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ON THE LIFESPAN OF STRONG SOLUTIONS TO THE PERIODIC DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

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Abstract. An explicit lifespan estimate is presented for the derivative Schrödinger equations with periodic boundary condition.

1. **Introduction.** We consider the Cauchy problem for the following derivative nonlinear Schrödinger (DNLS) equation:

$$\begin{cases} i\partial_t u + \partial_x^2 u = \lambda \partial_x (|u|^{p-1} u), & t \in [0,T), \quad x \in \mathbb{T}, \\ u(0) = u_0, & x \in \mathbb{T} \end{cases} \tag{1}$$

on one-dimensional torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, where $p > 1$ and $\lambda \in \mathbb{C}\backslash\{0\}$. The aim of this paper is to study an explicit upper bound of lifespan of solutions for (1) in terms of the data u_0 in the case $\mathrm{Re}\lambda \neq 0$.

The original DNLS equation is DNLS on $\mathbb{R}$ with $p = 3$ and $\lambda = -i$ with additional terms, which was derived in plasma physics for a model of Alfvén wave (see [13, 18]). By a simple computation, if $\lambda \in i\mathbb{R}$, then we have the charge (L^2) conservation law for solutions of DNLS with any $p > 1$ in the case of both torus and Euclidean space. The well-posedness for the original DNLS without additional terms has been studied, for example, in [2, 3, 7, 8, 9, 10, 12, 17, 20, 24]. Furthermore, the Cauchy problem (1) with $p = 3$ and $\lambda \in i\mathbb{R}$ has also been studied, for example, in [1, 6, 11, 15, 21, 23, 24]. Especially, the global well-posedness of (1) in the frame work of $H^s(\mathbb{T})$ with $s \geq 1/2$ has been studied by Win [25] and Mosincat [14] in the case where the charge of initial data is sufficiently small. Here $H^s(\mathbb{T})$ denotes the standard Sobolev space defined by $H^s(\mathbb{T}) = (1-\Delta)^{-s/2} L^2(\mathbb{T})$. However, the blowup problem for DNLS is still open in a general setting, where the conservation

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law is insufficient or fails, for examples, in the case where $p = 3$, $\lambda = i\mathbb{R}$, and the charge of initial data is large. Partial results have been obtained in [22].

On the other hand, Sunagawa [19] studied the finite time blow-up of

$$\begin{cases} i\partial_t u + \partial_x^2 u = \lambda |u|^{p-1}\partial_x u, & t \in [0,T), \quad x \in \mathbb{T}, \\ u(0) = u_0, & x \in \mathbb{T} \end{cases} \tag{2}$$

when $p = 3$ and $\mathrm{Re}\lambda \neq 0$, where (2) is connected with the gauge transformation of (1). Indeed let v be a solution for

$$\begin{cases} i\partial_t v + \partial_x^2 v = -i\partial_x(|v|^2 v), & t \in [0,T), \quad x \in \mathbb{R}, \\ v(0) = v_0, & x \in \mathbb{R}. \end{cases}$$

Then the gauge transformed solution w defined by

$$w(t,x) = v(t,x)\exp\left(\frac{i}{2}\int_{-\infty}^{x} |v(t,y)|^2 dy\right)$$

satisfies

$$i\partial_t w + \partial_x^2 w = -i|w|^2\partial_x w, \qquad t \in [0,T), \quad x \in \mathbb{R}.$$

For related subjects, we refer the reader [8, 9, 10, 11]. He showed that solutions of (2) blow up when

$$-\mathrm{sgn}(\mathrm{Re}\lambda)\cdot \mathrm{Im}\int_{\mathbb{T}} u_0(x)\overline{\partial_x u_0(x)}dx > 0.$$

Namely, in the case where $\mathrm{Re}\lambda \neq 0$, solutions may blow up even when their charge is arbitrary small. We remark that the condition $p = 3$ plays a crucial role in his argument.

In this article, we study the finite time blowup of solutions for (1) in the case where $\mathrm{Re}\lambda \neq 0$ and $p > 1$ by using a simple ODE argument. We remark that in this case, the conservation law fails and will show that there exists no L^2 global solution in a certain case. For the ODE approach, we refer the reader [4, 5, 16].

An obvious global solution for (1) is $u(t,x) = C$ for $C \in \mathbb{C}$. So it is necessary to consider a set of initial data without constants in order to show the finite time blowup of (1). Here we consider the initial data and solutions with vanishing total density defined as follows:

Definition 1.1. For $u_0 \in H^2(\mathbb{T})$ satisfying $\int_{\mathbb{T}} u_0(x)dx = 0$, u is called a strong solution with vanishing total density of the Cauchy problem (1) if there exists $T \in (0,\infty]$ such that $u \in C^1([0,T); H^2(\mathbb{T}))$ satisfies (1) and $\int_{\mathbb{T}} u(t,x)dx = 0$ for any $t \in [0,T)$.

Remark 1. Formally,

$$\frac{d}{dt}\int_{\mathbb{T}} u(t,x)dx = (2\pi)^{1/2}\frac{d}{dt}\hat{u}(t,0) = -i(2\pi)^{1/2}\mathfrak{F}\big[-\partial_x^2 u + \lambda\partial_x(|u|^{p-1}u)\big](0) = 0.$$

This implies that if $\int_{\mathbb{T}} u_0(x)dx = 0$, then $\int_{\mathbb{T}} u(t,x)dx = 0$ for any $t \in [0,T)$.

In this article, for $H^2(\mathbb{T})$ initial data with vanishing total density, we assume the existence of strong solutions with vanishing total density. We define the lifespan T_0 of a strong solution u to the Cauchy problem (1) by

$$T_0 = \sup\{T > 0;\ u \text{ is a strong solution for } (1)\}.$$

Then, from the ordinary differential inequality for $\displaystyle\int_0^{2\pi}\int_0^x u(\cdot,x)\overline{u(\cdot,y)}dy\,dx$, we may obtain the equivalent conditions for the finite time blowup for (1) and estimate of lifespan.

Proposition 1. *Let $u_0 \in L^2(\mathbb{T})$ satisfy $\displaystyle\int_{\mathbb{T}} u_0(x)dx = 0$. Then the following statements are equivalent:*

(i) u_0 satisfies

$$\mathrm{Re}\lambda \cdot \mathrm{Im}\int_0^{2\pi}\int_0^x u_0(x)\overline{u_0(y)}dy\,dx > 0. \tag{3}$$

(ii) There exists $\alpha \in \mathbb{C}$ such that

$$\mathrm{Re}\,\alpha \cdot \mathrm{Re}\lambda > 0, \quad \mathrm{Im}\left(\alpha\int_0^{2\pi}\int_0^x u_0(x)\overline{u_0(y)}dy\,dx\right) > 0. \tag{4}$$

If u_0 satisfies one of the equivalent conditions above and $u_0 \in H^2(\mathbb{T})$, then the corresponding strong solution with vanishing total density of the Cauchy problem (1) blows up in finite time. Moreover, the associated lifespan is estimated by

$$T_0 \le \frac{(2\pi)^p}{(p-1)|\mathrm{Re}\,\lambda|}\left|\int_0^{2\pi}\int_0^x u_0(x)\overline{u_0(y)}dy\,dx\right|^{-\frac{p-1}{2}}.$$

Remark 2. For $f \in L^2(\mathbb{T})$ with vanishing total density,

$$\mathrm{Re}\int_0^{2\pi}\int_0^x f(x)\overline{f(y)}dy\,dx = \frac{1}{2}\int_0^{2\pi}\frac{d}{dx}\left|\int_0^x f(y)dy\right|^2 dx = \frac{1}{2}\left|\int_0^{2\pi} f(x)dx\right|^2 = 0.$$

This means

$$\int_0^{2\pi}\int_0^x f(x)\overline{f(y)}dy\,dx \in i\mathbb{R}.$$

Then, (3) and (4) can be rewritten by

$$-i\mathrm{Re}\,\lambda \cdot \int_0^{2\pi}\int_0^x u_0(x)\overline{u_0(y)}dy\,dx > 0,$$

$$\mathrm{Re}\,\alpha \cdot \mathrm{Re}\lambda > 0, \quad -i\mathrm{Re}\,\alpha \cdot \int_0^{2\pi}\int_0^x u_0(x)\overline{u_0(y)}dy\,dx > 0,$$

respectively. This rewriting implies the equivalence between (3) and (4).

Remark 3. In contrast to nonlinear Schrödinger equation of nonlinearity without derivative and $\mathrm{Re}\lambda \neq 0$, it is unknown that whether Virial type identity works or not. On the other hand, in order to show Proposition 1, we show that $\displaystyle\int_0^{2\pi}\int_0^x u(\cdot,x)\overline{u(\cdot,y)}dy\,dx$, is a super solution of an ordinary differential equation and which implies that the amount blows up at a finite time.

2. **Proof of Proposition 1.** Let $M_\alpha(t) = \operatorname{Im}\Big(\alpha \int_0^{2\pi}\int_0^x u(t,x)\overline{u(t,y)}dy\,dx\Big)$, where α satisfies (4). Then $M_\alpha(t) > 0$ for sufficiently small t. By a direct calculation, we have

$$\begin{aligned}\frac{d}{dt}M_\alpha(t) =&\operatorname{Im}\Big(\alpha \int_0^{2\pi}\int_0^x \partial_t u(t,x)\overline{u(t,y)}dy\,dx\Big)\\ &+\operatorname{Im}\Big(\alpha \int_0^{2\pi}\int_0^x u(t,x)\overline{\partial_t u(t,y)}dy\,dx\Big)\\ =&I_1+I_2.\end{aligned}$$

By the vanishing total density, I_1 and I_2 may be computed as follows:

$$\begin{aligned}I_1 =&-\operatorname{Re}\Big(\alpha \int_0^{2\pi} i\partial_t u(t,x)\int_0^x \overline{u(t,y)}dy\,dx\Big)\\ =&-\operatorname{Re}\Big(\alpha \int_0^{2\pi} \partial_x(-\partial_x u(t,x)+\lambda(|u(t,x)|^{p-1}u(t,x)))\int_0^x \overline{u(t,y)}dy\,dx\Big)\\ =&-\operatorname{Re}\Big(\alpha(-\partial_x u(t,2\pi)+\lambda(|u(t,2\pi)|^{p-1}u(t,2\pi)))\int_0^{2\pi}\overline{u(t,y)}dy\Big)\\ &+\operatorname{Re}\Big(\alpha\int_0^{2\pi} -\overline{u(t,x)}\partial_x u(t,x)+\lambda|u(t,x)|^{p+1}dx\Big)\\ =&\operatorname{Re}\Big(\alpha\int_0^{2\pi} -\overline{u(t,x)}\partial_x u(t,x)+\lambda|u(t,x)|^{p+1}dx\Big),\\ I_2 =&\operatorname{Re}\Big(\alpha\int_0^{2\pi} u(t,x)\int_0^x \overline{i\partial_t u(t,y)}dy\,dx\Big)\\ =&\operatorname{Re}\Big(\alpha\int_0^{2\pi} u(t,x)\overline{(-\partial_x u(t,x)+\lambda(|u(t,x)|^{p-1}u(t,x)))}dx\Big)\\ &-\operatorname{Re}\Big(\alpha\overline{(-\partial_x u(t,0)+\lambda(|u(t,0)|^{p-1}u(t,0)))}\int_0^{2\pi} u(t,x)dx\Big)\\ =&\operatorname{Re}\Big(\alpha\int_0^{2\pi} u(t,x)\overline{(-\partial_x u(t,x)+\lambda(|u(t,x)|^{p-1}u(t,x)))}dx\Big).\end{aligned}$$

Then,

$$\begin{aligned}\frac{d}{dt}M_\alpha(t) =&-\operatorname{Re}\alpha\cdot\int_0^{2\pi} 2\operatorname{Re}(u(t,x)\overline{\partial_x u(t,x)})dx + 2\operatorname{Re}\alpha\cdot\operatorname{Re}\lambda\|u(t)\|_{L^{p+1}(\mathbb{T})}^{p+1}\\ =&-\operatorname{Re}\alpha\cdot\int_0^{2\pi}\partial_x|u(t,x)|^2dx + 2\operatorname{Re}\alpha\cdot\operatorname{Re}\lambda\|u(t)\|_{L^{p+1}(\mathbb{T})}^{p+1}\\ =&2\operatorname{Re}\alpha\cdot\operatorname{Re}\lambda\|u(t)\|_{L^{p+1}(\mathbb{T})}^{p+1}.\end{aligned}$$

Since

$$|M_\alpha(t)| \le |\operatorname{Re}\alpha|\|u(t)\|_{L^1(\mathbb{T})}^2 \le (2\pi)^{\frac{2p}{(p+1)}}|\operatorname{Re}\alpha|\|u(t)\|_{L^{p+1}(\mathbb{T})}^2,$$

we have

$$\frac{d}{dt}M_\alpha(t) \ge 2(2\pi)^{-p}|\operatorname{Re}\alpha|^{-\frac{p+1}{2}}\operatorname{Re}\alpha\cdot\operatorname{Re}\lambda M_\alpha(t)^{\frac{p+1}{2}}.$$

This and $M_\alpha(0) > 0$ implies

$$M_\alpha(t) \ge (M_\alpha(0)^{-\frac{p-1}{2}} - (p-1)(2\pi)^{-p}|\operatorname{Re}\alpha|^{-\frac{p+1}{2}}\operatorname{Re}\alpha\cdot\operatorname{Re}\lambda\, t)^{-\frac{2}{p-1}} > 0.$$

Therefore

$$T_0 \leq \inf\left\{ \frac{(2\pi)^p |\mathrm{Re}\,\alpha|^{\frac{p+1}{2}}}{(p-1)\mathrm{Re}\,\alpha\cdot\mathrm{Re}\,\lambda} M_\alpha(0)^{-\frac{p-1}{2}};\ \alpha\in\mathbb{C},\ \mathrm{Re}\,\alpha\cdot\mathrm{Re}\lambda>0\right\}$$
$$\leq \frac{(2\pi)^p}{(p-1)|\mathrm{Re}\,\lambda|}\left|\int_0^{2\pi}\int_0^x u_0(x)\overline{u_0(y)}dy\,dx\right|^{-\frac{p-1}{2}}.$$

Here the infimum may be attained with $\alpha=\lambda$.

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REFERENCES

[1] D. M. Ambrose and G. Simpson, Local existence theory for derivative nonlinear Schrödinger equations with noninteger power nonlinearities, *SIAM J. Math. Anal.*, **47** (2015), 2241–2264.
[2] H. A. Biagioni and F. Linares, Ill-posedness for the derivative Schrödinger and generalized Benjamin-Ono equations, *Trans. Amer. Math. Soc.*, **353** (2001), 3649–3659.
[3] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, A refined global well-posedness result for Schrödinger equations with derivative, *SIAM J. Math. Anal.*, **34** (2002), 64–86.
[4] K. Fujiwara and T. Ozawa, Finite time blowup of solutions to the nonlinear Schrödinger equation without gauge invariance, *J. Math. Phys.*, **57** (2016), 082103, 8pp.
[5] K. Fujiwara and T. Ozawa, Lifespan of strong solutions to the periodic nonlinear Schrödinger equation without gauge invariance, *J. Evol. Equ.*, **17** (2017), 1023–1030.
[6] A. Grünrock and S. Herr, Low regularity local well-posedness of the derivative nonlinear Schrödinger equation with periodic initial data, *SIAM J. Math. Anal.*, **39** (2008), 1890–1920.
[7] M. Hayashi and T. Ozawa, Well-posedness for a generalized derivative nonlinear Schrödinger equation, *J. Differential Equations*, **261** (2016), 5424–5445.
[8] N. Hayashi, The initial value problem for the derivative nonlinear Schrödinger equation in the energy space, *Nonlinear Anal.*, **20** (1993), 823–833.
[9] N. Hayashi and T. Ozawa, On the derivative nonlinear Schrödinger equation, *Phys. D*, **55** (1992), 14–36.
[10] N. Hayashi and T. Ozawa, Finite energy solutions of nonlinear Schrödinger equations of derivative type, *SIAM J. Math. Anal.*, **25** (1994), 1488–1503.
[11] S. Herr, On the Cauchy problem for the derivative nonlinear Schrödinger equation with periodic boundary condition, *Int. Math. Res. Not.*, **2006** (2006), Art. ID 96763, 33pp.
[12] X. Liu, G. Simpson and C. Sulem, Stability of solitary waves for a generalized derivative nonlinear Schrödinger equation, *J. Nonlinear Sci.*, **23** (2013), 557–583.
[13] K. Mio, T. Ogino, K. Minami and S. Takeda, Modified nonlinear Schrödinger equation for Alfvén waves propagating along the magnetic field in cold plasmas, *J. Phys. Soc. Japan*, **41** (1976), 265–271.
[14] R. Mosincat, Global well-posedness of the derivative nonlinear Schrödinger equation with periodic boundary condition in $H^{\frac{1}{2}}$, *J. Differential Equations*, **263** (2017), 4658–4722.
[15] A. R. Nahmod, T. Oh, L. Rey-Bellet and G. Staffilani, Invariant weighted Wiener measures and almost sure global well-posedness for the periodic derivative NLS, *J. Eur. Math. Soc. (JEMS)*, **14** (2012), 1275–1330.
[16] T. Ozawa and Y. Yamazaki, Life-span of smooth solutions to the complex Ginzburg-Landau type equation on a torus, *Nonlinearity*, **16** (2003), 2029–2034.
[17] G. d. N. Santos, Existence and uniqueness of solution for a generalized nonlinear derivative Schrödinger equation, *J. Differential Equations*, **259** (2015), 2030–2060.
[18] C. Sulem and P. Sulem, *The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse*, Applied Mathematical Sciences, Springer New York, 1999.
[19] H. Sunagawa, The lifespan of solutions to nonlinear Schrödinger and Klein-Gordon equations, *Hokkaido Math. J.*, **37** (2008), 825–838.
[20] H. Takaoka, Well-posedness for the one-dimensional nonlinear Schrödinger equation with the derivative nonlinearity, *Adv. Differential Equations*, **4** (1999), 561–580.
[21] H. Takaoka, A priori estimates and weak solutions for the derivative nonlinear Schrödinger equation on torus below $H^{1/2}$, *J. Differential Equations*, **260** (2016), 818–859.

[22] S. B. Tan, Blow-up solutions for mixed nonlinear Schrödinger equations, *Acta Math. Sin. (Engl. Ser.)*, **20** (2004), 115–124.
[23] L. Thomann and N. Tzvetkov, Gibbs measure for the periodic derivative nonlinear Schrödinger equation, *Nonlinearity*, **23** (2010), 2771–2791.
[24] M. Tsutsumi and I. Fukuda, On solutions of the derivative nonlinear Schrödinger equation. Existence and uniqueness theorem, *Funkcial. Ekvac.*, **23** (1980), 259–277.
[25] Y. Y. S. Win, Global well-posedness of the derivative nonlinear Schrödinger equations on T, *Funkcial. Ekvac.*, **53** (2010), 51–88.

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