} \quad F(y) = (4\pi)^{-\frac{N}{2}}\, \mathrm{e}^{-\frac{|y|^2}{4}} \tag{4.23}$$

is the Gaussian kernel and $H_\beta(y)$ are standard Hermite polynomials, which also define the adjoint eigenfunctions:

$$\psi_\beta^*(y) = b_\beta H_\beta(y) \equiv \tfrac{b_\beta}{F(y)}\, \psi_\beta(y), \tag{4.24}$$

where b_β are normalization constants. One knows that this is a result of the symmetry of the operator (4.22) in the weighted metric of $L^2_\rho(\mathbb{R}^N)$, where

$$\rho(y) = \mathrm{e}^{\frac{|y|^2}{4}} \sim \tfrac{1}{F(y)} \quad \Longrightarrow \quad \mathbf{B} = \tfrac{1}{\rho}\, \nabla \cdot (\rho \nabla) + \tfrac{N}{2}\, I, \quad \text{so} \quad (\mathbf{B})^*_{L^2_\rho} = \mathbf{B}.$$

In view of the relations (4.23) and (4.24) of the bi-orthonormal bases $\{\psi_\beta\}$ and $\{\psi^*_\beta\}$, the corresponding algebraic systems such as (4.20) can be variational. Moreover, even the original nonlinear elliptic equation similar to (2.7), where the 4th-order operator is replaced by a natural 2nd-order one of the porous medium type:

$$-\nabla(|f|^n \nabla \Delta f) \mapsto \nabla(|f|^n \nabla f),$$

then becomes variational itself. Thus, in this case, both branching (local phenomena) and global extensions of n-bifurcation branches can be performed on the basis of powerful Lusternik–Schnirel'man category variational theory from 1920s [32, § 56], so that existence and multiplicity (at least, not less than in the linear case $n = 0$) of solutions are guaranteed.

4.3. **Computations for branching of dipole solutions in 2D.** To avoid excessive computations and as a self-contained example, we now ascertain some expressions for those coefficients in the case when $|\beta| = 1$, $N = 2$, and $M_1 = 2$, so that, in our notations, $\{\psi_\beta\}_{|\beta|=1} = \{\hat\psi_1, \hat\psi_2\}$. Consequently, in this case, we obtain the following algebraic system: the expansion coefficients of $\psi_1 = c_1 \hat\psi_1 + c_2 \hat\psi_2$ satisfy

$$(4.25) \quad \begin{cases} c_1 \langle \hat\psi^*_1, h_1 \rangle - \frac{c_1 \alpha_1}{4} \langle \hat\psi^*_1, y \cdot \nabla \hat\psi_1 \rangle + c_1 \mu_{1,1} + c_2 \langle \hat\psi^*_1, h_2 \rangle - \frac{c_2 \alpha_1}{4} \langle \hat\psi^*_1, y \cdot \nabla \hat\psi_2 \rangle = 0, \\ c_1 \langle \hat\psi^*_2, h_1 \rangle - \frac{c_1 \alpha_1}{4} \langle \hat\psi^*_2, y \cdot \nabla \hat\psi_1 \rangle + c_2 \langle \hat\psi^*_2, h_2 \rangle - \frac{c_2 \alpha_1}{4} \langle \hat\psi^*_2, y \cdot \nabla \hat\psi_2 \rangle + c_2 \mu_{1,1} = 0, \\ c_1 + c_2 = 1, \end{cases}$$

where

$$h_1 := -\nabla \cdot [\ln(c_1 \hat\psi_1 + c_2 \hat\psi_2) \nabla \Delta \hat\psi_1], \;\; h_2 := -\nabla \cdot [\ln(c_1 \hat\psi_1 + c_2 \hat\psi_2) \nabla \Delta \hat\psi_2],$$

and, c_1, c_2, and $\mu_{1,1}$ are the coefficients that we want to calculate, α_1 is regarded as the value of the parameter α denoted by (4.1) and dependent on the eigenvalue λ_1, for which $\hat\psi_{1,2}$ are the associated eigenfunctions, and $\hat\psi^*_{1,2}$ the corresponding adjoint eigenfunctions. Hence, substituting the expression $c_2 = 1 - c_1$ from third equation into the other two, we have the following nonlinear algebraic system

$$(4.26) \quad \begin{cases} 0 = N_1(c_1, \mu_{1,1}) - c_1 \frac{\alpha_1}{4} \left[\langle \hat\psi^*_1, y \cdot \nabla \hat\psi_1 \rangle - \langle \hat\psi^*_1, y \cdot \nabla \hat\psi_2 \rangle \right], \\ 0 = N_2(c_1, \mu_{1,1}) - c_1 \frac{\alpha_1}{4} \left[\langle \hat\psi^*_2, y \cdot \nabla \hat\psi_1 \rangle - \langle \hat\psi^*_2, y \cdot \nabla \hat\psi_2 \rangle \right] + \mu_{1,1}, \end{cases}$$

where

$$N_1(c_1, \mu_{1,1}) := c_1 \langle \hat\psi^*_1, h_1 \rangle + \langle \hat\psi^*_1, h_2 \rangle - \tfrac{\alpha_1}{4} \langle \hat\psi^*_1, y \cdot \nabla \hat\psi_2 \rangle - c_1 \langle \hat\psi^*_1, h_2 \rangle + c_1 \mu_{1,1},$$

$$N_2(c_1, \mu_{1,1}) := c_1 \langle \hat\psi^*_2, h_1 \rangle + \langle \hat\psi^*_2, h_2 \rangle - \tfrac{\alpha_1}{4} \langle \hat\psi^*_2, y \cdot \nabla \hat\psi_2 \rangle - c_1 \langle \hat\psi^*_2, h_2 \rangle - c_1 \mu_{1,1}$$

represent the nonlinear parts of the algebraic system, with h_0 and h_1 depending on c_1.

Subsequently, to guarantee existence of solutions of the system (4.25), we apply the Brouwer Fixed Point Theorem to (4.26) by supposing that the values c_1 and $\mu_{1,1}$ are the unknowns, in a disc sufficiently big $D_R(\hat{c}_1,\hat{\mu}_{1,1})$ centered in a possible nondegenerate zero $(\hat{c}_1,\hat{\mu}_{1,1})$. Thus, we write the system (4.26) in the matrix form

$$\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}-\frac{\alpha_1}{4}\left[\langle\hat{\psi}_1^*,y\cdot\nabla\hat{\psi}_1\rangle-\langle\hat{\psi}_1^*,y\cdot\nabla\hat{\psi}_2\rangle\right] & 0\\ -\frac{\alpha_1}{4}\left[\langle\hat{\psi}_2^*,y\cdot\nabla\hat{\psi}_1\rangle-\langle\hat{\psi}_2^*,y\cdot\nabla\hat{\psi}_2\rangle\right] & 1\end{pmatrix}\begin{pmatrix}c_1\\ \mu_{1,1}\end{pmatrix}+\begin{pmatrix}N_1(c_1,\mu_{1,1})\\ N_2(c_1,\mu_{1,1})\end{pmatrix}.$$

Hence, we have that the zeros of the operator

$$\mathcal{F}(c_1,\mu_{1,1}):=\mathfrak{M}\begin{pmatrix}c_1\\ \mu_{1,1}\end{pmatrix}+\begin{pmatrix}N_1(c_1,\mu_{1,1})\\ N_2(c_1,\mu_{1,1})\end{pmatrix}$$

are the possible solutions of (4.26), where $\mathfrak{M}$ is the matrix corresponding to the linear part of the system, while

$$(N_1(c_1,\mu_{1,1}),N_2(c_1,\mu_{1,1}))^T,$$

corresponds to the nonlinear part. The application $\mathcal{H}:\mathcal{A}\times[0,1]\to\mathbb{R}$, defined by

$$\mathcal{H}(c_1,\mu_{1,1},t):=\mathfrak{M}\begin{pmatrix}c_1\\ \mu_{1,1}\end{pmatrix}+t\begin{pmatrix}N_1(c_1,\mu_{1,1})\\ N_2(c_1,\mu_{1,1})\end{pmatrix},$$

provides us with a homotopy transformation from the function $\mathcal{F}(c_1,\mu_{1,1})=\mathcal{H}(c_1,\mu_{1,1},1)$ to its linearization

$$\mathcal{H}(c_1,\mu_{1,1},0):=\mathfrak{M}\begin{pmatrix}c_1\\ \mu_{1,1}\end{pmatrix}. \tag{4.27}$$

Thus, the system (4.26) possesses a nontrivial solution if (4.27) has a nondegenerate zero, in other words, if the next condition is satisfied

$$\langle\hat{\psi}_1^*,y\cdot\nabla\hat{\psi}_1\rangle-\langle\hat{\psi}_1^*,y\cdot\nabla\hat{\psi}_2\rangle\neq 0. \tag{4.28}$$

Note that, if the substitution would have been $c_1=1-c_2$, the condition might also be

$$\langle\hat{\psi}_2^*,y\cdot\nabla\hat{\psi}_2\rangle-\langle\hat{\psi}_2^*,y\cdot\nabla\hat{\psi}_1\rangle\neq 0.$$

Then, under condition (4.28), the system (4.26) can be written in the form

$$\begin{pmatrix}c_1-\hat{c}_1\\ \mu_{1,1}-\hat{\mu}_{1,1}\end{pmatrix}=-\mathcal{M}^{-1}\begin{pmatrix}N_1(c_1,\mu_{1,1})-\hat{c}_1\\ N_2(c_1,\mu_{1,1})-\hat{\mu}_{1,1}\end{pmatrix}, \tag{4.29}$$

which can be interpreted as a fixed point equation. Moreover, applying Brouwer's Fixed Point Theorem, we have that

$$\begin{aligned}\mathrm{Ind}((\hat{c}_1,\hat{\mu}_{1,1}),\mathcal{H}(\cdot,\cdot,0))&=\mathcal{Q}_{C_R(\hat{c}_1,\hat{\mu}_{1,1})}(\mathcal{H}(\cdot,\cdot,0))\\ &=\mathrm{Deg}(\mathcal{H}(\cdot,\cdot,0),D_R(\hat{c}_1,\hat{\mu}_{1,1}))\\ &=\mathrm{Deg}(\mathcal{F}(c_1,\mu_{1,1}),D_R(\hat{c}_1,\hat{\mu}_{1,1})),\end{aligned}$$

where $\mathcal{Q}_{C_R(\hat{c}_1,\hat{\mu}_{1,1})}(\mathcal{H}(\cdot,\cdot,0))$ defines the number of rotations of the function $\mathcal{H}(\cdot,\cdot,0)$ around the curve $C_R(\hat{c}_1,\hat{\mu}_{1,1})$ and $\mathrm{Deg}(\mathcal{H}(\cdot,\cdot,0),D_R(\hat{c}_1,\hat{\mu}_{1,1}))$ denotes the topological degree of $\mathcal{H}(\cdot,\cdot,0)$ in $D_R(\hat{c}_1,\hat{\mu}_{1,1})$. Owing to classical topological methods, both are equal.

Thus, once we have proved the existence of solutions, we achieve some expressions for the coefficients required:

$$\begin{cases} \mu_{1,1} = c_2(\langle \hat{\psi}_1^* + \hat{\psi}_2^*, h_1 - h_2 \rangle - \frac{\alpha_1}{4} \langle \hat{\psi}_1^* + \psi_2^*, y \cdot \nabla \hat{\psi}_1 - y \cdot \nabla \hat{\psi}_2 \rangle) \\ \qquad\quad - \langle \hat{\psi}_1^* + \hat{\psi}_2^*, h_1 \rangle + \frac{\alpha_1}{4} \langle \hat{\psi}_1^* + \hat{\psi}_2^*, y \cdot \nabla \hat{\psi}_1 \rangle, \\ c_1 = 1 - c_2. \end{cases}$$

The expressions for the coefficients in a general case might be accomplished after some tedious calculations, otherwise similar to those performed above. Note that, in general, those nonlinear finite-dimensional algebraic problems are rather complicated, and the problem of an optimal estimate of the number of different solutions remains open. Moreover, reliable multiplicity results are very difficult to obtain. We expect that this number should be somehow related (and even sometimes coincides) with the dimension of the corresponding eigenspace of the linear operators $\mathbf{B} + \frac{k}{4} I$, for any $k = 0, 1, 2, \ldots$. This is a conjecture only that may be too illusive; see further supportive analysis presented below.

However, we devote the remaining of this section to a possible answer to that conjecture, which is not totally complete though, since we are imposing some conditions.

Thus, in order to detect the number of solutions of the nonlinear algebraic system (4.25), we proceed to reduce this system to a single equation for one of the unknowns. As a first step, integrating by parts in the terms in which h_1 and h_2 are involved and rearranging terms in the first two equations of the system (4.25), we arrive at

$$\begin{cases} \int\limits_{\mathbb{R}^N} \nabla \psi_1^* \cdot \ln(c_1 \hat{\psi}_1 + c_2 \hat{\psi}_2) \nabla \Delta (c_1 \hat{\psi}_1 + c_2 \hat{\psi}_2) \\ -c_1 \frac{\alpha_1}{4} \int\limits_{\mathbb{R}^N} \hat{\psi}_1^* y \cdot \nabla \hat{\psi}_1 + c_1 \mu_{1,1} - c_2 \frac{\alpha_1}{4} \int\limits_{\mathbb{R}^N} \hat{\psi}_1^* y \cdot \nabla \hat{\psi}_2 = 0, \\ \int\limits_{\mathbb{R}^N} \nabla \hat{\psi}_2^* \cdot \ln(c_1 \hat{\psi}_1 + c_2 \hat{\psi}_2) \nabla \Delta (c_2 \hat{\psi}_1 + c_2 \hat{\psi}_2) \\ -c_1 \frac{\alpha_1}{4} \int\limits_{\mathbb{R}^N} \hat{\psi}_2^* y \cdot \nabla \hat{\psi}_1 + c_2 \mu_{1,1} - c_2 \frac{\alpha_1}{4} \int\limits_{\mathbb{R}^N} \hat{\psi}_2^* y \cdot \nabla \hat{\psi}_2 = 0. \end{cases}$$

By the third equation, we have that $c_1 = 1 - c_2$, and hence, setting $c_1 \hat{\psi}_1 + c_2 \hat{\psi}_2 = \hat{\psi}_1 + (\hat{\psi}_2 - \hat{\psi}_1) c_2$ and substituting these into those new expressions for the first two equations of the system, we find that

$$(4.30) \qquad \begin{cases} \int\limits_{\mathbb{R}^N} \nabla \hat{\psi}_1^* \cdot \ln(\hat{\psi}_1 + (\hat{\psi}_2 - \hat{\psi}_1) c_2) \nabla \Delta (\hat{\psi}_1 + (\hat{\psi}_2 - \hat{\psi}_1) c_2) + \mu_{1,1} - c_2 \mu_{1,1} \\ -\frac{\alpha_1}{4} \int\limits_{\mathbb{R}^N} \hat{\psi}_1^* y \cdot \nabla \hat{\psi}_1 + c_2 \frac{\alpha_1}{4} \int\limits_{\mathbb{R}^N} \hat{\psi}_1^* y \cdot (\nabla \hat{\psi}_1 - \nabla \hat{\psi}_2) = 0, \\ \int\limits_{\mathbb{R}^N} \nabla \hat{\psi}_2^* \cdot \ln(\hat{\psi}_1 + (\hat{\psi}_2 - \hat{\psi}_1) c_2) \nabla \Delta (\hat{\psi}_1 + (\hat{\psi}_2 - \hat{\psi}_1) c_2) + c_2 \mu_{1,1} \\ -\frac{\alpha_1}{4} \int\limits_{\mathbb{R}^N} \hat{\psi}_2^* y \cdot \nabla \hat{\psi}_1 + c_2 \frac{\alpha_1}{4} \int\limits_{\mathbb{R}^N} \hat{\psi}_2^* y \cdot (\nabla \hat{\psi}_1 - \nabla \hat{\psi}_2) = 0. \end{cases}$$

Subsequently, adding both equations, we have that

$$\mu_{1,1} = -\int\limits_{\mathbb{R}^N} (\nabla\hat{\psi}_1^* + \nabla\hat{\psi}_2^*) \cdot \ln(\hat{\psi}_1 + (\hat{\psi}_2 - \hat{\psi}_1)c_2)\nabla\Delta(\hat{\psi}_1 + (\hat{\psi}_2 - \hat{\psi}_1)c_2)$$
$$+\tfrac{\alpha_1}{4} \int\limits_{\mathbb{R}^N} (\psi_1^* + \psi_2^*)y \cdot \nabla\hat{\psi}_1 - c_2\tfrac{\alpha_1}{4} \int\limits_{\mathbb{R}^N} (\hat{\psi}_1^* + \hat{\psi}_2^*)y \cdot (\nabla\hat{\psi}_2 - \nabla\hat{\psi}_1).$$

Thus, substituting it into the second equation of (4.30), we obtain the following equation with the single unknown c_2:

(4.31)

$$-c_2^2\tfrac{\alpha_1}{4} \int\limits_{\mathbb{R}^N} (\hat{\psi}_1^* + \hat{\psi}_2^*)y \cdot (\nabla\hat{\psi}_2 - \nabla\hat{\psi}_1) + c_2\tfrac{\alpha_1}{4}(\int\limits_{\mathbb{R}^N} (\hat{\psi}_1^* + 2\hat{\psi}_2^*)y \cdot \nabla\hat{\psi}_1 - \int\limits_{\mathbb{R}^N} \hat{\psi}_2^* y \cdot \nabla\hat{\psi}_2)$$
$$-\tfrac{\alpha_1}{4} \int\limits_{\mathbb{R}^N} \hat{\psi}_2^* y \cdot \nabla\hat{\psi}_1 + \int\limits_{\mathbb{R}^N} \nabla\psi_2^* \cdot \ln(\hat{\psi}_1 + (\hat{\psi}_2 - \hat{\psi}_1)c_2)\nabla\Delta(\hat{\psi}_1 + (\hat{\psi}_2 - \hat{\psi}_1)c_2)$$
$$-c_2 \int\limits_{\mathbb{R}^N} (\nabla\hat{\psi}_1^* + \nabla\hat{\psi}_2^*) \cdot \ln(\hat{\psi}_1 + (\hat{\psi}_2 - \hat{\psi}_1)c_2)\nabla\Delta(\hat{\psi}_1 + (\hat{\psi}_2 - \hat{\psi}_1)c_2) = 0,$$

which can be written in the following way:

$$c_2^2 A + c_2 B + C + \omega(c_2) \equiv \mathfrak{F}(c_2) + \omega(c_2) = 0. \tag{4.32}$$

Here, $\omega(c_2)$ can be considered as perturbation of the quadratic form $\mathfrak{F}(c_2)$ with the coefficients defined by

$$A := -\tfrac{\alpha_1}{4} \int\limits_{\mathbb{R}^N} (\hat{\psi}_1^* + \hat{\psi}_2^*)y \cdot (\nabla\hat{\psi}_2 - \nabla\hat{\psi}_1),$$
$$B := \tfrac{\alpha_1}{4}(\int\limits_{\mathbb{R}^N} (\hat{\psi}_1^* + 2\hat{\psi}_2^*)y \cdot \nabla\hat{\psi}_1 - \int\limits_{\mathbb{R}^N} \hat{\psi}_2^* y \cdot \nabla\hat{\psi}_2),, \quad C := -\tfrac{\alpha_1}{4} \int\limits_{\mathbb{R}^N} \hat{\psi}_2^* y \cdot \nabla\hat{\psi}_1,$$
$$\omega(c_2) := \int\limits_{\mathbb{R}^N} \nabla\hat{\psi}_2^* \cdot \ln(\hat{\psi}_1 + (\hat{\psi}_2 - \hat{\psi}_1)c_2)\nabla\Delta(\hat{\psi}_1 + (\hat{\psi}_2 - \hat{\psi}_1)c_2)$$
$$-c_2 \int\limits_{\mathbb{R}^N} (\nabla\hat{\psi}_1^* + \nabla\hat{\psi}_2^*) \cdot \ln(\hat{\psi}_1 + (\hat{\psi}_2 - \hat{\psi}_1)c_2)\nabla\Delta(\hat{\psi}_1 + (\hat{\psi}_2 - \hat{\psi}_1)c_2).$$

Since, due to the normalizing constraint (4.19), $c_2 \in [0, 1]$, solving the quadratic equation $\mathfrak{F}(c_2)$ yields:

(i) $c_2 = 0 \Longrightarrow \mathfrak{F}(0) = C$;
(ii) $c_2 = 1 \Longrightarrow \mathfrak{F}(1) = A + B + C$; and
(iii) differentiating $\mathfrak{F}$ with respect to c_2, we obtain that $\mathfrak{F}'(c_2) = 2c_2 A + B$. Then, the critical point of the function $\mathfrak{F}$ is $c_2^* = -\frac{B}{2A}$ and its image is $\mathfrak{F}(c_2^*) = -\frac{B}{4A} + C$.

Consequently, the conditions that must be imposed in order to have more than one solution (we already know the existence of at least one solution) are as follows:

(a) $C(A + B + C) > 0$;
(b) $C\big(-\frac{B}{4A} + C\big) < 0$; and
(c) $0 < -\frac{B}{2A} < 1$.

Note that, for $-\frac{B}{4A} + C = 0$, we have just a single solution.

Hence, considering the equation again in the form

$$\mathfrak{F}(c_2) + \omega(c_2) = 0,$$

where $\omega(c_2)$ is a perturbation of the quadratic form $\mathfrak{F}(c_2)$, and bearing in mind that the objective is to detect the number of solutions of the system (4.25), we need to control somehow this perturbation. Under the conditions (a), (b), and (c), $\mathfrak{F}(c_2)$ possesses exactly two solutions. Therefore, controlling the possible oscillations of the perturbation $\omega(c_2)$ in such a way that

$$\|\omega(c_2)\|_{L^\infty} \le \mathfrak{F}(c_2^*),$$

we can assure that the number of solutions for (4.25) is exactly two. This is the dimension of the kernel of the operator $\mathbf{B} + \frac{1}{4} I$ (as we expected in our more general conjecture).

The above particular example shows how difficult are the questions on existence and multiplicity of solutions for such non-variational branching problems. Recall that the actual values of the coefficients A, B, C, and others, which the number of solutions crucially depend on, is difficult even estimate numerically in view of a complicated nature of the eigenfunctions (3.14) involved, to say nothing of the nonlinear perturbation $\omega(c_2)$.

4.4. **Branching computations for $|\beta| = 2$.** Overall, the above analysis provides us with some expressions for the solutions for the self-similar equation (2.7) depending on the value of k. Actually, we can achieve those expressions for every critical value α_k, but again the calculus gets rather difficult. For the sake of completeness, we now analyze the case $|\beta| = 2$ and $M_2 = 3$, so that $\{\psi_\beta\}_{|\beta|=2} = \{\hat\psi_1, \hat\psi_2, \hat\psi_3\}$ stands for a basis of the eigenspace $\ker\left(\mathbf{B} + \frac{1}{2} I\right)$, with $k = 2$ ($\lambda_k = -\frac{k}{4}$).

Thus, in this case, performing in a similar way as was done for (4.25) with $\psi_2 = c_1\hat\psi_1 + c_2\hat\psi_2 + c_3\hat\psi_3$, we arrive at the following algebraic system:

$$(4.33)\quad \begin{cases} c_1\langle\hat\psi_1^*, h_1\rangle + c_2\langle\hat\psi_1^*, h_2\rangle + c_3\langle\hat\psi_1^*, h_3\rangle - \frac{c_1\alpha_2}{4}\langle\hat\psi_1^*, y\cdot\nabla\hat\psi_1\rangle - \frac{c_2\alpha_2}{4}\langle\hat\psi_1^*, y\cdot\nabla\hat\psi_2\rangle \\ \qquad - \frac{c_3\alpha_2}{4}\langle\hat\psi_1^*, y\cdot\nabla\hat\psi_3\rangle + c_1\mu_{1,2} = 0, \\ c_1\langle\hat\psi_2^*, h_1\rangle + c_2\langle\hat\psi_2^*, h_2\rangle + c_2\langle\hat\psi_2^*, h_3\rangle - \frac{c_1\alpha_2}{4}\langle\hat\psi_2^*, y\cdot\nabla\hat\psi_1\rangle - \frac{c_2\alpha_2}{4}\langle\hat\psi_2^*, y\cdot\nabla\hat\psi_2\rangle \\ \qquad - \frac{c_3\alpha_2}{4}\langle\hat\psi_2^*, y\cdot\nabla\hat\psi_3\rangle + c_2\mu_{1,2} = 0, \\ c_1\langle\hat\psi_3^*, h_1\rangle + c_2\langle\hat\psi_3^*, h_2\rangle + c_2\langle\hat\psi_3^*, h_3\rangle - \frac{c_1\alpha_2}{4}\langle\hat\psi_3^*, y\cdot\nabla\hat\psi_1\rangle - \frac{c_2\alpha_2}{4}\langle\hat\psi_3^*, y\cdot\nabla\hat\psi_2\rangle \\ \qquad - \frac{c_3\alpha_2}{4}\langle\hat\psi_3^*, y\cdot\nabla\hat\psi_3\rangle + c_3\mu_{1,2} = 0, \\ c_1 + c_2 + c_3 = 1, \end{cases}$$

where

$$\begin{aligned} h_1 &:= -\nabla\cdot[\ln(c_1\hat\psi_1 + c_2\hat\psi_2 + c_3\hat\psi_3)\nabla\Delta\hat\psi_1], \\ h_2 &:= -\nabla\cdot[\ln(c_1\hat\psi_1 + c_2\hat\psi_2 + c_3\hat\psi_3)\nabla\Delta\hat\psi_2], \\ h_3 &:= -\nabla\cdot[\ln(c_1\hat\psi_1 + c_2\hat\psi_2 + c_3\hat\psi_3)\nabla\Delta\hat\psi_3], \end{aligned}$$

and c_1, c_2, c_3, and $\mu_{1,2}$ are the unknowns to be evaluated. Also, α_2 is regarded as the value of the parameter α denoted by (4.1) and is dependent on the eigenvalue λ_2 with $\hat\psi_1, \hat\psi_2, \hat\psi_3$ representing the associated eigenfunctions and $\hat\psi_1^*, \hat\psi_2^*, \hat\psi_3^*$ the corresponding adjoint eigenfunctions.

Subsequently, substituting $c_3 = 1 - c_1 - c_2$ into the first three equations and performing an argument based upon the Brouwer Fixed Point Theorem and the topological degree as the one done above for the case $|\beta| = 1$, we ascertain the existence of a nondegenerate solution of the algebraic system if the following condition is satisfied:

$$\langle \hat{\psi}_1^*, y\cdot\nabla(\hat{\psi}_3 - \hat{\psi}_1)\rangle\langle \hat{\psi}_2^*, y\cdot\nabla(\hat{\psi}_3 - \hat{\psi}_2)\rangle - \langle \hat{\psi}_1^*, y\cdot\nabla(\hat{\psi}_3 - \hat{\psi}_2)\rangle\langle \hat{\psi}_2^*, y\cdot\nabla(\hat{\psi}_3 - \hat{\psi}_1)\rangle \neq 0. \tag{4.34}$$

Note that, by similar substitutions, other conditions might be obtained.

Furthermore, once we know the existence of at least one solution, we proceed now with a possible way of computing the number of solutions of the nonlinear algebraic system (4.33). Obviously, since the dimension of the eigenspace is bigger than that in the case when $|\beta| = 1$, the difficulty to obtain multiplicity results increases.

Firstly, integrating by parts in the nonlinear terms, in which h_1, h_2 and h_3 are involved, and rearranging terms in the first three equations gives

$$\int_{\mathbb{R}^N} \nabla\psi_1^* \cdot \ln(c_1\hat{\psi}_1 + c_2\hat{\psi}_2 + c_3\hat{\psi}_3)\nabla\Delta(c_1\hat{\psi}_1 + c_2\hat{\psi}_2 + c_3\hat{\psi}_3) - c_1\tfrac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_1^* y\cdot\nabla\hat{\psi}_1 + c_1\mu_{1,2}$$
$$- c_2\tfrac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_1^* y\cdot\nabla\hat{\psi}_2 - c_3\tfrac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_1^* y\cdot\nabla\hat{\psi}_3 = 0,$$

$$\int_{\mathbb{R}^N} \nabla\hat{\psi}_2^* \cdot \ln(c_1\hat{\psi}_1 + c_2\hat{\psi}_2 + c_3\hat{\psi}_3)\nabla\Delta(c_2\hat{\psi}_1 + c_2\hat{\psi}_2 + c_3\hat{\psi}_3) - c_1\tfrac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_2^* y\cdot\nabla\hat{\psi}_1 + c_2\mu_{1,2}$$
$$- c_2\tfrac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_2^* y\cdot\nabla\hat{\psi}_2 - c_3\tfrac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_2^* y\cdot\nabla\hat{\psi}_3 = 0,$$

$$\int_{\mathbb{R}^N} \nabla\hat{\psi}_3^* \cdot \ln(c_1\hat{\psi}_1 + c_2\hat{\psi}_2 + c_3\hat{\psi}_3)\nabla\Delta(c_2\hat{\psi}_1 + c_2\hat{\psi}_2 + c_3\hat{\psi}_3) - c_1\tfrac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_3^* y\cdot\nabla\hat{\psi}_1 + c_3\mu_{1,2}$$
$$- c_2\tfrac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_3^* y\cdot\nabla\hat{\psi}_2 - c_3\tfrac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_3^* y\cdot\nabla\hat{\psi}_3 = 0.$$

By the fourth equation, we have that $c_1 = 1 - c_2 - c_3$. Then, setting

$$c_1\hat{\psi}_1 + c_2\hat{\psi}_2 + c_3\hat{\psi}_3 = \hat{\psi}_1 + c_2(\hat{\psi}_2 - \hat{\psi}_1) + c_3(\hat{\psi}_3 - \hat{\psi}_1)$$

and substituting it into the expressions obtained above for the first three equations of the system yield

$$\int_{\mathbb{R}^N} \nabla\hat{\psi}_1^* \cdot \ln(\hat{\psi}_1 + (\hat{\psi}_2 - \hat{\psi}_1)c_2 + (\hat{\psi}_3 - \hat{\psi}_1)c_3)\nabla\Delta(\hat{\psi}_1 + (\hat{\psi}_2 - \hat{\psi}_1)c_2 + (\hat{\psi}_3 - \hat{\psi}_1)c_3)$$
$$+ \mu_{1,2} - c_2\mu_{1,2} - c_3\mu_{1,2} - \tfrac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_1^* y\cdot\nabla\hat{\psi}_1$$
$$+ \tfrac{\alpha_2}{4}\int_{\mathbb{R}^N} \hat{\psi}_1^* y\cdot((\nabla\hat{\psi}_1 - \nabla\hat{\psi}_2)c_2 + (\nabla\hat{\psi}_1 - \nabla\hat{\psi}_3)c_3) = 0,$$

$$\int_{\mathbb{R}^N} \nabla\hat{\psi}_2^* \cdot \ln(\hat{\psi}_1+(\hat{\psi}_2-\hat{\psi}_1)c_2+(\hat{\psi}_3-\hat{\psi}_1)c_3)\nabla\Delta(\hat{\psi}_1+(\hat{\psi}_2-\hat{\psi}_1)c_2+(\hat{\psi}_3-\hat{\psi}_1)c_3)$$

$$(4.35) \qquad +c_2\mu_{1,2}-\tfrac{\alpha_2}{4}\int_{\mathbb{R}^N}\hat{\psi}_2^* y\cdot\nabla\hat{\psi}_1$$

$$+\tfrac{\alpha_2}{4}\int_{\mathbb{R}^N}\hat{\psi}_2^* y\cdot((\nabla\hat{\psi}_1-\nabla\hat{\psi}_2)c_2+(\nabla\hat{\psi}_1-\nabla\hat{\psi}_3)c_3)=0,$$

$$\int_{\mathbb{R}^N} \nabla\hat{\psi}_3^* \cdot \ln(\hat{\psi}_1+(\hat{\psi}_2-\hat{\psi}_1)c_2+(\hat{\psi}_3-\hat{\psi}_1)c_3)\nabla\Delta(\hat{\psi}_1+(\hat{\psi}_2-\hat{\psi}_1)c_2+(\hat{\psi}_3-\hat{\psi}_1)c_3)$$

$$+c_3\mu_{1,2}-\tfrac{\alpha_2}{4}\int_{\mathbb{R}^N}\hat{\psi}_3^* y\cdot\nabla\hat{\psi}_1$$

$$+\tfrac{\alpha_2}{4}\int_{\mathbb{R}^N}\hat{\psi}_3^* y\cdot((\nabla\hat{\psi}_1-\nabla\hat{\psi}_2)c_2+(\nabla\hat{\psi}_1-\nabla\hat{\psi}_3)c_3)=0.$$

Now, adding the first equation of (4.35) to the other two, we have that

$$\int_{\mathbb{R}^N} (\nabla\hat{\psi}_1^*+\nabla\hat{\psi}_2^*) \cdot \ln(\hat{\psi}_1+(\hat{\psi}_2-\hat{\psi}_1)c_2+(\hat{\psi}_3-\hat{\psi}_1)c_3)\nabla\Delta(\hat{\psi}_1+(\hat{\psi}_2-\hat{\psi}_1)c_2+(\hat{\psi}_3-\hat{\psi}_1)c_3)$$

$$+\mu_{1,2}-c_3\mu_{1,2}-\tfrac{\alpha_2}{4}\int_{\mathbb{R}^N}(\hat{\psi}_1^*+\hat{\psi}_2^*) y\cdot\nabla\hat{\psi}_1$$

$$+\tfrac{\alpha_2}{4}\int_{\mathbb{R}^N}(\hat{\psi}_1^*+\hat{\psi}_2^*) y\cdot((\nabla\hat{\psi}_1-\nabla\hat{\psi}_2)c_2+(\nabla\hat{\psi}_1-\nabla\hat{\psi}_3)c_3)=0,$$

$$\int_{\mathbb{R}^N} (\nabla\hat{\psi}_1^*+\nabla\hat{\psi}_3^*) \cdot \ln(\hat{\psi}_1+(\hat{\psi}_2-\hat{\psi}_1)c_2+(\hat{\psi}_3-\hat{\psi}_1)c_3)\nabla\Delta(\hat{\psi}_1+(\hat{\psi}_2-\hat{\psi}_1)c_2+(\hat{\psi}_3-\hat{\psi}_1)c_3)$$

$$+\mu_{1,2}-c_2\mu_{1,2}-\tfrac{\alpha_2}{4}\int_{\mathbb{R}^N}(\hat{\psi}_1^*+\hat{\psi}_3^*) y\cdot\nabla\hat{\psi}_1$$

$$+\tfrac{\alpha_2}{4}\int_{\mathbb{R}^N}(\hat{\psi}_1^*+\hat{\psi}_3^*) y\cdot((\nabla\hat{\psi}_1-\nabla\hat{\psi}_2)c_2+(\nabla\hat{\psi}_1-\nabla\hat{\psi}_3)c_3)=0.$$

Subsequently, subtracting those equations yields

$$\mu_{1,2}=\tfrac{1}{c_3-c_2}\Big[\int_{\mathbb{R}^N}(\nabla\hat{\psi}_2^*-\nabla\hat{\psi}_3^*)\cdot\ln\Psi\nabla\Delta\Psi-\tfrac{\alpha_2}{4}\int_{\mathbb{R}^N}(\hat{\psi}_2^*-\hat{\psi}_3^*)y\cdot\nabla\hat{\psi}_1$$

$$+\tfrac{\alpha_2}{4}\int_{\mathbb{R}^N}(\hat{\psi}_2^*-\hat{\psi}_3^*)y\cdot((\nabla\hat{\psi}_1-\nabla\hat{\psi}_2)c_2+(\nabla\hat{\psi}_1-\nabla\hat{\psi}_3)c_3)\Big],$$

where $\Psi=\hat{\psi}_1+(\hat{\psi}_2-\hat{\psi}_1)c_2+(\hat{\psi}_3-\hat{\psi}_1)c_3$. Thus, substituting it into (4.35) (note that, from the substitution into one of the last two equations, we obtain the same equation), we arrive at the following system, with c_2 and c_3 as the unknowns:

$$
\begin{aligned}
c_3 \int\limits_{\mathbb{R}^N} (\nabla\hat{\psi}_1^* - \nabla\hat{\psi}_2^* + \nabla\hat{\psi}_3^*) \cdot \ln\Psi\nabla\Delta\Psi - c_2 \int\limits_{\mathbb{R}^N} (\nabla\hat{\psi}_1^* + \nabla\hat{\psi}_2^* - \nabla\hat{\psi}_3^*) \cdot \ln\Psi\nabla\Delta\Psi \\
+ \int\limits_{\mathbb{R}^N} (\nabla\hat{\psi}_2^* - \nabla\hat{\psi}_3^*) \cdot \ln\Psi - \tfrac{\alpha_2}{4} \int\limits_{\mathbb{R}^N} (\hat{\psi}_2^* - \hat{\psi}_3^*) y \cdot \nabla\hat{\psi}_1 \\
+ c_2 \tfrac{\alpha_2}{4} [\int\limits_{\mathbb{R}^N} (\hat{\psi}_2^* - \hat{\psi}_3^*) y \cdot \nabla(2\hat{\psi}_1 - \hat{\psi}_2) - \int\limits_{\mathbb{R}^N} \hat{\psi}_1^* y \cdot \nabla\hat{\psi}_1] \\
+ c_3 \tfrac{\alpha_2}{4} [\int\limits_{\mathbb{R}^N} (\hat{\psi}_2^* - \hat{\psi}_3^*) y \cdot \nabla(2\hat{\psi}_1 - \hat{\psi}_3) - \int\limits_{\mathbb{R}^N} \hat{\psi}_1^* y \cdot \nabla\hat{\psi}_1]
\end{aligned}
$$

$$
\begin{aligned}
& + c_2 c_3 \tfrac{\alpha_2}{4} [\int\limits_{\mathbb{R}^N} \hat{\psi}_1^* y \cdot (\nabla\hat{\psi}_3 - \nabla\hat{\psi}_2) - \int\limits_{\mathbb{R}^N} (\nabla\hat{\psi}_2^* - \nabla\hat{\psi}_3^*) y \cdot (2\nabla\hat{\psi}_1 - \nabla\hat{\psi}_2 - \nabla\hat{\psi}_3)] \\
& + c_3^2 \tfrac{\alpha_2}{4} \int\limits_{\mathbb{R}^N} (\hat{\psi}_1^* - \hat{\psi}_2^* + \hat{\psi}_3^*) y \cdot (\nabla\hat{\psi}_1 - \nabla\hat{\psi}_3) \\
& - c_2^2 \tfrac{\alpha_2}{4} \int\limits_{\mathbb{R}^N} (\hat{\psi}_1^* + \hat{\psi}_2^* - \hat{\psi}_3^*) y \cdot (\nabla\hat{\psi}_1 - \nabla\hat{\psi}_2) = 0,
\end{aligned}
$$

$$
\begin{aligned}
c_3 \int\limits_{\mathbb{R}^N} \nabla\hat{\psi}_2^* \cdot \ln\Psi\nabla\Delta\Psi - c_2 \int\limits_{\mathbb{R}^N} \nabla\hat{\psi}_3^* \cdot \ln\Psi\nabla\Delta\Psi - c_3 \tfrac{\alpha_2}{4} \int\limits_{\mathbb{R}^N} \hat{\psi}_2^* y \cdot \nabla\hat{\psi}_1 + c_2 \tfrac{\alpha_2}{4} \int\limits_{\mathbb{R}^N} \hat{\psi}_3^* y \cdot \nabla\hat{\psi}_1 \\
+ c_3 \tfrac{\alpha_2}{4} \int\limits_{\mathbb{R}^N} \hat{\psi}_2^* y \cdot ((\nabla\hat{\psi}_1 - \nabla\hat{\psi}_2) c_2 + (\nabla\hat{\psi}_1 - \nabla\hat{\psi}_3) c_3) \\
- c_2 \tfrac{\alpha_2}{4} \int\limits_{\mathbb{R}^N} \hat{\psi}_3^* y \cdot ((\nabla\hat{\psi}_1 - \nabla\hat{\psi}_2) c_2 + (\nabla\hat{\psi}_1 - \nabla\hat{\psi}_3) c_3) = 0.
\end{aligned}
$$

These can be re-written in the following form:

$$
\begin{aligned}
A_1 c_2^2 + B_1 c_3^2 + C_1 c_2 + D_1 c_3 + E_1 c_2 c_3 + \omega_1(c_2, c_3) = 0, \\
A_2 c_2^2 + B_2 c_3^2 + C_2 c_2 + D_2 c_3 + E_2 c_2 c_3 + \omega_2(c_2, c_3) = 0,
\end{aligned} \tag{4.36}
$$

where

$$
\begin{aligned}
\omega_1(c_2, c_3) := c_3 \int\limits_{\mathbb{R}^N} (\nabla\hat{\psi}_1^* - \nabla\hat{\psi}_2^* + \nabla\hat{\psi}_3^*) \cdot \ln\Psi\nabla\Delta\Psi \\
- c_2 \int\limits_{\mathbb{R}^N} (\nabla\hat{\psi}_1^* + \nabla\hat{\psi}_2^* - \nabla\hat{\psi}_3^*) \cdot \ln\Psi\nabla\Delta\Psi \\
+ \int\limits_{\mathbb{R}^N} (\nabla\hat{\psi}_2^* - \nabla\hat{\psi}_3^*) \cdot \ln\Psi - \tfrac{\alpha_2}{4} \int\limits_{\mathbb{R}^N} (\hat{\psi}_2^* - \hat{\psi}_3^*) y \cdot \nabla\hat{\psi}_1
\end{aligned}
$$

and

$$
\omega_2(c_2, c_3) := c_3 \int\limits_{\mathbb{R}^N} \nabla\hat{\psi}_2^* \cdot \ln\Psi\nabla\Delta\Psi - c_2 \int\limits_{\mathbb{R}^N} \nabla\hat{\psi}_3^* \cdot \ln\Psi\nabla\Delta\Psi
$$

are the perturbations of the quadratic polynomials

$$
\mathfrak{F}_i(c_2, c_3) := A_i c_2^2 + B_i c_3^2 + C_i c_2 + D_i c_3 + E_i c_2 c_3, \quad \text{with} \quad i = 1, 2.
$$

The coefficients of those quadratic expressions are given by

$$A_1 := -\tfrac{\alpha_2}{4} \int\limits_{\mathbb{R}^N} (\hat{\psi}_1^* + \hat{\psi}_2^* - \hat{\psi}_3^*) y \cdot (\nabla\hat{\psi}_1 - \nabla\hat{\psi}_2),$$
$$B_1 := \tfrac{\alpha_2}{4} \int\limits_{\mathbb{R}^N} (\hat{\psi}_1^* - \hat{\psi}_2^* + \hat{\psi}_3^*) y \cdot (\nabla\hat{\psi}_1 - \nabla\hat{\psi}_3),$$
$$C_1 := \tfrac{\alpha_2}{4} [\int\limits_{\mathbb{R}^N} (\hat{\psi}_2^* - \hat{\psi}_3^*) y \cdot \nabla(2\hat{\psi}_1 - \hat{\psi}_2) - \int\limits_{\mathbb{R}^N} \hat{\psi}_1 y \cdot \nabla\hat{\psi}_1],$$
$$D_1 := \tfrac{\alpha_2}{4} [\int\limits_{\mathbb{R}^N} (\hat{\psi}_2^* - \hat{\psi}_3^*) y \cdot \nabla(2\hat{\psi}_1 - \hat{\psi}_3) - \int\limits_{\mathbb{R}^N} \hat{\psi}_1 y \cdot \nabla\hat{\psi}_1],$$
$$E_1 := \tfrac{\alpha_2}{4} [\int\limits_{\mathbb{R}^N} \hat{\psi}_1^* y \cdot (\nabla\hat{\psi}_3 - \nabla\hat{\psi}_2) - \int\limits_{\mathbb{R}^N} (\nabla\hat{\psi}_2^* - \nabla\hat{\psi}_3^*) y \cdot (2\nabla\hat{\psi}_1 - \nabla\hat{\psi}_2 - \nabla\hat{\psi}_3)],$$

$$A_2 := -\tfrac{\alpha_2}{4} \int\limits_{\mathbb{R}^N} \hat{\psi}_3^* y \cdot (\nabla\hat{\psi}_1 - \nabla\hat{\psi}_2),$$
$$B_2 := \tfrac{\alpha_2}{4} \int\limits_{\mathbb{R}^N} \hat{\psi}_2^* y \cdot ((\nabla\hat{\psi}_1 - \nabla\hat{\psi}_3),$$
$$C_2 := \tfrac{\alpha_2}{4} \int\limits_{\mathbb{R}^N} \hat{\psi}_3^* y \cdot \nabla\hat{\psi}_1,$$
$$D_2 := -\tfrac{\alpha_2}{4} \int\limits_{\mathbb{R}^N} \hat{\psi}_2^* y \cdot \nabla\hat{\psi}_1,$$
$$E_2 := \tfrac{\alpha_2}{4} \int\limits_{\mathbb{R}^N} \hat{\psi}_2^* y \cdot (\nabla\hat{\psi}_1 - \nabla\hat{\psi}_2) - \hat{\psi}_3^* y \cdot (\nabla\hat{\psi}_1 - \nabla\hat{\psi}_3).$$

Therefore, using the conic classification to solve (4.36), we will have the number of solutions through the intersection of two conics. Then, depending on the type of conic, we shall always obtain one to four possible solutions for our system. Hence, somehow, the number of solutions depends on the coefficients we have for the system and, at the same time, on the eigenfunctions that generate the subspace $\ker\left(\mathbf{B} + \frac{k}{4}\right)$. Thus, we have the following conditions, which will provide us with the conic section of each equation of the system (4.36):

(i) If $B_i^2 - 4A_iE_i < 0$, the equation represents an *ellipse*, unless the conic is degenerate, for example $c_2^2 + c_3^2 + k = 0$ for some positive constant k. So, if $A_i = B_j$ and $E_i = 0$, the equation represents a *circle*;

(ii) If $B_i^2 - 4A_iE_i = 0$, the equation represents a *parabola*;

(iii) If $B_i^2 - 4A_iE_i > 0$, the equation represents a *hyperbola*. If we also have $A_i + E_i = 0$ the equation represents a hyperbola (a rectangular hyperbola).

Consequently, the zeros of the system (4.36) and, hence, of the system (4.33), adding the "normalizing" constraint (4.19), are ascertained by the intersection of those two conics in (4.36) providing us with the number of possible n-branches between one and four. Note that in case those conics are two circles we only have two intersection points at most. Moreover, due to the dimension of the eigenspaces it looks like in this case that we have four possible intersection points two of them will coincide. However, the justification for this is far from clear.

Moreover, as was done for the previous case when $|\beta| = 1$, we need to control the oscillations of the perturbation functions in order to maintain the number of solutions. Therefore, imposing that

$$\|\omega_i(c_2, c_3)\|_{L^\infty} \le \mathfrak{F}_i(c_2^*, c_3^*), \quad \text{with} \quad i = 1, 2,$$

we ascertain that the number of solutions must be between one and four. This again gives us an idea of the difficulty of more general multiplicity results.

4.5. **Further comments on mathematical justification of existence.** We return to the self-similar nonlinear eigenvalue problem (2.7), associated with (1.1), which can be written in the form

$$\mathcal{L}(\alpha, n)f + \mathcal{N}(n, f) = 0, \quad \text{where} \quad \mathcal{N}(n, f) := \nabla \cdot ((1 - |f|^n)\nabla\Delta f)\,. \tag{4.37}$$

As we have seen, the main difficulty in justifying the n-branching behaviour concerns the distribution and "transversal topology" of zero surfaces of solutions close to finite interface hyper-surfaces.

Recall that, as in classic nonlinear operator theory [11, 32, 43], our analysis above always assumed that we actually dealt with and performed computations for the integral equation:

$$f = -\mathcal{L}^{-1}(\alpha, n)\mathcal{N}(n, f) \equiv \mathcal{G}(n, f), \quad \mathcal{L}(\alpha, n) := -\Delta^2 + \tfrac{1-\alpha n}{4}\, y \cdot \nabla + \alpha I, \tag{4.38}$$

where $\mathcal{L}(\alpha, n)$ is invertible in L^2_ρ (this is directly checked via Section 3) and, hence compact, for a fixed α, and $f \in C_0(\mathbb{R}^N)$ for small $n > 0$. This confirms that the zeros of the function $\mathcal{F}(n, f)$ are fixed points of the map $\mathcal{G}(n, f)$. Note again that (4.38) is an eigenvalue problem, where admissible real values of α are supposed to be defined together with its solvability. This makes existence/multiplicity questions for (4.38) extremely difficult.

There are two cases of this problem. The first and simpler one occurs when the eigenvalue α is determined *a priori*, e.g., in the case $k = 0$, where $\alpha_0(0) = \frac{N}{4}$ denoted as $\alpha_0(0) = \alpha_0$, and where, for $n > 0$, the first nonlinear eigenvalue is given explicitly (see (2.10)):

$$\alpha_0(n) = \tfrac{N}{4+Nn}.$$

Then (4.38) with $\alpha = \alpha_0(n)$ for $n > 0$ becomes a standard nonlinear integral equation with, however, a quite curious and hard-to-detect functional setting. Indeed, the right-hand side in (4.38), where the nonlinearity is not in a fully divergent form, assumes the extra regularity at least such as

$$f \in H^3_\rho. \tag{4.39}$$

In view of the known good properties of the compact resolvent $(\mathcal{L} - \lambda I)^{-1}$, it is clear that the action of the inverse one $\mathcal{L}^{-1}$ is sufficient to restore the regularity, since locally in $\mathbb{R}^N$ this acts like Δ^{-2}. Therefore, it is plausible that

$$\mathcal{G} : H^3_\rho \to H^3_\rho, \tag{4.40}$$

and it is not difficult to get an *a priori* bound at least for small enough f's. The accompanying analysis as $y \to \infty$ (due to the unbounded domain) assumes no novelties or special difficulties and is standard for such weighted L^2 and Sobolev spaces.

Therefore, application of Schauder's Fixed Point Theorem (see e.g., [4, p. 90]) to (4.38) is a most powerful tool to imply existence of a solution, and moreover a continuous curve of fixed points $\Gamma_n = \{f,\ n > 0 \text{ small}\}$. By scaling invariance of the similarity equation, we are obliged to impose the normalization condition, say,

$$f(0) = \delta_0 > 0 \quad \text{sufficiently small.} \tag{4.41}$$

Uniqueness remains a completely open problem. However, studying the behaviour of the solution curve Γ_n as $n \to 0$ and applying (under suitable hypothesis) the branching techniques developed above, we may conclude that any such continuous curve must be originated at a properly scaled eigenfunction $\psi_0 = F$, so that such a curve is unique due to well-posedness of all the asymptotic expansions.

A possibility of extension of Γ_n for larger values of $n > 0$ represents an essentially more difficult nonlocal open problem. Indeed, via compactness of linear operators involved in (4.38), it is easy to expect that such a curve can end up at a bifurcation point only (unless blows up). However, nonexistence of turning saddle-node points at some $n_* > 0$ (meaning that the n-branch is nonexistent for some $n > n_*$) is not that easy to rule out. Moreover, such turning points with thin film operators involved are actually possible, [20].

After establishing existence of such solutions for small $n > 0$, we face the next problem on their asymptotic properties including the fact that these are compactly supported. On a qualitative level, these questions were discussed in [13].

In the case of higher-order nonlinear eigenfunctions of (4.38) for $k \geq 1$ including the dipole case $k = 1$, the parameter α becomes an eigenvalue that is essentially involved into the problem setting. This assumes to consider the equation (4.38) in the extended space

$$(f, \alpha) \in X = H^3_\rho \times \{\alpha \in \mathbb{R}\} \ \text{ and } \ \mathcal{G} : X \to X, \tag{4.42}$$

where proving the latter mapping for some compact subsets becomes a hard open problem. Note that here even the necessary convexity issue for applying Schauder's Theorem can be hard. We still do not know whether the representation such as (4.42) may lead to any rigorous treatment of the nonlinear eigenvalue problem (4.38) for $k \geq 1$.

5. General Cauchy problem: a homotopic approach

5.1. **Key concepts to justify: a first discussion.** We now discuss some related properties of more general solutions of the CP for the TFE (1.1) using a homotopic approach when the parameter n approaches zero. As shown in Section 3, we already know the similarity expression for the solutions of the "limiting" bi-harmonic equation (1.3). This fundamental solution has also a self-similar structure thanks to the scaling invariance and the uniqueness of the fundamental solution of the equation (1.3), denoted by (3.2).

The idea is to perform a homotopic approach from (1.1) to (1.3) in order to reveal important (and still obscure in general) properties of the Cauchy problem. The reason is that the bi-harmonic equation (1.3) i.e., (1.1) when $n = 0$, with the same initial data, admits the unique classic solution given by the convolution (3.1), where $b(x,t)$ is the fundamental solution (3.2) of the operator $\frac{\partial}{\partial t} + \Delta^2$, and the oscillatory rescaled kernel $F(y)$ is the unique solution of the problem (3.3). Hence, we expect that the knowledge of the solutions of (1.3) can be extended to (1.1) at least for sufficiently small $n > 0$. In other words, we claim that the "fundamental" solutions for $n = 0$ and small $n > 0$ exhibit several similar properties, excluding, on the other hand, some others such as the compact support one for $n > 0$. In addition, the homotopic path $n \to 0^+$ can be used for a proper definition of the solutions of the Cauchy problem for the TFE–4 (1.1).

Thus, we assume that $n > 0$ is sufficiently small. We define some "homotopic classes" of degenerate parabolic PDE's saying that the TFE (1.1) is homotopic to the linear PDE (1.3) if there exists a family of uniformly parabolic equations (a *homotopic deformation*) with coefficient $\phi_\varepsilon(u)$ analytic in both variables $u \in \mathbb{R}$ and $\varepsilon \in (0,1]$,

$$u_t = -\nabla \cdot (\phi_\varepsilon(u) \nabla \Delta u)\,, \tag{5.1}$$

such that $\phi_1(u) = 1$ and

$$\phi_\varepsilon(u) \to |u|^n \quad \text{as} \quad \varepsilon \to 0 \quad \text{uniformly on compact subsets}\,. \tag{5.2}$$

We should point out that such a limit for *nonnegative* and not changing sign solutions, with various non-analytic (and non-smooth) regularizations has been widely used before in TFE–FBP theory as a key foundation; cf. [3], [5], and [7].

A possible homotopic path can be

$$\phi_\varepsilon(u) := \varepsilon^n + (1-\varepsilon)(\varepsilon^2 + u^2)^{\frac{n}{2}}, \quad \varepsilon \in (0,1].$$

For any $\varepsilon \in (0,1]$, denote by $u_\varepsilon(x,t)$ the unique solution of the CP for the regularized non-degenerate equation (5.1) with same data u_0. By classic parabolic theory, u_ε is continuous (and analytic) in $\varepsilon \in (0,1]$ in any natural functional topology. The main problem is the behaviour as $\varepsilon \to 0$, where the regularized PDE loses its uniform parabolicity. For second-order parabolic equations obeying the Maximum Principle, such regularization-continuity approaches are typical for constructing unique solutions with singularities (finite time blow-up, extinction, finite interfaces, etc.); see [18] as a source of key references and basic results. However, for higher-order degenerate parabolic flows admitting strongly oscillatory solutions of changing sign, such a homotopy-continuity approach generates a number of difficult problems. In fact, despite the fact that the passage to the limit as $\varepsilon \to 0$ looks like a reasonable way to define a proper solution of the TFE, we expect that there are always special classes of compactly supported initial data, for which such a limit is non-existent and, moreover, there are many partial limits, thus defining a variety of different solutions (meaning nonuniqueness), as we show below.

5.2. **Preliminary estimates.** To ascertain such a limit for (5.1) when $\varepsilon \to 0$, we firstly obtain some estimations for its regularized solutions $\{u_\varepsilon(x,t)\}$. Here, by Ω we denote either $\mathbb{R}^N$, or, equivalently, the bounded domain $\Gamma_0 \cap \{t\}$, i.e., the section of the support.

Proposition 5.1. *Let $u_\varepsilon(x,t)$ be the unique global solution of the CP for the regularized nondegenerate equation* (5.1) *with the initial data u_0. Then, for any $t \in [0,T]$, the following is satisfied:*

(i) $u_\varepsilon(\cdot,t) \in H_0^1(\Omega)$;
(ii) $u_\varepsilon(\cdot,t) \in L^p(\Omega)$, *with* $p = 1, 2, \infty$; *and*
(iii) $h_\varepsilon \in L^2(\Omega \times [0,T])$, *with* $h_\varepsilon := \phi_\varepsilon(u_\varepsilon)\nabla\Delta u_\varepsilon$.

Proof. Firstly, multiplying (5.1) by Δu_ε, integrating in $\Omega \times [0,t]$ for any $t \in [0,T]$, and applying the formula of integration by parts yield

$$\tfrac{1}{2} \int_\Omega |\nabla u_\varepsilon(x,t)|^2 + \int_0^t \int_\Omega \phi_\varepsilon(u)|\nabla\Delta u_\varepsilon|^2 = \tfrac{1}{2} \int_\Omega |\nabla u_\varepsilon(x,0)|^2, \tag{5.3}$$

thanks to the boundary conditions (2.1). Note that

$$\begin{aligned} &\int_\Omega [|\nabla u_\varepsilon(x,t+h)|^2 - |\nabla u_\varepsilon(x,t)|^2] \\ &= -\int_\Omega [\Delta u_\varepsilon(x,t+h) + \Delta u_\varepsilon(x,t)][u_\varepsilon(x,t+h) - u_\varepsilon(x,t)]. \end{aligned}$$

Then, dividing that equality by h, passing to the limit as $h \downarrow 0$, and integrating between 0 and any $t \in [0,T]$, we find that

$$\int_0^t \int_\Omega \Delta u_\varepsilon u_t = \tfrac{1}{2} \int_\Omega |\nabla u_\varepsilon(x,t)|^2 - \tfrac{1}{2} \int_\Omega |\nabla u_\varepsilon(x,0)|^2,$$

which provides us with the necessary expression to obtain (5.3). Thus, from (5.3), we have that (in fact, this is true from the beginning for classic C^∞-smooth solutions of (5.1), but we will need those manipulations in what follows)

$$\int_\Omega |\nabla u_\varepsilon(x,t)|^2 \le K \quad \text{and} \quad \int_0^t \int_\Omega \phi_\varepsilon(u)|\nabla\Delta u_\varepsilon|^2 \le K, \tag{5.4}$$

since both terms of the left-hand side in (5.3) are always positive and the right-hand side is bounded by (2.2), for some positive constant $K > 0$ that is independent of ε. Then,

$$\nabla u_\varepsilon(\cdot,t) \in L^2(\Omega) \quad \text{for any} \quad t \in [0,T].$$

Moreover, by Poincaré's inequality, $u_\varepsilon(\cdot,t) \in L^2(\Omega)$, and hence,

$$u_\varepsilon(\cdot,t) \in H_0^1(\Omega) \quad \text{for any} \quad t \in [0,T]. \tag{5.5}$$

In fact, we may assume that

$$u_\varepsilon(\cdot,t) \in L^\infty(\Omega) \quad \text{for all} \quad t \in (0,T). \tag{5.6}$$

Indeed, for $N = 1$, this follows from (5.4) by Sobolev's embedding. For $N \ge 2$, this is a natural assumption inherited from the smooth analytic parabolic flow (5.1), though its full proof sometimes can be a difficult issue; we refer to scaling and other techniques that may be convenient, [23]. From the conservation of mass assumption, we can also assure that $u_\varepsilon(\cdot,t) \in L^1(\Omega)$. Also, expression (5.3) combined with $u_\varepsilon(\cdot,t) \in L^1(\Omega)$ provides us with the estimate

$$h_\varepsilon \in L^2(\Omega \times [0,T]) \quad (h_\varepsilon = \phi_\varepsilon(u_\varepsilon)\nabla\Delta u_\varepsilon). \tag{5.7}$$

Indeed, from (5.4) we find that

$$\int_0^t \int_\Omega [\varepsilon^n + (1-\varepsilon)(\varepsilon^2+u^2)^{\frac{n}{2}}]|\nabla\Delta u_\varepsilon|^2 \le K, \quad \text{so that}$$

$$\varepsilon^n \int_0^t \int_\Omega |\nabla\Delta u_\varepsilon|^2 \le K \quad \text{and} \quad \int_0^t \int_\Omega (\varepsilon^2+u^2)^{\frac{n}{2}}|\nabla\Delta u_\varepsilon|^2 \le K, \tag{5.8}$$

since $\varepsilon \in (0,1)$ with a constant $K > 0$ independent of ε. Now, using Hölder's inequality,

$$\int_0^t \int_\Omega |h_\varepsilon|^2 \le 2\varepsilon^{2n} \int_0^t \int_\Omega |\nabla\Delta u_\varepsilon|^2 + 2\int_0^t \int_\Omega (\varepsilon^2+u^2)^{\frac{n}{2}}(\varepsilon^2+u^2)^{\frac{n}{2}}|\nabla\Delta u_\varepsilon|^2,$$

by (5.8) and (5.6) (note also that $u_\varepsilon(\cdot,t) \in L^1(\Omega)$), we obtain (5.7). □

Furthermore, the following estimates are also ascertained:

Lemma 5.1. *Let $u_\varepsilon(x,t)$ the unique global solution of the CP for the regularized uniformly parabolic equation* (5.1) *with the initial data u_0. Then, there exists some positive constant $K > 0$ such that, for $x_1, x_2 \in \Omega$, independently of ε and t,*

$$|u_\varepsilon(x_1,t) - u_\varepsilon(x_2,t)| \le K|x_1 - x_2|^{\frac{1}{2}} \quad \textit{for odd } N, \quad \textit{and}$$

$$|u_\varepsilon(x_1,t) - u_\varepsilon(x_2,t)| \le K|x_1 - x_2| \quad \textit{for even } N.$$

Proof. Thanks to Sobolev's inequality, we have that, for the odd dimension $N \ge 1$,

$$\|u_\varepsilon(\cdot,t)\|_{C^{0,\frac{1}{2}}(\Omega)} \le C\|u_\varepsilon(\cdot,t)\|_{H_0^{N-j_N}(\Omega)}, \quad \text{with} \quad j_1 = 0, j_3 = 1, j_5 = 2, \cdots,$$

for a positive constant $C > 0$. On the other hand, when the dimension $N \ge 1$ is even,

$$\|u_\varepsilon(\cdot,t)\|_{C^{0,1}(\Omega)} \le C\|u_\varepsilon(\cdot,t)\|_{H_0^{N-j_N}(\Omega)}, \quad \text{with} \quad j_2 = 0, j_4 = 1, j_6 = 2, \cdots,$$

for some positive constant $C > 0$. Hence, since $C_0^\infty(\Omega)$ is dense in $W_0^{k,2}(\Omega) = H_0^k(\Omega)$ for $k \ge 1$, by the analytic smoothness of the solutions of the uniformly parabolic PDE (5.1) with analytic coefficients, for some positive constant K independently of ε and t,

$$|u_\varepsilon(x_1,t) - u_\varepsilon(x_2,t)| \le K|x_1 - x_2|^{\frac{1}{2}} \quad \text{for odd } N, \quad \text{and}$$

$$|u_\varepsilon(x_1,t) - u_\varepsilon(x_2,t)| \le K|x_1 - x_2| \quad \text{for even } N. \quad \square$$

Moreover, as was noted in [3, 5] and [7], $u_\varepsilon(x,t)$ is also Hölder continuous in time with exponent $\frac{1}{8}$, i.e., $u_\varepsilon(\cdot,t) \in C^{0,\frac{1}{8}}([0,T])$ for the one dimensional case. However, we provide a new version for the N-dimensional case.

Lemma 5.2. *Let $u_\varepsilon(x,t)$ be the unique global solution of the CP for the regularized equation* (5.1) *with the initial data u_0. Then, there exists a positive constant $\tau > 0$ such that*

$$|u_\varepsilon(x,t_2) - u_\varepsilon(x,t_1)| \le \tau |t_2 - t_1|^{\frac{N}{4(N+1)}},$$

if the dimension N is odd, and

$$|u_\varepsilon(x,t_2) - u_\varepsilon(x,t_1)| \le \tau |t_2 - t_1|^{\frac{N}{3N+2}},$$

if the dimension N is even, for any $t_1, t_2 \in [0,T]$, independently of ε and x.

Proof. First, consider a non-negative cut-off function $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}(\varphi) \subset \Omega$ and $\int_\Omega \varphi = 1$. Subsequently, multiplying (5.1) by a test function $\varphi_\gamma \in C_0^\infty(\mathbb{R}^N)$, where

$$\varphi_\gamma(x) := \tfrac{1}{\gamma}\, \varphi\big(\tfrac{x-x_0}{\gamma^{1/N}}\big),$$

with some $x_0 \in \Omega$ and a constant $\gamma > 0$ to be properly chosen later on, integrating over $\Omega \times [t_1,t_2]$ and applying the formula of integration by parts, we find that

$$
\begin{aligned}
&-\int_{t_1}^{t_2}\int_\Omega \varphi_\gamma u_{\varepsilon,t} - \int_{t_1}^{t_2}\int_\Omega \nabla\varphi_\gamma \cdot (\phi_\varepsilon(u_\varepsilon)\nabla\Delta u_\varepsilon) = 0, \\
&\text{where} \quad -\int_{t_1}^{t_2}\int_\Omega \varphi_\gamma u_{\varepsilon,t} \equiv \int_\Omega \varphi_\gamma (u_\varepsilon(t_2) - u_\varepsilon(t_1)).
\end{aligned}
\tag{5.9}
$$

On the other hand, by the present choice of φ_γ, we know that $\int_\Omega \varphi_\gamma = 1$, since the Jacobian of $\frac{x-x_0}{\gamma^{1/N}}$ is $\frac{1}{\gamma}$. Then, we have that

$$
\begin{aligned}
&u_\varepsilon(x_0,t_2) - u_\varepsilon(x_0,t_1) \equiv \int_\Omega \varphi_\gamma(x)(u_\varepsilon(x_0,t_2) - u_\varepsilon(x_0,t_1)) \\
&\equiv \int_\Omega \varphi_\gamma(x)(u_\varepsilon(x_0,t_2) - u_\varepsilon(x,t_2) + u_\varepsilon(x,t_2) - u_\varepsilon(x,t_1) + u_\varepsilon(x,t_1) - u_\varepsilon(x_0,t_1)) \\
&\le \int_\Omega \varphi_\gamma(x)|u_\varepsilon(x,t_2) - u_\varepsilon(x_0,t_2)| + \Big|\int_\Omega \varphi_\gamma(x)(u_\varepsilon(x,t_2) - u_\varepsilon(x,t_1))\Big| \\
&+ \int_\Omega \varphi_\gamma(x))|u_\varepsilon(x_0,t_1) - u_\varepsilon(x,t_1)|.
\end{aligned}
$$

Owing to Lemma 5.1 and (5.9) on the time interval (t_1,t_2), taking into account in the last inequality, we obtain

$$|u_\varepsilon(x_0,t_2) - u_\varepsilon(x_0,t_1)| \le 2K|x-x_0|^{\frac{1}{2}} + \Big|\int_{t_1}^{t_2}\int_\Omega \nabla\varphi_\gamma \cdot (\phi_\varepsilon(u)\nabla\Delta u_\varepsilon)\Big|, \quad \text{with} \quad N \ \text{odd},$$

$$|u_\varepsilon(x_0,t_2) - u_\varepsilon(x_0,t_1)| \le 2K|x-x_0| + \Big|\int_{t_1}^{t_2}\int_\Omega \nabla\varphi_\gamma \cdot (\phi_\varepsilon(u)\nabla\Delta u_\varepsilon)\Big|, \quad \text{with} \quad N \ \text{even}.$$

Moreover, by Hölder's inequality and the choice of φ_γ,

$$\Big|\int_{t_1}^{t_2}\int_\Omega \nabla\varphi_\gamma \cdot (\phi_\varepsilon(u)\nabla\Delta u_\varepsilon)\Big| \le \Big(\int_{t_1}^{t_2}\int_\Omega |\phi_\varepsilon(u)\nabla\Delta u_\varepsilon|^2\Big)^{\frac{1}{2}} \Big(\int_{t_1}^{t_2}\int_\Omega |\nabla\varphi_\gamma|^2\Big)^{\frac{1}{2}},$$

and, hence, thanks also to Proposition 5.1,

$$\Big|\int_{t_1}^{t_2}\int_{\Omega} \nabla\varphi_\gamma\cdot(\phi_\varepsilon(u)\nabla\Delta u_\varepsilon)\Big| \le \tau\gamma^{-\frac{N+2}{2N}}|t_2-t_1|^{\frac12}.$$

Therefore, overall, interchanging x and x_0 yields

$$|u_\varepsilon(x,t_2)-u_\varepsilon(x,t_1)| \le 2K|x-x_0|^{\frac12}+\tau\gamma^{-\frac{N+2}{2N}}|t_2-t_1|^{\frac12}, \quad \text{with} \quad N \text{ odd},$$

$$|u_\varepsilon(x,t_2)-u_\varepsilon(x,t_1)| \le 2K|x-x_0|+\tau\gamma^{-\frac{N+2}{2N}}|t_2-t_1|^{\frac12}, \quad \text{with} \quad N \text{ even}.$$

Thus, taking $2K<\tau$, $|x-x_0|<\gamma$, and $\gamma<|t_2-t_1|^\beta$, we obtain that

$$|u_\varepsilon(x,t_2)-u_\varepsilon(x,t_1)| \le \tau|t_2-t_1|^{\frac{\beta}{2}}+\tau|t_2-t_1|^{-\frac{(N+2)\beta}{2N}+\frac12}, \quad \text{with} \quad N \text{ odd},$$

$$|u_\varepsilon(x,t_2)-u_\varepsilon(x,t_1)| \le \tau|t_2-t_1|^{\beta}+\tau|t_2-t_1|^{-\frac{(N+2)\beta}{2N}+\frac12}, \quad \text{with} \quad N \text{ even}.$$

Consequently, taking $\beta=\frac{N}{2(N+1)}$, if N is odd, and $\beta=\frac{N}{3N+2}$ when N is even, completes the proof. □

5.3. **Passing to the limit.** To conclude this section, we show existence of weak solutions for the degenerate parabolic problem (1.1) passing to the limit as ε goes to zero. However, we must admit from the beginning that, from the analysis performed below, it is not possible to assure which the limit will be (the solution of some CP or maybe the solution of some FBP).

By Proposition 5.1, since, for bounded supports Ω, the embedding $H_0^1(\Omega)\hookrightarrow L^2(\Omega)$ is compact, we can extract a convergent subsequence in $L^2(\Omega)$ as $\varepsilon\downarrow 0$ for the solutions of (5.1) labelled again $u_\varepsilon(x,t)$ such that

$$\lim_{\varepsilon\to 0}\big\|u_\varepsilon(\cdot,t)-U(\cdot,t)\big\|_{L^2(\Omega)}=0. \tag{5.10}$$

Consequently, the convergence of the non-degenerate solutions of the problem (5.1) is strong in $L^2(\Omega)$.

Moreover, thanks to the Hölder continuity proved in Lemmas 5.1 and 5.2, we have a strong convergence as $\varepsilon\downarrow 0$ in $C^{0,\frac12,\frac{N}{4(N+1)}}(\bar\Omega\times[0,T])$, when N is odd, and in $C^{0,1,\frac{N}{3N+2}}(\bar\Omega\times[0,T])$, when N is even. This is possible after applying the Ascoli–Arzelá Theorem, since $\{u_\varepsilon\}$ is uniformly bounded and equicontinuous in $\bar\Omega\times[0,T]$. Of course, these estimates can imply other even stronger convergence results, which are not treated below in detail.

Note that one difficulty we face is whether this limit depends on the taken subsequence or not. In other words, this analysis does not include any uniqueness result, which is expected to be a more difficult open problem for such nonlinear degenerate parabolic TFEs in non-fully divergence form and with non-monotone operators. However, the principal issue of the analytic regularization via (5.1) is that it is expected to lead to a smoother solution at the interface than those for the standard FBP. The difference is that the analytic regularized family $\{u_\varepsilon\}$, in addition to (2.1), is assumed to guarantee that, a.e. on the interface (assumed now sufficiently smooth),

$$\frac{\partial^2 u}{\partial \mathbf{n}^2}=0. \tag{5.11}$$

In fact, proper oscillatory solutions of the CP are assumed to exhibit even more regularity at smooth interfaces [14]:

$$\frac{\partial^l u}{\partial \mathbf{n}^l} = 0, \quad \text{where} \quad l = \left[\frac{3}{n}\right] - 1. \tag{5.12}$$

Therefore, as $n \to 0^+$, the smoothness of such solutions at the interfaces increases without bounds. Obviously, this is not the case for the FBP (a "positive obstacle" one) with standard conditions (2.1) and a usual quadratic ("parabolic") decay at the interfaces.

Thus, as above and customary, multiplying (5.1) by a test function $\varphi \in C_0^\infty(\bar{\Omega} \times (0,T))$ and integrating by parts in $\Omega \times [0,T]$ gives

$$-\int_0^T \int_\Omega \varphi_t u_\varepsilon - \int_0^T \int_\Omega \nabla\varphi \cdot (\phi_\varepsilon(u) \nabla \Delta u_\varepsilon) = 0.$$

Next, operating with this equality, we find that

$$\int_0^T \int_\Omega \varphi_t u_\varepsilon + \varepsilon^n \int_0^T \int_\Omega \nabla\varphi \cdot \nabla\Delta u_\varepsilon + (1-\varepsilon) \int_0^T \int_\Omega \nabla\varphi \cdot ((\varepsilon^2 + u^2)^{\frac{n}{2}} \nabla\Delta u_\varepsilon) = 0. \tag{5.13}$$

Applying Hölder's inequality, it is clear from (5.3) that there exists a subsequence labeled by $\{\varepsilon_k\}$ such that the second term of (5.13) approximates zero as $\varepsilon_k \downarrow 0$ for a sufficiently small, $n \approx 0$, $n > 0$,

$$\left|\varepsilon_k^n \int_0^T \int_\Omega \nabla\varphi \cdot \nabla\Delta u_{\varepsilon_k}\right| \le \varepsilon_k \Big(\varepsilon_k^{2(n-1)} \int_0^T \int_\Omega (\nabla\Delta u_{\varepsilon_k})^2\Big)^{\frac{1}{2}} \Big(\int_0^T \int_\Omega |\nabla\varphi|^2\Big)^{\frac{1}{2}} \le K\varepsilon_k \downarrow 0,$$

as $\varepsilon_k \downarrow 0$, for some positive constant $K > 0$.

Moreover, on the subset $\mathcal{P} := \{(x,t) \in \Omega \times [0,T]\,;\, |u(x,t)| > \delta > 0\}$, for any arbitrarily small $\delta > 0$, it is clear that the limiting solution as $n \to 0$ is a weak solution of the bi-harmonic equation (1.3). Indeed, by the regularity of the uniformly parabolic equation (5.1) and the uniformly Hölder continuity of its solutions proved in Lemmas 5.1 and 5.2, we obtain that $u_{\varepsilon,t}$, ∇u_ε, Δu_ε, $\nabla\Delta u_{\varepsilon,x}$, and $\Delta^2 u_\varepsilon$ converge uniformly on compact subsets of $\mathcal{P}$. In general, it is not that difficult to see that, as $\varepsilon = \varepsilon_k \to 0$ (along the lines of classic results in [5] and related others), we obtain a weak solution of the TFE–4, i.e.,

$$\int_0^T \int_{\mathcal{P}} \varphi_t U + \int_0^T \int_{\mathcal{P}} \nabla\varphi \cdot |U|^n \nabla\Delta U = 0, \tag{5.14}$$

where $U(x,t)$ is the limit obtained through (5.10). We naturally assume that $\varphi \in C_0^\infty(\mathcal{P})$.

However, in the "bad" subset $\{|u| \le \delta\}$, for any sufficiently small $\delta \ge 0$, we must take $\varepsilon > 0$ sufficiently small and depending on δ. Indeed, we take ε such that $0 < \varepsilon \le \delta$. Thus, applying Hölder's inequality to the third term in (5.13) over the subspace where $|u| \le \delta$, we have that

$$\Big|\int_0^T \int_{\{|u|\le\delta\}} \nabla\varphi \cdot ((1-\varepsilon)(\varepsilon^2 + u_\varepsilon^2)^{\frac{n}{2}} \nabla\Delta u_\varepsilon)\Big|$$

$$\le \Big(\int_0^T \int_{\{|u|\le\delta\}} |\nabla\varphi|^2\Big)^{\frac{1}{2}} \Big(\int_0^T \int_{\{|u|\le\delta\}} (1-\varepsilon)^2 (\varepsilon^2 + u_\varepsilon^2)^n |\nabla\Delta u_\varepsilon|^2\Big)^{\frac{1}{2}}.$$

Then, since $\varphi \in C_0^\infty(\bar\Omega \times (0,\infty))$ and $\varepsilon \in (0,1)$, we get

$$\Big|\int_0^T \int_{\{|u|\le\delta\}} \nabla\varphi \cdot ((1-\varepsilon)(\varepsilon^2+u_\varepsilon^2)^{\frac{n}{2}} \nabla\Delta u_\varepsilon)\Big| \le C\Big(\int_0^T \int_{\{|u|\le\delta\}} (1-\varepsilon)(\varepsilon^2+u_\varepsilon^2)^n |\nabla\Delta u_\varepsilon|^2\Big)^{\frac12}$$

for some positive constant $C > 0$. Making use of the fact that $|u| \le \delta$ and by (5.4), we find that

$$\Big|\int_0^T \int_{\{|u|\le\delta\}} \nabla\varphi \cdot ((1-\varepsilon)(\varepsilon^2+u_\varepsilon^2)^{\frac{n}{2}} \nabla\Delta u_\varepsilon)\Big| \le$$
$$C\Big(\int_0^T \int_{\{|u|\le\delta\}} (1-\varepsilon)(\varepsilon^2+\delta^2)^{\frac{n}{2}} (\varepsilon^2+u_\varepsilon^2)^{\frac{n}{2}} |\nabla\Delta u_\varepsilon|^2\Big)^{\frac12},$$

and, hence, using (5.8),

$$\Big|\int_0^T \int_{\{|u|\le\delta\}} \nabla\varphi \cdot ((1-\varepsilon)(\varepsilon^2+u_\varepsilon^2)^{\frac{n}{2}} \nabla\Delta u_\varepsilon)\Big| \le C_1 \delta^{\frac{n}{2}} \sim C_1 \varepsilon^{\frac{n}{2}}, \tag{5.15}$$

for some constant $C_1 > 0$ and taking $\varepsilon \sim \delta$ sufficiently small.

Finally, the estimate (5.15) shows the actual rate of the limit as $n \to 0$, together with $\varepsilon \to 0$, in the analytic approximating flow (5.1) to get in this limit weak (and hence classic by standard parabolic theory) solutions of the bi-harmonic equation (1.3). Namely, one has to have that

$$n = n(\varepsilon) \to 0 \ \text{ such that } \ \ \varepsilon^{\frac{n(\varepsilon)}{2}} \to 0 \implies n(\varepsilon) \gg \tfrac{1}{|\ln \varepsilon|} \ \text{as } \varepsilon \to 0^+. \tag{5.16}$$

However, this is not the end of the problem: indeed, under the condition (5.16) on the parameters, we definitely arrive at the limit ε, $n(\varepsilon) \to 0$ to the weak solution of the bi-harmonic equation written in the following "mild" form:

$$\textstyle\int_0^T \int_\Omega \varphi_t U + \int_0^T \int_\Omega \nabla\varphi \cdot \nabla\Delta U = 0. \tag{5.17}$$

This is not a full definition of weak solutions, since it assumes just a single integration by parts, so allows us also positive solutions of the "obstacle" FBP for (1.3) with the corresponding conditions (2.1) (with $n = 0$), which can be constructed by "singular" regularization as in [5].

Thus, unfortunately, our analysis still does not recognize the desired difference between oscillatory solutions of the CP and others (possibly positive ones) of the standard FBP and others that can be posed for the TFE–4 (1.1). Nevertheless, this first step in a homotopy analysis declares useful estimates and bounds on the parameters of regularization such as (5.16), which are absolutely necessary for passing to the limit to get sign changing solutions of the linear bi-harmonic flow.

Then, the homotopy concept as a connection to the linear PDE (5.1) can describe the origin (at $n = 0$) of the oscillatory solutions of TFEs and hence establish a transition to the *maximal regularity* of the solutions of (1.1). Indeed, inevitably, bearing in mind the oscillatory character of the kernel $F(|y|)$ of the fundamental solution, the proper solutions of the CP are going to be oscillatory near finite interfaces at least for small $n > 0$.

5.4. **Final remark: towards a full homotopy approach.** The main obstacle faced in obtaining a full exhibition of such a homotopic approach, when $n \to 0^+$, is the study of difficult pointwise limits near interfaces, where key singularities (a kind of Riemann's problems) occur. In particular, as a clue, let us mention a proper topology, in which we can define the homotopy. This is clear when we transform (1.1) (here, for simplicity, avoiding ε-regularization as in (5.1)) into a perturbation of the bi-harmonic equation (1.3):

$$u_t = -\Delta^2 u + g_n(u), \quad \text{where} \quad g_n(u) := \nabla \cdot ((1 - |u|^n)\nabla\Delta u)\,, \tag{5.18}$$

which we write down as an integral equation using the compact resolvent $(\mathbf{B} - \lambda I)^{-1}$ and the semigroup from Section 3. We then deal with the integral equation

$$u(t) = b(t) * u_0 + \textstyle\int_0^t b(t-s) * g_n(u(s))\,\mathrm{d}s, \tag{5.19}$$

where $b(t)$ is the fundamental solution (3.2). As usual, the integral form (5.19) allows us to weaken the necessary treatment of the perturbation $g_n(u)$, which is assumed to be small as $n \to 0$. However, this does not rule out the principal difficulty concerning such an unusual and very sensitive perturbation $g_n(u)$.

Thus, the main open problem of the homotopy issues is as follows: under which "topology-functional-geometric" setting for admitted solutions $u(x,t)$,

$$\boxed{\textstyle\int_0^t b(t-s) * g_n(u(s))\,\mathrm{d}s \to 0 \quad \text{a.e. as} \quad n \to 0^+.} \tag{5.20}$$

In particular, it is not difficult to see that, for sufficiently smooth functions u with a finite number of transversal zero surfaces uniformly in small $n \geq 0$, we have that

$$g_n(u) \to 0 \quad \text{as} \quad n \to 0^+\,.$$

at least a.e., and in other natural (weighted) topologies associated with the operator $\mathbf{B}$ and/or others. This can even be true uniformly on compact subsets, if the differential operators in u are bounded on such special functions, whose regularity and the "geometric transversal structure" near zero surfaces well-correspond to the desired maximal regularity/structure that are generic for the TFE solutions. But such a detailed *a priori* information on solutions seems to be excessive and not required. Taking the full homotopic approach as $\varepsilon\,, n \to 0$, we desperately need to understand the structure of the zero surfaces, because of the oscillatory behaviour of these solutions of changing sign. Away from small neighbourhoods of such zeros (zero curves or surfaces), there is no any essential problem. The structural properties of zeros of solutions of the TFE–4 (1.1) and the TFE–6 have been discussed in [14, 16], and the results therein inspire us with a certain optimism concerning the correctness of the general homotopy approach to the CP for the TFEs, though difficulties are far away from being properly settled in a general setting.

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Centro di Ricerca Matematica Ennio De Giorgi, Scuola Normale Superiore, 56100-Pisa, ITALY

E-mail address: alvcau.pablo@gmail.com

Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK

E-mail address: vag@maths.bath.ac.uk