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ESTIMATES OF THE DERIVATIVES OF MINIMIZERS OF A SPECIAL CLASS OF VARIATIONAL INTEGRALS

Maria Alessandra Ragusa

Dipartimento di Matematica, Università di Catania,
Università di Catania,
Viale A. Doria, 6, 95125 Catania, Italia

Atsushi Tachikawa

Department of Mathematics, Faculty of Science and Technology,
Tokyo University of Science,
Noda, Chiba, 278-8510, Japan

Abstract. The note concerns on some estimates in Morrey Spaces for the derivatives of local minimizers of variational integrals of the form

$$\int_\Omega F(x,u,Du)dx$$

where the integrand has the following special form

$$F(x,u,Du) \;=\; A(x,u,g^{\alpha\beta}(x)h_{ij}(u)\frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^i}{\partial x^\beta}),$$

where $(g^{\alpha\beta})$ and (h_{ij}) symmetric positive definite matrices. We are not assuming the continuity of A and g with respect to x. We suppose that $A(\cdot,u,t)/(1+t)$ and $g(\cdot)$ are in the class $L^\infty \cap VMO$.

1. Introduction.

Partial regularity results for solutions of nonlinear elliptic systems has been well studied by Morrey [32], Giusti [23], Giusti and Miranda [25] using an indirect argument similar to that one introduced by De Giorgi and Almgren in the regularity theory of parametric minimal surfaces. On the other hand, in Giaquinta-Ginsti [15] and Giaquinta-Modica [20] higher integrability of solutions has been proved and used to implement perturbation arguments to study regularity of solutions. Using this perturbation, or direct, argument in the present note the authors prove partial regularity for minimizers of the following variational integrals

$$\mathcal{A}(u;\Omega) := \int_\Omega F(x,u,Du)dx$$

where Ω is a domain of $\mathbb{R}^m$, $u:\Omega\to\mathbb{R}^n,\ Du=(D_\alpha u^i),\ \alpha=1,\dots,m,\ i=1,\dots,n.$ Throughout this paper we assume that the integrand $F(x,u,\xi):\Omega\times\mathbb{R}^n\times\mathbb{R}^{mn}$ is *p-growth,* namely, for some $0<\lambda_0<\Lambda_0$ and $\mu\in\mathbb{R}$, F satisfies

$$\lambda_0(\mu^2+|\xi|^2)^{p/2} \le F(x,u,\xi) \le \Lambda_0(\mu^2+|\xi|^2)^{p/2} \tag{1.1}$$

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for all $(x, u, \xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$. Moreover, we assume the following convexity condition

$$\lambda_0(\mu^2 + |\xi|^2)^{p/2}|\eta|^2 \leq \frac{\partial^2 F(x, u, \xi)}{\partial \xi^j_\beta \partial \xi^i_\alpha} \eta^i_\alpha \eta^j_\beta \leq \Lambda_1(\mu^2 + |\xi|^2)^{p/2}|\eta|^2 \tag{1.2}$$

for all $(x, u, \xi, \eta) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \times R^{mn}$.

When $F(x, u, \xi)$ is sufficiently smooth, especially continuous with respect to x, many regularity results are already known. In this paper, we consider the case that $F(\cdot, u, \xi)$ is in the class so-called VMO for every $(u, \xi) \in \mathbb{R}^n \times \mathbb{R}^{mn}$.

Here we should mention that if $\mu = 0$ the variational feature of the functional is very different from one for $\mu \neq 0$. If $\mu = 0$, the Euler-Lagrange equation of the functional is degenerate at a point $x \in \Omega$ where $Du(x) = 0$. In the previous paper [34], the authors proved partial regularity results for the case that $\mu \neq 0$. On the other hand, one of the most typical example of p-growth functional

$$\mathcal{D}_p(u) := \int_\Omega |Du|^p dx \tag{1.3}$$

does not satisfy the convexity condition (1.2) for $\mu \neq 0$.

In this paper, assuming that the integrand of the functional has special form

$$F(x, u, Du) = A(x, u, g^{\alpha\beta}(x)h_{ij}(u)D_\alpha u^i D_\beta u^j), \tag{1.4}$$

we will show a partial regularity result which holds even if $\mu = 0$. Here, $(g^{\alpha\,\beta})$ and (h_{ij}) are symmetric positive definite matrices having smooth coefficients. This kind of functionals are extensions of $\mathcal{D}_p(u)$ and arise as $p-$energy of maps between Riemannian manifolds. From this point of view, the geometric interest may occur on the above functionals. Moreover we observe that some methods of proofs of regularity for classes of nonlinear elliptic systems can also be applied to the equations of nonlinear Hodge theory, studied by a lot of people, for instance by L. M. Sibner and R. B. Sibner in [36]. In [18] Giaquinta and Giusti considered the quadratic functionals

$$\int_\Omega g^{\alpha\,\beta}(x)\, h_{ij}(x, u)\, D_\alpha u^i\, D_\beta u^j\, dx.$$

As an important example of the above type of functionals, the *energy functional* between Riemannian manifolds is known. Let (M, g) and (N, h) are Riemannian m- and n-manifolds respectively and let Ω be a bounded domain of M. Then, the *energy* of a map $u : \Omega \to N$ is defined as

$$\mathcal{E}(u; \Omega) := \int_\Omega g^{\alpha\,\beta}(x)\, h_{ij}(u)\, D_\alpha u^i\, D_\beta u^j \sqrt{g} dx,$$

where $(\, g^{\alpha\beta}\,)$ is the inverse matrix of the metric tensor $(\, g_{\alpha\beta}\,)$ of (M, g) and $g = \det(g_{\alpha\beta})$.

For example, the energy of a map from a unit disk $D^m \subset \mathbb{R}^m$ into a unit sphere S^m is given by

$$\int_{D^m} \frac{|Du|^2}{(1 + |u|^2)} dx$$

in local coordinates. A solution of the Euler-Lagrange equation of the Energy is called a *harmonic map* and has attracted great interests since a famous paper of Eells-Sampson [10] was published.

More generally, the *p-energy* of a map $u : \Omega \to N$ is given by

$$\mathcal{E}^p(u;\Omega) := \int_\Omega \left\{ g^{\alpha\,\beta}(x)\, h_{ij}(u)\, D_\alpha u^i\, D_\beta u^j \right\}^{p/2} \sqrt{g} dx,$$

and its critical point is called a *p-harmonic map.*

We mention the study of Giaquinta and Modica in [22] where partial regularity in the vector valued case and everywhere regularity in the scalar case for minimizers of variational integrals $\int F(x,u,Du)\,dx$ has proved if the integrands has the special structure

$$F(x,u,\xi) = A(x,u,|\xi|^2) \tag{1.5}$$

or, more generally,

$$F(x,u,\xi) = A(x,\ u,\ a^{\alpha\beta}(x,u)b_{ij}(x,u)\xi^i_\alpha \xi^j_\beta)$$

where $a^{\alpha\beta}$ and b_{ij} are symmetric positive definite matrices and $A(x,u,t)$ is of class C^2 with respect to t. For vector valued case, they proved that a minimizer of the functional is in the class $C^{0,\alpha}(\Omega_0)$ for some open subset $\Omega_0 \subset \Omega$ and that the Hausdorff dimension of the singular set $\Omega \setminus \Omega_0$, $\dim\mathcal{H}(\Omega \setminus \Omega_0)$, is less than $m-p-\varepsilon$ for some $\varepsilon \in (0,1)$ ([22, Theorem 4.2]). As a special case, their results give partial regularity of minimizers for p-energy of maps between Riemannian manifolds. About the regularity results for minimizers of p-enegy, see also a work by Hardt-Lin [26].

We should mention also the paper of Fusco-Hutchinson [13]. They treated functional of the form

$$\int \left\{ g^{\alpha\beta}(x,u) h_{ij}(x,u) D_\alpha u^i D_\beta u^j \right\}^{p/2} dx,$$

and got partial regularity result similar to the one of [22]. It is remarkable that in [13], for the case that $g^{\alpha\beta}$ depend only on x and are Hölder continuous, the estimate of the Hausdorff dimension of the singular set is improved. They proved that, for such a case, a minimizer u is in the class $C^{0,\alpha}(\Omega_0)$ with $\dim\mathcal{H}(\Omega\setminus\Omega_0) \le m-[p]-1$ where $[p]$ is the integer part of p ([13, Theorem 8.1]).

Under similar assumptions to that in the previous mentioned paper [22] it is proved by Ivert, Giaquinta, Giusti and Modica in [17], [19] and [21] that minimizers have Hölder continuous derivatives in an open set Ω_0 contained in Ω such that $\mathrm{meas}(\Omega\backslash\Omega_0) = 0$.

Without assuming some special structure such as (1.4), non-degeneracy is always assumed to get regularity results, as far as the authors know. Moreover, to get partial Hölder regularity, a condition on the relation between the dimension m and the growth order p, $p+2 \ge m$ is necessary. (See Corollary 3.2.) On the other hand, assuming the special structure (1.4), as one sees in the Theorem 4.1, we can show the partial Hölder regularity without any restriction on the dimension m. These differences arise from the difference for the fundamental regularity results for the functionals of types $\int F(Dv)dx$ and $\int g(|Dv|)dx$. In [7] Campanato proved that a weak solution $v : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$ of the Euler-Lagrange equation of a non-degenerate p-growth functional

$$\int_\Omega F(Dv)dx$$

satisfies for every $x \in \Omega$ and $R \le \mathrm{dist}(x,\partial\Omega)$

$$\int_{B(x,tR)} |Dv|^p dx \le Ct^\lambda \int_{B(x,R)} |Dv|^p dx,\ \ \lambda = \min\{2+\varepsilon, m\} \tag{1.6}$$

for some $\varepsilon > 0$. So, by Morrey's Dirichlet growth theorem or Campanato's theorem, we can deduce Hölder regularity if $m \le 2+p$. On the other hand, in [39], Uhlenbeck proved that a minimizer u of a functional of p-growth

$$\int_\Omega |Du(x)|^p dx$$

where $p \ge 2$, and more generally for local minimizers of

$$\int_\Omega g(|Du(x)|^2) dx$$

where $g(t^2)$ behaves like t^p, is in the class $C^{1,\alpha}$ for some some $\alpha \in (0,1)$, even if the functional is degenerate as $|Du|^p$ (for precise statement, see Theorem 2.5 below).

This fundamental result has been generalized in two different ways: in [13] and [22] dependence on the integrand (x,u) is allowed, and in [9], [30], [37] and [38] the case $1 < p < 2$ is studied. Under this assumption, regularity is proved in [30], [37] and [38] only for $n = 1$, which is the smooth case corresponds to a partial differential equation instead of a system, and in [30] and [37] only for quasilinear systems. Regularity results in the nonlinear case with $n > 1$, $1 < p < 2$ and dependence also on the variables (x,u) are obtained by Acerbi and Fusco in [1].

In this note we prove partial regularity for local minimiziers keeping in mind the results obtained for $p \ge 2$ by Uhlenbeck.

In general, when we try to deduce regularity of minimizers u of $\int F(x,u,Du)dx$ or $\int g(x,u,|Du|)dx$ using direct approach, we compare minimizers v of so-called frozen functionals $\int F(x_0,u_0,Dv)dx$ or $\int g(x_0,u_0,|Dv|)$ for some fixed x_0 and u_0. So, a similar difference occurs.

About p-harmonic maps between Riemannian manifolds Fuchs proved partial regularity in [12]. Moreover, in [11], he proved everywhere regularity of a energy minimizing map under the condition that its image is contained in the *regular ball.* A geodesic ball $B(p_0,R)$ on a Riemannian manifold N is called to be *regular* if it does not meet the cut locus of p_0 and $R < \pi/2\sqrt{\kappa}$, where κ denotes the supremum of the sectional curvatures of N. If the sectional curvatures of N are non positive the second condition is not necessary. Here, we should mention that in [11] the blow-up argument is employed. On the other hand, for the case that the coefficients of the considered equation are not continuous the blow-up argument does not work. Namely, in the cases that we treat here, we can not use the blow-up method.

The hypothesis we consider in this note has been inspired by the papers [6] and the subsequent [7], where Campanato obtains deep Hölder regularity results using $\mathcal{L}^{p,\lambda}$ spaces (see the definition e. g. in [5]) for solutions of elliptic systems having nonlinearity greater or equal to 2. In these notes the coefficients of the second order elliptic differential operators are supposed continuous, the VMO assumption is a more recent idea that cast one's mind back to the papers by Caffarelli [3], [4], Chiarenza, Frasca and Longo [8] and later object of interest for an ever increasing number of authors. See for example, Acerbi-Mingione [2], Kinnunen-Zhou [28].

About recent progress on the regularity theory, see, for example, a review by Mingione [31] and the references therein.

2. **Preliminaries and definitions.**

Let us suppose $r > 0$, $\mu \ge 0$, $p \ge 2$ and define by

$$B(x,r) = \{y \in \mathbb{R}^m : |y^\tau - x^\tau| < r,\ \tau = 1\dots,m\}$$

a generic ball in $\mathbb{R}^m$ centered at x with radius $2r$. Let us now give the definition of the Morrey spaces $L^{p,\lambda}$, in the sequel we are interested in the Morrey regularity of the gradient of u in these spaces

Definition 2.1. (see e.g. [33]). Let $1 < p < \infty, 0 \leq \lambda < m$. A measurable function $f \in L^1_{\text{loc}}(\Omega, \mathbb{R}^n)$ is in the Morrey class $L^{p,\lambda}(\Omega, \mathbb{R}^n)$ if the following norm is finite

$$\|f\|_{L^{p,\lambda}(\Omega)} = \sup_{\substack{0<\rho<\operatorname{diam}\Omega \\ x\in\Omega}} \frac{1}{\rho^\lambda} \int\limits_{\Omega\cap B(x,\rho)} |f(y)|^p dy,$$

where $B(x,\rho)$ ranges in the class of the balls above defined.

Definition 2.2. Let $f \in L^1(\Omega, \mathbb{R}^n)$ we set the integral mean $f_{x,R}$ by

$$f_{x,R} = \fint_{\Omega\cap B(x,R)} f(y)dy = \frac{1}{|\Omega \cap B(x,R)|} \int\limits_{\Omega\cap B(x,R)} f(y)dy$$

where $|\Omega \cap B(x,R)|$ is the Lebesgue measure of $\Omega \cap B(x,R)$.

If we are not interested in specifying which the center is, we only set f_R.

Let us now give the definition of bounded mean oscillation function (BMO) that appear at first in the note by John and Nirenberg [27].

Definition 2.3. Let $f \in L^1_{\text{loc}}(\mathbb{R}^m)$. We say that f belongs to $BMO(\mathbb{R}^m)$ if the seminorm

$$\|f\|_* \equiv \sup_{B(x,R)} \frac{1}{|B(x,R)|} \int\limits_{B(x,R)} |f(y) - f_{x,R}|dy < \infty$$

where $B(x,R)$ is defined as above.

Let us recall the definition of the space of vanishing mean oscillation functions, given at first by Sarason in [35].

Definition 2.4. Let $f \in BMO(\mathbb{R}^m)$ and

$$\eta(f,R) = \sup_{\rho\leq R} \frac{1}{|B(x,\rho)|} \int\limits_{B(x,\rho)} |f(y) - f_\rho|dy$$

where $B(x,\rho)$ ranges over the class of the balls of $\mathbb{R}^m$ of radius ρ. We say that $f \in VMO(\Omega)$ if

$$\lim_{R\to 0} \eta(f,R) = 0.$$

It is useful in the sequel to mention the following significant regularity result obtained by Uhlenbeck.

Theorem 2.5 ([39, Main Theorem], see also [22, Theorem 3.1])**.** *Let $v \in W^{1,p}_{loc}(\Omega, \mathbb{R}^n)$ be a local minimizer of the functional*

$$F(v,\Omega) = \int_\Omega f(Dv)dx$$

with integrand

$$f(Dv) = g(|Dv|^2)$$

satisfying the following assumptions:

(U-1) for some positive λ and Λ and for all ξ we have

$$\lambda(\mu^2 + |\xi|^2)^{\frac{p}{2}} \le f(\xi) \le \Lambda(\mu^2 + |\xi|)^{\frac{p}{2}}$$

where $\mu \ge 0$ and $p \ge 2$;

(U-2) the function $f(\xi)$ is of class C^2 and

$$|f_{\xi\xi}(\xi)| \le c_1(\mu^2 \, + \, |\xi|^2)^{\frac{p-2}{2}}$$

$$|f_{\xi\xi}(\xi) - f_{\xi\xi}(\tau)| \le c_2(\mu^2 \, + \, |\xi|^2 + |\tau|^2)^{\frac{p-2}{2} - \frac{\alpha}{2}} |\xi - \tau|^{\alpha}$$

for some positive α;

(U-3) the integrand $f(\xi)$ is elliptic in the sense that

$$f_{\xi^i_\alpha \xi^j_\beta}(\xi) \, \gamma^i_\alpha \gamma^j_\beta \ge (\mu^2 + |\xi|^2)^{\frac{p-2}{2}} \, |\gamma|^2 \qquad \forall \gamma \in \mathbb{R}^{mn}.$$

Then Dv is locally Hölder continuous with some exponent $\sigma \in (0, 1)$. Moreover for every $x_0 \in \Omega$, for all ρ, R with $0 < \rho < R < \operatorname{dist}(x_0, \partial\Omega)$ we have the estimates

$$\sup_{B(x_0,R/2)} |Dv|^p \le C \Big[\fint_{B(x_0,R)} |Dv|^p + \mu^p \Big], \tag{2.7}$$

$$\Phi(x_0, \rho) \le C \left(\frac{\rho}{R} \right)^{2\sigma} \Phi(x_0, R), \tag{2.8}$$

where

$$\Phi(x_0, r) := \fint_{B(x_0,r)} \Big| V(Dv) - \big(V(Dv)\big)_{x_0,r} \Big|^2 dx,$$

$$V(\xi) := \big(\mu^2 + |\xi|^2\big)^{(m-2)/4} \xi$$

Remark 2.6. We mention that for a minimizer v of $\int F(Dv)dx$ the following estimate can be deduce immediately from (2.7)

$$\fint_{B(x_0,\rho)} \big(\mu^2 + |Dv|^2\big)^{p/2} dx \le C \fint_{B(x_0,R)} \big(\mu^2 + |Dv|^2\big)^{p/2} dx \tag{2.9}$$

for any $x_0 \in \Omega$ and $0 < \rho < R < \operatorname{dist}(x_0, \partial\Omega)$. We will use the above estimate to prove our main result.

3. **General but non-degenerate case.** In this section we state partial Morrey- and Hölder-regularity results given in [34].

(F-1) There exist constants $\Lambda_1 > \lambda_1 > 0$ and $\mu \ne 0$ such that

$$\lambda_1(\mu^2 + |\xi|)^p \le F(x, u, \xi) \le \Lambda_1(\mu^2 + |\xi|)^p$$

$$\lambda_1(\mu + |\xi|^2)^{p/2-1}|\eta|^2 \le \frac{\partial^2 F(x, u, \xi)}{\partial \xi^i_\alpha \partial \xi^j_\beta} \eta^i_\alpha \eta^j_\beta \le \Lambda_1(\mu^2 + |\eta|^2)^{p/2-1}|\xi|^2$$

for all $(x, u, \xi, \eta) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \times \mathbb{R}^{mn}$;

(F-2) for every $(u, \xi) \in \mathbb{R}^n \times \mathbb{R}^{mn}$, $F(\cdot, u, \xi) \in VMO(\Omega)$ and the mean oscillation of $F(\cdot, u, \xi)/(\mu^2 + |\xi|^2)^{2/p}$ vanishes uniformly with respect to u, ξ in the following sense: there exist a positive number ρ_0 and a function $\sigma(z, \rho) : \mathbb{R}^m \times [0, \rho_0) \to [0, \infty)$ with

$$\lim_{R \to 0} \sup_{\rho < R} \fint_{B(0,\rho) \cap \Omega} \sigma(z, \rho) dz = 0, \tag{3.1}$$

such that $F(\cdot, u, \xi)$ satisfies for every $x \in \overline{\Omega}$ and $y \in B(x, \rho_0) \cap \Omega$

$$\left|F(y,u,\xi) - F_{x,\rho}(u,\xi)\right| \le \sigma(x-y,\rho)(\mu^2 + |\xi|^2)^{p/2}\ \forall (u,\xi) \in \mathbb{R}^n \times \mathbb{R}^{mn}, \tag{3.2}$$

where

$$F_{x,\rho}(u,\xi) = \fint_{B(x,\rho)\cap\Omega} F(y,u,\xi)dy;$$

(F-3) for every $x \in \Omega$, $\xi \in \mathbb{R}^{mn}$ and $u, v \in \mathbb{R}^n$

$$\left|F(x,u,\xi) - F(x,v,\xi)\right| \le (1 + |\xi|^2)^{\frac{p}{2}} \omega(|u-v|^2)$$

where ω is some monotone increasing concave function with $\omega(0) = 0$;

(F-4) for almost all $x \in \Omega$ and all $u \in \mathbb{R}^n$, $F(x, u, \cdot) \in C^2(\mathbb{R}^{mn})$.

As mentioned in the first section, in [34] we compare a minimizer u of

$$\int_\Omega F(x,u,Du)dx$$

with a minimizer v of a frozen functional

$$\int_{B(x_0,r)} F(x_0,u_r,Dv)dx,$$

where $x_0 \in \Omega$ with $B(x_0, r) \subset\subset \Omega$ and u_r is the integral mean of u on $B(x_0, r)$. By virtue of Campanato's result in [7], v satisfies (1.6). So, estimating $\int |Du - Dv|^p dx$, we get the following partial Morrey-regularity of u.

For the case that the integrand $F(x, u, p)$ is continuous in x, see Kristensen-Mingione [29].

Theorem 3.1 ([34, Theorem 2.5]). *Assume that $\Omega \subset \mathbb{R}^m$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$ and that $p \ge 2$. Let $u \in H^{1,p}(\Omega, \mathbb{R}^n)$ a minimizer of the functional*

$$\mathcal{F}(u,\Omega) = \int_\Omega F(x,u,Du)dx$$

in the class

$$X_g(\Omega) = \{u \in H^{1,p}(\Omega)\ ;\ u - g \in H_0^{1,p}(\Omega)\}$$

for a given boundary data $g \in H^{1,s}(\Omega)$ with $s > p$. Suppose that assumptions (F-1), (F-2), (F-3)*and* (F-4) *are satisfied. Then, for some positive ε, for every $0 < \tau < \min\{2 + \varepsilon, m(1 - \frac{p}{s})\}$ we have*

$$Du \in L^{p,\tau}(\Omega_0, \mathbb{R}^{mn}) \tag{3.3}$$

where Ω_0 is a relatively open subset of $\overline{\Omega}$ which satisfies

$$\overline{\Omega} \setminus \Omega_0 = \{x \in \Omega \colon \liminf_{R\to 0} \frac{1}{R^{m-p}} \int_{\Omega\cap B(x,R)} |Du(y)|^p dy > 0\}.$$

Moreover, we have

$$\mathcal{H}^{m-p-\delta}(\overline{\Omega} \setminus \Omega_0) = 0$$

for some $\delta > 0$, where $\mathcal{H}^r$ denotes the r-dimensional Hausdorff measure.

As a corollary of the above theorem we have the following partial Hölder regularity result.

Corollary 3.2 ([34, Corollary 2.6]). *Let g, u and Ω_0 be as in Theorem 3.1. Assume that $p + 2 \ge m$ and that $s > \max\{m, p\}$. Then, for some $\alpha \in (0, 1)$, we have*

$$u \in C^{0,\alpha}(\Omega_0, \mathbb{R}^n). \tag{3.4}$$

Moreover, as a corollary of the proof of Theorem 3.1, we have the following full-regularity result for the case that F does not depend on u.

Corollary 3.3. *Assume that F and g satisfy all assumptions of Theorem 3.1 and that F does not depend on u. Let u be a minimizer of $\mathcal{F}$ in the class X_g then*

$$Du \in L^{p,\tau}(\Omega, \mathbb{R}^{mn}). \tag{3.5}$$

Moreover, if $p+2 \geq m$ and $s > \max\{m,p\}$, we have full-Hölder regularity of u. Namely we have

$$u \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^n).$$

4. **Degenerate case.** Let the integrand function $A(x,u,t)$ be defined on $\Omega \times \mathbb{R}^n \times \mathbb{R}$, in the sequel we assume that it satisfies the following assumptions.

(A-1) There exist a constant $\mu \in \mathbb{R}$ and positive constants C, λ, Λ, $\lambda \leq \Lambda$ such that

$$\lambda(\mu^2 + t)^{\frac{p}{2}} \leq A(x,u,t) \leq \Lambda(\mu^2 + t)^{\frac{p}{2}}$$
$$\lambda(\mu^2 + t)^{\frac{p}{2}-1} \leq A_t(x,u,t) \leq \Lambda(\mu^2 + t)^{\frac{p}{2}-1}$$
$$\lambda(\mu^2 + t)^{\frac{p}{2}-2} \leq A_{tt}(x,u,t) \leq \Lambda(\mu^2 + t)^{\frac{p}{2}-2}$$

for all $(x,u,t) \in \Omega \times \mathbb{R}^n \times \mathbb{R}$, where the subscript t signifies differentiation with respect to t :

(A-2) for every $(u,t) \in \mathbb{R}^n \times \mathbb{R}$, $A(\cdot,u,t) \in VMO(\Omega)$ and the mean oscillation of $A(\cdot,u,t)/(\mu^2+|t|)^{(p/2)}$ vanishes uniformly with respect to u,t in the following sense: there exist a positive number ρ_0 and a function $\sigma(z,\rho) : \mathbb{R}^m \times [0,\rho_0[\to [0,+\infty[$ with

$$\lim_{r\to 0} \sup_{\rho<r} \fint_{B(0,\rho)\cap\Omega} \sigma(z,\rho)dz = 0, \tag{4.1}$$

such that $A(\cdot,u,t)$ satisfies, for every $x \in \Omega$ and $y \in B(x,\rho_0) \cap \Omega$,

$$\left|A(y,u,t) - A_{x,\rho}(u,t)\right| \leq \sigma(x-y,\rho)(\mu^2 + t)^{\frac{p}{2}}, \ \forall (u,t) \in \mathbb{R}^n \times \mathbb{R} \tag{4.2}$$

where

$$A_{x,\rho}(u,t) = \fint_{B(x,\rho)\cap\Omega} A(z,u,t)dz;$$

(A-3) for every $x \in \Omega$, $t \in \mathbb{R}$ and $u, v \in \mathbb{R}^n$,

$$\left|A(x,u,t) - A(x,v,t)\right| \leq \omega(|u-v|^2)(\mu^2 + t)^{\frac{p}{2}},$$

where $\omega : [0,\infty) \to [0,\infty)$ is some continuous monotone increasing concave function with $\omega(0) = 0$;

(A-4) for almost all $x \in \Omega$ and all $u \in \mathbb{R}^n$ $A(x,u,\cdot) \in C^2(\mathbb{R})$, and A_{tt} satisfies for some positive α :

$$\left|A_{tt}(x,u,t)t - A_{tt}(x,u,s)s\right| \leq c(\mu^2 + t + s)^{\frac{p-2}{2}-\alpha}|t-s|^\alpha$$

for some positive α;

(A-5) there exist constants $\lambda_0, \Lambda_0, \lambda_1, \Lambda_1$ $(0 < \lambda_i < \Lambda_i, \ i = 0,1)$ such that

$$\lambda_0|\zeta|^2 \leq g^{\alpha\beta}(x)\zeta_\alpha\zeta_\beta \leq \Lambda_0|\zeta|^2, \ \lambda_1|\eta|^2 \leq h_{ij}(u)\eta^i\eta^j \leq \Lambda_1|\eta|^2$$

for all $x \in \Omega$, $u, \zeta \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^n$;

(A-6) for every $u, v \in \mathbb{R}^n$

$$\left|h_{ij}(u) - h_{ij}(v)\right| \leq \omega(|u - v|^2)$$

where ω is some monotone increasing concave function such that $\omega(0) = 0$;

(A-7) $g^{\alpha\beta}$ are in the class $L^\infty \cap VMO(\Omega)$.

A *local minimizer* of the functional $\mathcal{A}$ is a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ which satisfies

$$\mathcal{A}(u; \mathrm{supp}\varphi) \le \mathcal{A}(u+\varphi; \mathrm{supp}\varphi)$$

for every $\varphi \in W^{1,p}_0(\Omega, \mathbb{R}^n)$.

Under the above assumptions we have the following partial regularity result which is an extension of the result of [22] to the case that the coefficient of the integrand $A(x,u,p)$ is VMO with respect to x.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$ and $p \ge 2$. Let also $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ be a minimizer of the functional*

$$\mathcal{A}(u, \Omega) = \int_\Omega F(x, u, Du)\, dx$$

being the integrand of the form

$$F(x,u,Du) = A(x, u, g^{\alpha\beta}(x) h_{ij}(u) D_\alpha u^i D_\beta u^j)\,.$$

Suppose that $A(x,u,t)$, $g^{\alpha\beta}$ and h_{ij} satisfy the assumptions (A-1) – (A-7) *. Then there exists an open set $\Omega_0 \subset \Omega$ such that $u \in C^{0,\alpha}(\Omega_0)$ for any $\alpha \in (0,1)$. Moreover, we have $\Omega \setminus \Omega_0 \subset \Sigma_1 \cup \Sigma_2$, where*

$$\Sigma_1 = \{x \in \Omega\ ;\ \sup_{R>0} |u_{x,R}| = +\infty\}$$
$$\Sigma_2 = \{x \in \Omega\ ;\ \liminf_{R\to+0} R^{p-m} \int_{B(x,R)} |Du|^p dx > 0\}$$

Furthermore, for some positive $\delta > 0$,

$$\mathcal{H}^{m-p-\delta}(\Omega \setminus \Omega_0) = 0$$

where $\mathcal{H}^r$ denotes the r-dimensional Hausdorff measure.

Example. By Theorem 4.1 we can treat p-energy for maps from a Riemmanian manifold whose metric has singularity. A simple example is given as follows. Let $h = (h_{ij}(u))$ be a smooth Riemmanian metric satisfying (A-6), $f : \mathbb{R}^m \to \mathbb{R}$ the function defined by

$$f(x) = \begin{cases} 2 & \text{for} \quad -(x^{m-1})^2 \le x^m \le (x^{m-1})^2, \\ \text{of class } C^2 & \text{for} \quad x \notin \{x \in \mathbb{R}^m\ ;\ x^{m-1} = x^m = 0\}, \\ 1 & \text{for} \quad x^m > 2(x^{m-1})^2,\ x^m < -2(x^{m-1})^2 \end{cases}$$

and $g^{\alpha\beta}(x) = f(x)\delta^{\alpha\beta}$, where $\delta^{\alpha\beta}$ is Kronecher's delta. It is easy to see that $g^{\alpha\beta}(x)$ is discontinuous at $\{x \in \mathbb{R}^m\ ;\ x^{m-1} = x^m = 0\}$. Put

$$F(x,u,\xi) = \left(f(x)\delta^{\alpha\beta} h_{ij}(u) \xi^i_\alpha \xi^j_\beta\right)^{\frac{p}{2}},$$

then $F(x,u,\xi)$ satisfy all assumptions of Theorem 4.1. On the other hand, because of the discontinuity of f, known results (see e.g. [22]) are not valid for F defined above.

In order to show the above theorem we prepare the following so-called *reverse Hölder inequality with increasing domain* which has been proved in [16], which always plays important role for getting regularity.

Proposition 4.2. *Let u be a minimizer of $\mathcal{A}(\cdot,\Omega)$, then there exists a constant $q_0 > 1$ such that for any $q \in (1, q_0)$ we have $Du \in L^q_{loc}(\Omega)$. Moreover for every $x_0 \in \Omega$ and R with $0 < R < \operatorname{dist}(x_0, \partial\Omega)$, the following reverse Hölder inequality holds:*

$$\Big(⨍_{B(x_0,R/2)} H(Du)^q dx\Big)^{\frac{1}{q}} \le c ⨍_{B(x_0,R)} H(Du)dx, \tag{4.3}$$

where c does not depend on R and x_0.

Now, we can prove our main theorem proceeding as in [22].

Proof of Theorem 4.1. Let us set $x_0 \in \Omega$, $R > 0$ $B(R) = B(x_0, R)$ and $B(2R) = B(x_0, 2R) \subset\subset \Omega$. For every $(u, t) \in \mathbb{R}^n \times \mathbb{R}^{mn}$ we define

$$A_R(u,t) \;=\; ⨍_{B(R)} A(y,u,t)dy, \quad u_R ⨍_{B(R)} u(y)dy, \quad g_R \;=\; ⨍_{B(R)} g(y)dy,$$

and

$$A_0(\zeta) \;=\; A_R(u_R,\, g_R\, h(u_R)\, \zeta\, \zeta)\,.$$

Here and in the sequel, we omit indices α, β, i, j when there is no doubt of confusion.

Now, let us consider the following "frozen functional".

$$\mathcal{A}_0(u) \;=\; \int_{B(R)} A_0(Du)\, dx \;=\; \int_{B(R)} A_R(u_R, g_R h(u_R)\, Du\, Du)\, dx.$$

Let also $v \in H^{1,p}(B(R))$ be a minimizer of $\mathcal{A}_0(\overline{v},\, B(R))$ in the set of functions

$$\{\overline{v} \in H^{1,p}(B(R))\;;\; u \,-\, \overline{v} \in H^{1,p}_0(B(R))\}$$

and $\omega \,=\, u \,-\, v$.

Moreover, as in [22], we put

$$H(\xi) = (\mu^2 + |\xi|^2)^{\frac{p}{2}} \tag{4.4}$$

Let $r < \frac{R}{2}$, in order to estimate $\int_{B(R)} H(Du)dx$, we observe that

$$H(Du) \le c(p)H(Dv) + c(p)\big\{H^{1/p}(Du) + H^{1/p}(Dv)\big\}^p$$

as in [22, p.79].

Since the conditions (A-1) and (A-4) imply (U-2), using the above mentioned theorem by Uhlenbeck for minimizers of functionals of the form

$$\mathcal{F}(v) = \int F(|Dv|)dx$$

we have that

$$\int_{B(R)} H(Dv)dx \le c\left(\frac{r}{R}\right)^m \int_{B(R/2)} H(Dv)dx \tag{4.5}$$

where c does not depend on r, R, x_0 (see (4.6) of [22]). From [22] formula (4.8) (see also [17] formula (2.9) for $p = 2$), we have

$$\int_{B(R)} |Dw|^p dx \le c\big\{\mathcal{A}_0(u) - \mathcal{A}_0(v)\big\} \;=$$

$$=c \int_{B(R)} \big[A_R(u_R, g_R h(u_R) Du\, Du) - A_R(u_R, g_R h(u_R) Dv Dv)\big]\, dx \;=$$

$$=c \int_{B(R)} \big[A_R(u_R, g_R\, h(u_R) Du\, Du) - A(x, u_R, g_R\, h(u_R) Du Du)\big]\, dx +$$

$$+c\int_{B(R)}\big[A(x,u_R,g_R\,h(u_R)Du\,Du)-A(x,u,g_R\,h(u_R)DuDu)\big]dx+$$

$$+c\int_{B(R)}\big[A(x,u,g_R\,h(u_R)Du\,Du)-A(x,u,g(x)\,h(u_R)DuDu)\big]dx+$$

$$+c\int_{B(R)}\big[A(x,u,g(x)\,h(u_R)Du\,Du)-A(x,u,g(x)\,h(u)DuDu)\big]dx+$$

$$+c\int_{B(R)}\big[A(x,u,g(x)\,h(u)Du\,Du)-A(x,u,g(x)\,h(v)DvDv)\big]dx+$$

$$+c\int_{B(R)}\big[A(x,v,g(x)\,h(v)Dv\,Dv)-A(x,v,g(x)\,h(u_R)DvDv)\big]dx+$$

$$+c\int_{B(R)}\big[A(x,v,g(x)\,h(u_R)Dv\,Dv)-A(x,v,g_R\,h(u_R)DvDv)\big]dx+$$

$$+c\int_{B(R)}\big[A(x,v,g_R\,h(u_R)Dv\,Dv)-A(x,u_R,g_R\,h(u_R)DvDv)\big]dx+$$

$$+c\int_{B(R)}\big[A(x,u_R,g_R\,h(u_R)Dv\,Dv)-A_R(u_R,g_R\,h(u_R)DvDv)\big]dx\,.$$

$$=(1)+(2)+(3)+(4)+(5)+(4')+(3')+(2')+(1')$$

The term (5) is less or equals zero because of u is a minimizer. In (1) and $(1')$ we use assumption (A-2), then

$$|(1)|+|(1')|\le\int_{B(R)}2\sigma(x-x_0,R)\big\{H(Du)+H(Dv)\big\}dx\,.$$

Using the minimality of u, the Hölder inequality and (4.3), we get

$$|(1)|+|(1')|\le c\cdot\left(\int_{B(R)}H(Du)dx\right)\cdot\left(\fint_{B(R)}\sigma(x-x_0,R)^{q'}dx\right)^{\frac{1}{q'}}\,. \tag{4.6}$$

The integrals in (2) and $(2')$ can be estimate using assumption (A-3), then

$$|(2)|+|(2')|\le\int_{B(R)}\big\{H(Du)\omega(|u_R-u|^2)+H(Dv)\omega(|u_R-v|^2)\big\}dx\,.$$

As for the estimates on $|(1)|+|(1')|$, we get

$$\begin{aligned}&|(2)|+|(2')|\le\\ \le\quad& c\cdot\left(\int_{B(R)}H(Du)dx\right)\\ &\cdot\left\{\Big(\fint_{B(R)}\omega(|u_R-u|^2)^{q'}dx\Big)^{\frac{1}{q'}}+\Big(\fint_{B(R)}\omega(|u_R-v|^2))^{q'}dx\Big)^{\frac{1}{q'}}\right\}.\end{aligned} \tag{4.7}$$

To estimate the term (3) and $(3')$ let us observe that

$$A(t)\,-\,A(s)=A_t\big((1-\theta)t\,+\,\theta\,s\big)\cdot(t-s),$$

where $t = g_R\, h(u_R)\, Du\, Du$ and $s = g(x)\, h(u_R)\, Du\, Du$. Then, using hypotheses (A-1), we obtain

$$
\begin{aligned}
&|A(x,u,g_R h(u_R)DuDu) - A(x,u,g(x)h(u_R)DuDu)| \le \\
\le\ & \Big|A_t\Big(x,u,\big[(1-\theta)g_R + \theta\cdot g(x)\big]h(u_R)DuDu\Big)\Big||h(u_R)||g_R - g(x)|\cdot|Du|^2 \le \\
\le\ & |g_R - g(x)|\cdot(\mu^2+|Du|^2)^{\left(\frac{p}{2}-1\right)}|Du|^2 = c\cdot|g_R - g(x)|\cdot H(Du).
\end{aligned}
$$

Using the above estimate and reverse Hölder inequality (4.3), we can estimate (3), and similarly (3′), as follows

$$
\begin{aligned}
&\int_{B(R)}\big[A(x,u,g_R\,h(u_R)Du\,Du) - A(x,u,g(x)\,h(u_R)DuDu)\big]dx \le \\
\le\ & c\cdot\int_{B(R)}|g_R - g(x)|\,H(Du)dx \le \\
\le\ & c\cdot\Big(\int_{B(R)}H(Du)^q dx\Big)^{\frac{1}{q}}\cdot\Big(\int_{B(R)}|g_R - g(x)|^{q'}\,dx\Big)^{\frac{1}{q'}} \le \\
\le\ & c\cdot\Big(\int_{B(2R)}H(Du)dx\Big)\cdot\Big(⨍_{B(R)}|g_R - g(x)|^{q'}\,dx\Big)^{\frac{1}{q'}}
\end{aligned}
$$

for some $q > 1$. Let us finally consider the term (4), similarly (4′). Using (A-6), (A-7) and the boundedness of g, we can estimate them as follows.

$$
\begin{aligned}
&\Big|A\big(x,u,g(x)\,h(u_R)Du\,Du\big) - A\big(x,u,g(x)\,h(u)DuDu\big)\Big| \le \\
\le\ & \Big|A_t\big(x,u,\big[(1-\theta)\,h(u_R) + \theta h(u)\big]\,g(x)Du\,Du\big)\Big|\times \\
& \times|g(x)|\,|h(u_R) - h(u)|\cdot|Du|^2 \le \\
\le\ & c\,|h(u_R) - h(u)|\,\big(\mu^2+|Du|^2\big)^{\left(\frac{p}{2}-1\right)}|Du|^2 \\
\le\ & c\,\omega\,(\,|u_R - u|^2\,)\,H(Du)\,.
\end{aligned}
$$

So we obtain, using the assumptions on A :

$$\begin{aligned}
&\int_{B(R)} |Dw|^p \, dx \le \\
\le \quad & c \int_{B(2R)} H(Du)dx \left[\Big(\fint_{B(R)} \sigma(x-x_0,R)^{q'} dx \Big)^{\frac{1}{q'}} + \right. \\
& + \Big(\fint_{B(R)} \omega(|u_R-u|^2)^{q'} dx \Big)^{\frac{1}{q'}} + \Big(\fint_{B(R)} \omega(|u_R-v|^2)^{q'} dx \Big)^{\frac{1}{q'}} + \\
& \left. + \Big(\fint_{B(R)} |g_R - g(x)|^{q'} \, dx \Big)^{\frac{1}{q'}} \right] \\
= \quad & I + II + III + IV.
\end{aligned} \tag{4.8}$$

Let us use (4.8) and (4.5), then in a similar way of (3.14) in [34], using Hölder inequality, Jensen and Poincare' inequality in II and III, and the assumption on g in IV, we have

$$\begin{aligned}
&\int_{B(R)} |Du|^p \, dx \le \\
\le \quad & C \Big\{ \Big(\frac{r}{R} \Big)^{\lambda} + \left(\fint_{B(R)} \sigma(x,R) dx \right)^{\frac{q-1}{q}} + \\
& + \omega \left(R^{p-m} \int_{Q(R)} |D\,u|^p dx \right)^{\frac{q-1}{q}} + \eta(g,R) \Big\} \cdot \int_{B(2R)} H(Du) dx.
\end{aligned} \tag{4.9}$$

Furthermore recalling hypotheses (A-2) we have

$$\fint_{B(R)} \sigma(x,R)dx \to 0, \;\; \eta(g,R) \to 0 \text{ as } R \to 0.$$

Finally, using a well-known iteration argument (see, for example [24, pp.317—318]) and [14, Theorem 6.2], we conclude the proof. □

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E-mail address: maragusa@dmi.unict.it
E-mail address: tachikawa_atsushi@ma.noda.tus.ac.jp