Two-Scale Convergence: a bright idea from Yaounde

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After Nguetseng and Allaire,

$$u_\varepsilon \underset{2}{\rightharpoonup} u \qquad \underset{\text{def}}{\Longleftrightarrow} \qquad \{u_\varepsilon\} \text{ is bounded in } L^2(\mathbf{R}^N), \text{ and}$$

$$\lim_{\varepsilon\to 0} \int_{\mathbf{R}^N} u_\varepsilon(x)\, \psi\Big(x, \frac{x}{\varepsilon}\Big)\, dx = \iint_{\mathbf{R}^N\times[0,1[^N} u(x,y)\, \psi(x,y)\, dx dy,$$

for any smooth ψ that is $[0,1[^N$-periodic w.r.t. y.

Canonic example: $\psi(x, x/\varepsilon) \underset{2}{\rightharpoonup} \psi(x,y)$ for any function ψ as above. E.g.:

$$x \sin(2\pi x/\varepsilon) \underset{2}{\rightharpoonup} x \sin(2\pi y).$$

Set $\mathcal{Y} := [0,1[^N$: N-dimensional torus, and identify any function on $\mathcal{Y}$ with its periodic extension to $\mathbf{R}^N$. Set

$$\hat{n}(x) := \max\{n \in \mathbf{Z} : n \le x\}, \quad \hat{r}(x) := x - \hat{n}(x) \ (\in [0,1[) \qquad \forall x \in \mathbf{R}$$

$$\mathcal{N}(x) := (\hat{n}(x_1), ..., \hat{n}(x_N)) \in \mathbf{Z}^N, \quad \mathcal{R}(x) := x - \mathcal{N}(x) \in \mathcal{Y} \qquad \forall x \in \mathbf{R}^N.$$

Two-scale decomposition (unfolding):

$$x = \varepsilon[\mathcal{N}(x/\varepsilon) + \mathcal{R}(x/\varepsilon)] \qquad \forall x \in \mathbf{R}^N, \forall \varepsilon > 0;$$

$$\begin{cases} \varepsilon\mathcal{N}(x/\varepsilon): & \text{coarse-scale variable} \\ \mathcal{R}(x/\varepsilon): & \text{fine-scale variable.} \end{cases}$$

Two-scale composition (folding):

$$S_\varepsilon(x,y) := \varepsilon\mathcal{N}(x/\varepsilon) + \varepsilon y \qquad \forall (x,y) \in \mathbf{R}^N \times \mathcal{Y}, \forall \varepsilon > 0.$$

Lemma 1.1 *Let $f : \mathbf{R}^N \times \mathcal{Y} \to \mathbf{R}$ be integrable and of Caratheodory. Then*

$$\int_{\mathbf{R}^N} f(x, x/\varepsilon)\, dx = \iint_{\mathbf{R}^N \times \mathcal{Y}} f(S_\varepsilon(x,y), y)\, dxdy \qquad \forall \varepsilon > 0.$$

For any $p \in [1, +\infty]$, $v \mapsto v \circ S_\varepsilon$ is then a linear isometry $L^p(\mathbf{R}^N) \to L^p(\mathbf{R}^N \times \mathcal{Y})$.

Proof. As $\mathbf{R}^N = \cup_{m \in \mathbf{Z}^N} (\varepsilon m + \varepsilon \mathcal{Y})$ and $\mathcal{N}(x/\varepsilon) = m$ for any $x \in \varepsilon m + \varepsilon \mathcal{Y}$,

$$\int_{\mathbf{R}^N} f(x, x/\varepsilon)\, dx = \sum_{m \in \mathbf{Z}^N} \int_{\varepsilon m + \varepsilon Y} f(x, x/\varepsilon)\, dx = \sum_{m \in \mathbf{Z}^N} \varepsilon^N \int_Y f(\varepsilon[m+y], y)\, dy$$

$$= \sum_{m \in \mathbf{Z}^N} \int_{\varepsilon m + \varepsilon Y} dx \int_Y f(\varepsilon[\mathcal{N}(x/\varepsilon) + y], y)\, dy = \int_{\mathbf{R}^N} dx \int_Y f(S_\varepsilon(x,y), y)\, dy. \qquad \square$$

Two-Scale Convergence

By ε we represent the generic element of an arbitrary but prescribed, positive and vanishing sequence of real numbers; e.g., $\varepsilon = \{1, 1/2, 1/3, ..., 1/n, ...\}$.

For any sequence of measurable functions, $u_\varepsilon : \mathbf{R}^N \to \mathbf{R}$, and any measurable function, $u : \mathbf{R}^N \times \mathcal{Y} \to \mathbf{R}$, we say that u_ε *two-scale converges* to u (w.r.t. the prescribed sequence $\{\varepsilon_n\}$) in some specific sense, whenever $u_\varepsilon \circ S_\varepsilon \to u$ in the corresponding standard (i.e., *one-scale)* sense.

In this way, for any $p \in [1, +\infty]$ we define strong and weak (weak star for $p = \infty$) two-scale convergence in $L^p(\mathbf{R}^N \times \mathcal{Y})$; we then write

$$u_\varepsilon \underset{2}{\to} u, \qquad u_\varepsilon \underset{2}{\rightharpoonup} u, \qquad u_\varepsilon \underset{2}{\overset{*}{\rightharpoonup}} u \qquad \text{(resp.).}$$

This can be extended to two-scale convergence in $C^0(\mathbf{R}^N \times \mathcal{Y})$.

Proposition 2.1 *Let* $p \in [1, +\infty[$ *and* $\{u_\varepsilon\}$ *be a sequence in* $L^p(\mathbf{R}^N)$. *Then:*

$$u_\varepsilon \to u \;\; \textit{in } L^p(\mathbf{R}^N) \quad \Leftrightarrow \quad \begin{cases} u_\varepsilon \underset{2}{\to} u \;\; \textit{in } L^p(\mathbf{R}^N \times \mathcal{Y}) \\ u \textit{ is independent of } y, \end{cases}$$

$$u_\varepsilon \underset{2}{\to} u \;\; \textit{in } L^p(\mathbf{R}^N \times \mathcal{Y}) \quad \Rightarrow \quad u_\varepsilon \underset{2}{\rightharpoonup} u \;\; \textit{in } L^p(\mathbf{R}^N \times \mathcal{Y}),$$

$$u_\varepsilon \underset{2}{\rightharpoonup} u \;\; \textit{in } L^p(\mathbf{R}^N \times \mathcal{Y}) \quad \Rightarrow \quad u_\varepsilon \rightharpoonup \int_{\mathcal{Y}} u(\cdot, y)\, dy \;\; \textit{in } L^p(\mathbf{R}^N).$$

Proposition 2.2 *Let $p \in [1, +\infty[$ and $\{u_\varepsilon\}$ be a sequence in $L^p(\mathbf{R}^N)$. Then*

$$u_\varepsilon \underset{2}{\rightharpoonup} u \quad \textit{in } L^p(\mathbf{R}^N \times \mathcal{Y}) \quad \Leftrightarrow \quad \{u_\varepsilon\} \textit{ is bounded in } L^p(\mathbf{R}^N) \textit{ and}$$
$$\int_{\mathbf{R}^N} u_\varepsilon(x)\, \psi(x, x/\varepsilon)\, dx \to \iint_{\mathbf{R}^N \times \mathcal{Y}} u(x,y)\, \psi(x,y)\, dxdy \qquad \forall \psi \in \mathcal{D}(\mathbf{R}^N \times \mathcal{Y}),$$

$$u_\varepsilon \underset{2}{\rightharpoonup} u \;\; \textit{in } L^p(\mathbf{R}^N \times \mathcal{Y}) \quad \Rightarrow$$
$$\liminf_{\varepsilon \to 0} \|u_\varepsilon\|_{L^p(\mathbf{R}^N)} \geq \|u\|_{L^p(\mathbf{R}^N \times \mathcal{Y})} \left(\geq \left\| \int_{\mathcal{Y}} u(\cdot, y)\, dy \right\|_{L^p(\mathbf{R}^N)} \right).$$

For $p \in]1, +\infty[$

$$u_\varepsilon \underset{2}{\to} u \;\; \textit{in } L^p(\mathbf{R}^N \times \mathcal{Y}) \;\Leftrightarrow\; \begin{cases} u_\varepsilon \underset{2}{\rightharpoonup} u \;\; \textit{in } L^p(\mathbf{R}^N \times \mathcal{Y}) \\ \|u_\varepsilon\|_{L^p(\mathbf{R}^N)} \to \|u\|_{L^p(\mathbf{R}^N \times \mathcal{Y})} \;. \end{cases}$$

Two-Scale Convergence of Derivatives

Theorem 1 *Let* $p \in]1, +\infty[$ *and* $\{u_\varepsilon\}$ *be a sequence such that* $u_\varepsilon \rightharpoonup u$ *in* $W^{1,p}(\mathbf{R}^N)$*. For any* ε *there exists a unique* $u^*_{1\varepsilon} \in \ell^p(W^{1,p}_*(\mathcal{Y}))$ *such that*

$$\int_{\mathcal{Y}} [\nabla u^*_{1\varepsilon}(m,y) - (\nabla u_\varepsilon)(\varepsilon(m+y))] \cdot \nabla\zeta \, dx = 0 \qquad \forall \zeta \in W^{1,p'}(\mathcal{Y}), \ \forall m \in \mathbf{Z}^N.$$

Moreover there exists $u_1 \in L^p\left(\mathbf{R}^N; W^{1,p}_*(\mathcal{Y})\right)$ *such that setting*

$$u_{1\varepsilon}(\varepsilon(m+y)) := u^*_{1\varepsilon}(m,y), \quad z_\varepsilon(\varepsilon(m+y)) := \nabla u^*_{1\varepsilon}(m,y) \quad \textit{for a.a. } y \in Y, \ \forall m \in \mathbf{Z}^N,$$

as $\varepsilon \to 0$ *along a suitable subsequence,*

$$u_{1\varepsilon} \underset{2}{\rightharpoonup} u_1 \ \textit{ in } L^p(\mathbf{R}^N \times \mathcal{Y}), \qquad z_\varepsilon \underset{2}{\rightharpoonup} \nabla_y u_1 \ \textit{ in } L^p(\mathbf{R}^N \times \mathcal{Y})^N.$$

This entails that, as $\varepsilon \to 0$ *along the extracted subsequence,*

$$\nabla u_\varepsilon \underset{2}{\rightharpoonup} \nabla u + \nabla_y u_1 \quad \textit{in } L^p(\mathbf{R}^N \times \mathcal{Y})^N.$$

Theorem 1' *Under the above assumptions, conversely, for any* $u \in W^{1,p}(\mathbf{R}^N)$ *and any* $u_1 \in L^p\big(\mathbf{R}^N; W^{1,p}_*(\mathcal{Y})\big)$ *there exists a sequence* $\{u_\varepsilon\}$ *of* $W^{1,p}(\mathbf{R}^N)$ *such that*

$$u_\varepsilon \to u \quad \textit{in } L^p(\mathbf{R}^N), \qquad \nabla u_\varepsilon \underset{2}{\to} \nabla u + \nabla_y u_1 \quad \textit{in } L^p(\mathbf{R}^N \times \mathcal{Y})^N.$$

Comparable results hold if the gradient (∇) is replaced either by the curl ($\nabla\times$), or by the divergence ($\nabla\cdot$).

Theorem 1 [c] is a reformulation of a fundamental result of Nguetseng [b]; see also [a]. Theorem 1' may also be found in [c].

[a] G. Allaire: *Homogenization and two-scale convergence.* S.I.A.M. J. Math. Anal. **23** (1992) 1482–1518

[b] G. Nguetseng: *A general convergence result for a functional related to the theory of homogenization.* S.I.A.M. J. Math. Anal. **20** (1989) 608–623

[c] A. V.: *Two-scale convergence of first-order operators.* Z. Anal. Anwendungen **26** (2007), 133-164

Denoting the weak one-scale (two-scale, resp.) limit by $\lim_{\varepsilon\to 0}{}^{(1)}$ ($\lim_{\varepsilon\to 0}{}^{(2)}$, resp.),

$$\lim_{\varepsilon\to 0}{}^{(2)}\nabla u_\varepsilon = \lim_{\varepsilon\to 0}{}^{(1)}\nabla u_\varepsilon + \nabla_y u_1 \ \Big(= \nabla \lim_{\varepsilon\to 0}{}^{(1)} u_\varepsilon + \nabla_y u_1\Big) \qquad \text{a.e. in } \mathbf{R}^N \times \mathcal{Y}.$$

For $p = 2$, this decomposition is orthogonal in $L^2(\mathbf{R}^N \times \mathcal{Y})^N$.

An example. Let

$$N = 1, \qquad u_\varepsilon(x) := x + \varepsilon \sin(2\pi x/\varepsilon) \quad \forall x \in [0,1].$$

Then, for any $p \in]1,+\infty[$,

$$\left.\begin{aligned} u_{1\varepsilon}(x) = \sin(2\pi x/\varepsilon) &\underset{2}{\rightharpoonup} \sin(2\pi y) = u_1(y), \\ \varepsilon D u_{1\varepsilon}(x) = 2\pi\cos(2\pi x/\varepsilon) &\underset{2}{\rightharpoonup} 2\pi\cos(2\pi y) = D_y u_1(y) \end{aligned}\right. \qquad \text{in } L^p(]0,1[\times\mathcal{Y}),$$

whence

$$D u_\varepsilon \underset{2}{\rightharpoonup} Du + D_y u_1 = x + 2\pi\cos(2\pi y) \qquad \text{in } L^p(]0,1[\times\mathcal{Y}).$$

An Example: Elliptic Homogenization

The ε-Problem. For any $\varepsilon > 0$, let

$$f_\varepsilon \in L^2(\Omega)^N, \qquad A_\varepsilon = \{a_{\varepsilon,ij}\} \in L^2(\Omega)^{N^2},$$

$$\exists C > 0 : \forall\varepsilon, \forall x \in \mathbf{R}^N, \forall\xi \in \mathbf{R}^N, \quad \sum_{i,j=1}^{N} a_{\varepsilon,ij}(x)\xi_i\xi_j \geq C|\xi|^2;$$

e.g., $a_{\varepsilon,ij} = a_{ij}(x, x/\varepsilon)$ for some Caratheodory function a_{ij}.

Problem 1$_\varepsilon$ *Find* $u_\varepsilon \in H_0^1(\Omega)^N$ *such that*

$$\int_\Omega (A_\varepsilon \cdot \nabla u_\varepsilon) \cdot \nabla v \, dx = \int_\Omega f_\varepsilon \cdot \nabla v \, dx \qquad \forall v \in H_0^1(\Omega)^N.$$

$\forall\varepsilon$, this problem is obviously well-posed.

The Two-Scale Problem. Now we assume that

$$A_\varepsilon \underset{2}{\rightarrow} A \quad \text{in } L^2(\Omega \times \mathcal{Y})^{N^2}, \qquad f_\varepsilon \underset{2}{\rightharpoonup} f \quad \text{in } L^2(\Omega \times \mathcal{Y})^N.$$

Problem 1 *Find $u \in H_0^1(\Omega)^N$ and $u_1 \in L^2\big(\Omega; H^1(\mathcal{Y})^N\big)$ such that $\int_{\mathcal{Y}} u_1(\cdot, y)\, dy = 0$ a.e. in Ω, and*

$$\iint_{\Omega\times\mathcal{Y}} [A\cdot(\nabla u + \nabla_y u_1)]\cdot(\nabla v + \nabla_y v_1)\, dxdy = \iint_{\Omega\times\mathcal{Y}} f\cdot(\nabla v + \nabla_y v_1)\, dxdy$$
$$\forall v \in H_0^1(\Omega)^N, \forall v_1 \in L^2\big(\Omega; H^1(\mathcal{Y})^N\big).$$

This equation is equivalent to a coarse-scale equation coupled with a fine-scale one:

$$\nabla\cdot \int_{\mathcal{Y}} A\cdot(\nabla u + \nabla_y u_1)\, dy = \nabla\cdot \int_{\mathcal{Y}} f\, dy \qquad \text{in } H^{-1}(\Omega)^N,$$
$$\nabla_y\cdot[A\cdot(\nabla u + \nabla_y u_1)] = \nabla_y\cdot f \qquad \text{in } H^{-1}(\mathcal{Y})^N, \text{ a.e. in } \Omega.$$

The Classic Approach: Tartar's Energy Method. For f independ of y:

(i) solve the *cell problem* for the unknown functions $w_1, ..., w_N \in L^2\big(\Omega; H^1(\mathcal{Y})^N\big)$:

$$\nabla_y \cdot [A(x, y) \cdot (\nabla_y w_i + e_i)] = 0 \qquad \text{in } H^{-1}(\mathcal{Y})^N, \text{ for a.a. } x;$$

(ii) define the homogenized matrix

$$\begin{aligned} A^*_{ij}(x) &= \int_{\mathcal{Y}} A(x, y) \cdot (\nabla_y w_i + e_i) \cdot (\nabla_y w_j + e_j)\, dy \\ &= \int_{\mathcal{Y}} \Big(\sum_{m=1}^{N} A_{\ell m}(x, y) \cdot (\nabla_y w_i)_\ell \cdot (\nabla_y w_j)_m + A_{ij}(x, y) \Big)\, dy \qquad \text{for a.a. } x; \end{aligned}$$

(iii) formulate the (one-scale) homogenized equation

$$\nabla \cdot [A^* \cdot \nabla u] = \nabla f \qquad \text{in } H^{-1}(\Omega)^N.$$

The functions u_1 (cf. Theorem 1 above) and $(w_1, ..., w_3)$ are then related by

$$u_1 = \textstyle\sum_{i=1}^{N} w_i\, (\nabla u)_i \qquad \text{a.e. in } \Omega \times \mathcal{Y}.$$

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Two-scale convergence

and

Homogenization of a nonlinear PDE problem

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The Problem

Ampère's and Faraday's equations coupled with nonlinear constitutive relations, for a metal occupying a (possibly multi-connected) domain $\Omega \subset \mathbf{R}^3$:

$$\begin{cases} \nabla\times\vec{H} = \vec{J} + (1-\chi)\frac{\partial\vec{E}}{\partial t} + \vec{J}_e & \text{in } \mathbf{R}^3_T, \\ \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} & \text{in } \mathbf{R}^3_T, \\ \vec{B} \in \partial\varphi(\vec{H}, x) & \text{in } \Omega_T, \\ \vec{B} = \vec{H} & \text{in } \mathbf{R}^3_T \setminus \Omega_T, \\ \vec{J} = \vec{\alpha}(\vec{E}, \vec{H}, x) & \text{in } \Omega_T, \\ \vec{J} = \vec{0} & \text{in } \mathbf{R}^3_T \setminus \Omega_T, \\ \vec{E}(\cdot, 0) = \vec{E}^0 & \text{in } \mathbf{R}^3 \setminus \Omega, \\ \vec{B}(\cdot, 0) = \vec{B}^0 & \text{in } \mathbf{R}^3. \end{cases}$$

$\vec{\alpha}(\cdot,\cdot,x)$ globally continuous, and maximal monotone w.r.t. the first argument,
$\varphi(\cdot, x)$ convex lower semicontinuous, $\quad \vec{E}^0, \vec{B}^0, \vec{J}_e$ prescribed,
χ:= characteristic function of Ω; $\quad A_T := A\times]0, T[$ for any set A.

This problem is quasi-linear parabolic in Ω, linear hyperbolic in $\mathbf{R}^3 \setminus \Omega$.
(It would not be natural to confine it to Ω.)

Weak formulation for a heterogeneous periodic material, for any $\varepsilon > 0$.

Problem 1$_\varepsilon$. *Find* $\vec{B}_\varepsilon, \vec{H}_\varepsilon, \vec{E}_\varepsilon, \vec{J}_\varepsilon \in L^2(\mathbf{R}^3_T)^3$ *such that*

$$\begin{cases} \nabla \times \vec{H}_\varepsilon = \vec{J}_\varepsilon + (1-\chi)\frac{\partial \vec{E}_\varepsilon}{\partial t} + \vec{J}_e & \textit{in } \mathcal{D}'(\mathbf{R}^3_T)^3, \\ \nabla \times \vec{E}_\varepsilon = -\frac{\partial \vec{B}_\varepsilon}{\partial t} & \textit{in } \mathcal{D}'(\mathbf{R}^3_T)^3, \\ \vec{B}_\varepsilon \in \partial\varphi(\vec{H}_\varepsilon, x/\varepsilon) & \textit{a.e. in } \Omega_T, \\ \vec{B}_\varepsilon = \vec{H}_\varepsilon & \textit{a.e. in } \mathbf{R}^3_T \setminus \Omega_T, \\ \vec{J}_\varepsilon = \vec{\alpha}(\vec{E}_\varepsilon, \vec{H}_\varepsilon, x/\varepsilon) & \textit{a.e. in } \Omega_T, \\ \vec{J}_\varepsilon = \vec{0} & \textit{a.e. in } \mathbf{R}^3_T \setminus \Omega_T, \\ (1-\chi)\vec{E}_\varepsilon(\cdot, 0) = (1-\chi)\vec{E}^0, & \textit{in } \mathcal{D}'(\mathbf{R}^3)^3, \\ \vec{B}_\varepsilon(\cdot, 0) = \vec{B}^0 & \textit{in } \mathcal{D}'(\mathbf{R}^3)^3. \end{cases}$$

Theorem 5. *Assume that*

$$\vec{E}^0 \in L^2(\mathbf{R}^3 \setminus \Omega)^3, \quad \vec{B}^0 \in L^2(\mathbf{R}^3)^3, \quad \nabla \cdot \vec{B}^0 = 0 \ \textit{ in } \mathcal{D}'(\mathbf{R}^3),$$

$$\vec{J}_e \in L^2(\mathbf{R}^3_T)^3, \quad \nabla \cdot \vec{J}_e(\cdot, t) = 0 \ \textit{ in } \mathcal{D}'(\mathbf{R}^3), \ \textit{for a.a. } t \in]0, T[,$$

$$\exists c > 0, h \in L^1(\Omega): \quad \varphi(\vec{v}, x) \geq c|\vec{v}|^2 + h(x),$$

$$\vec{v} \mapsto \partial\varphi(\vec{v}, \cdot) \ \textit{ is strictly monotone and has affine growth at } \infty,$$

$$\vec{v} \mapsto |\vec{\alpha}(\vec{v}, \cdot, \cdot)| \ \textit{ is monotone and has affine growth at } \infty.$$

Then for any $\varepsilon > 0$ there exists a solution of Problem 1_ε such that

$$\vec{B}_\varepsilon \in L^\infty\big(0, T; L^2(\mathbf{R}^3)^3\big) \cap H^1\big(0, T; (L^2_{\mathrm{rot}}(\mathbf{R}^3)^3)'\big),$$

$$\vec{H}_\varepsilon \in L^\infty\big(0, T; L^2(\mathbf{R}^3)^3\big) \cap L^2\big(0, T; L^2_{\mathrm{rot}}(\Omega)^3\big),$$

$$\vec{E}_\varepsilon \in L^2(\mathbf{R}^3_T)^3 \cap L^\infty\big(0, T; L^2(\mathbf{R}^3 \setminus \Omega)^3\big),$$

$$\vec{J}_\varepsilon \in L^2(\mathbf{R}^3_T)^3.$$

The solution is uniformly bounded w.r.t. ε in these spaces.

Outline of the Proof.

Approximation via implicit time-discretization (with time-step = T/m for $m \in \mathbf{N}$).

The energy estimate yields the uniform boundedness of the approximating fields $\vec{B}_{\varepsilon m}$, $\vec{H}_{\varepsilon m}$, $\vec{E}_{\varepsilon m}$, $\vec{J}_{\varepsilon m}$ in the above function spaces.

These fields then weakly star converge along a subsequence.

The two PDEs are retrieved by taking the limit in the time-discretized equations.

The nonlinear $\vec{B}_\varepsilon$ vs. $\vec{H}_\varepsilon$ relation is proved via *compensated compactness*.

Compactness by strict convexity then yields $\vec{H}_{\varepsilon m} \to \vec{H}_\varepsilon$ strongly in $L^2(\Omega_T)^3$.

The nonlinear $\vec{J}_\varepsilon$ vs. $(\vec{E}_\varepsilon, \vec{H}_\varepsilon)$ relation follow via monotonicity and semicontinuity.

Free Boundaries. Discontinuities in the $\vec{B}_\varepsilon$ vs. $\vec{H}_\varepsilon$ constitutive relation are not excluded. This may correspond to the occurrence of *free boundaries*.

Lemma . *If*

$$\psi : \mathbf{R}^M \to \mathbf{R} \cup \{+\infty\} \text{ is strictly convex, lower semicontinuous,}$$

$$u_n \to u \qquad \text{weakly in } L^1(\Omega)^M,$$

$$\int_\Omega \psi(u_n)\,dx \to \int_\Omega \psi(u)\,dx \neq +\infty,$$

then

$$u_n \to u \qquad \text{strongly in } L^1(\Omega)^M,$$

$$\psi(u_n) \to \psi(u) \qquad \text{strongly in } L^1(\Omega).$$

A. V.: *Strong convergence results related to strict convexity.*
Communications in P.D.E.s **9** (1984) 439–466

Set

$$\hat{v} := \int_{\mathcal{Y}} v(\cdot, y)\, dy \qquad \forall v = v(x, y),$$

$$V := \left\{ \vec{v} \in L^2(\mathcal{Y})^3 : \hat{\vec{v}} = \vec{0}, \nabla \times \vec{v} = \vec{0} \ \text{ in } \mathcal{D}'(\mathcal{Y})^3 \right\},$$

$$W := \left\{ \vec{w} \in L^2(\mathcal{Y})^3 : \hat{\vec{w}} = \vec{0}, \nabla \cdot \vec{w} = 0 \ \text{ in } \mathcal{D}'(\mathcal{Y}) \right\}.$$

Problem 2. *Find* $\vec{B}, \vec{H}, \vec{E}, \vec{H}_1, \vec{E}_1, \vec{J} \in L^2(\mathbf{R}^3 \times \mathcal{Y})^3$ *such that*

$$\vec{B} \in L^2(\mathbf{R}^3_T; W) \cap H^1(0,T; L^2(\mathbf{R}^3; V')),$$
$$\vec{H}, \vec{E} \in L^2(\mathbf{R}^3_T; V), \quad \nabla_x \times \hat{\vec{H}}, \nabla_x \times \hat{\vec{E}} \in L^2(\mathbf{R}^3_T)^3,$$
$$\vec{H}_1, \vec{E}_1 \in L^2(\mathbf{R}^3_T; W), \quad \vec{J} \in L^2(\mathbf{R}^3_T \times \mathcal{Y})^3,$$

$$\hat{\vec{H}}_1 = \hat{\vec{E}}_1 = \vec{0} \qquad \text{a.e. in } \mathbf{R}^3_T,$$

$$\begin{cases}
\nabla_x \times \vec{H} + \nabla_y \times \vec{H}_1 = \vec{J} + (1-\chi)\frac{\partial \vec{E}}{\partial t} + \vec{J}_e & \text{in } \mathcal{D}'(\mathbf{R}^3_T \times \mathcal{Y})^3,\\
\nabla_x \times \vec{E} + \nabla_y \times \vec{E}_1 = -\frac{\partial \vec{B}}{\partial t} & \text{in } \mathcal{D}'(\mathbf{R}^3_T \times \mathcal{Y})^3,\\
\vec{B} \in \partial\varphi(\vec{H}, y) & \text{a.e. in } \Omega_T \times \mathcal{Y},\\
\vec{B} = \vec{H} & \text{a.e. in } (\mathbf{R}^3_T \setminus \Omega_T) \times \mathcal{Y},\\
\vec{J} = \vec{\alpha}(\vec{E}, \vec{H}, y) & \text{a.e. in } \Omega_T \times \mathcal{Y},\\
\vec{J} = \vec{0} & \text{a.e. in } (\mathbf{R}^3_T \setminus \Omega_T) \times \mathcal{Y},\\
(1-\chi)\vec{E}(\cdot, 0) = (1-\chi)\vec{E}^0 & \text{in } \mathcal{D}'(\mathbf{R}^3 \times \mathcal{Y})^3,\\
\vec{B}(\cdot, 0) = \vec{B}^0 & \text{in } \mathcal{D}'(\mathbf{R}^3 \times \mathcal{Y})^3.
\end{cases}$$

(Henceforth φ and $\vec{\alpha}$ are assumed $\mathcal{Y}$-periodic w.r.t. their final argument.)

Theorem 6. *There exist* $\vec{B}, \vec{H}, \vec{E}, \vec{J}$ *such that, as* $\varepsilon \to 0$ *along a suitable sequence,*

$$\vec{B}_\varepsilon \underset{2}{\overset{*}{\rightharpoonup}} \vec{B} \qquad \text{in } L^\infty\big(0,T;L^2(\mathbf{R}^3\times\mathcal{Y})^3\big),$$

$$\vec{H}_\varepsilon \underset{2}{\overset{*}{\rightharpoonup}} \vec{H} \qquad \text{in } L^\infty\big(0,T;L^2(\mathbf{R}^3\times\mathcal{Y})^3\big),$$

$$\vec{E}_\varepsilon \underset{2}{\overset{*}{\rightharpoonup}} \vec{E} \qquad \text{in } L^2(\Omega_T\times\mathcal{Y})^3 \cap L^\infty\Big(0,T;L^2\big((\mathbf{R}^3\setminus\Omega)\times\mathcal{Y}\big)^3\Big),$$

$$\vec{J}_\varepsilon \underset{2}{\rightharpoonup} \vec{J} \qquad \text{in } L^2(\Omega_T\times\mathcal{Y})^3.$$

This entails that $(\vec{B},\vec{H},\vec{E},\vec{J})$ *is a solution of Problem 2.*

Let us now assume that $\alpha = \partial\gamma$, with γ convex and lower semicontinuous, and set

$$\begin{aligned} \varphi_0(\vec{H}) &:= \inf_{\eta\in V} \int_{\mathcal{Y}} \varphi(\vec{H} + \vec{\eta}(y), y)\, dy \\ \gamma_0(\vec{E}, \vec{H}) &:= \inf_{\eta\in V} \int_{\mathcal{Y}} \gamma(\vec{E} + \vec{\eta}(y), \vec{H}, y)\, dy \end{aligned} \qquad \forall \vec{H}, \vec{E} \in \mathbf{R}^3.$$

Theorem 7. *If $(\vec{B}, \vec{H}, \vec{E}, \vec{J})$ is a solution of Problem 2 then $(\hat{\vec{B}}, \hat{\vec{H}}, \hat{\vec{E}}, \hat{\vec{J}})$ is a solution of the next problem.*

Single-Scale Formulation

Problem 3. *Find* $\hat{\vec{B}}, \hat{\vec{H}}, \hat{\vec{E}}, \hat{\vec{J}} \in L^2(\mathbf{R}^3_T)^3$ *such that*

$$\nabla \times \hat{\vec{H}}, \nabla \times \hat{\vec{E}} \in L^2(\mathbf{R}^3_T)^3,$$

$$\begin{cases} \nabla \times \hat{\vec{H}} = \hat{\vec{J}} + (1-\chi)\frac{\partial \hat{\vec{E}}}{\partial t} + \vec{J}_e & \textit{in } \mathcal{D}'(\mathbf{R}^3_T)^3, \\ \nabla \times \hat{\vec{E}} = -\frac{\partial \hat{\vec{B}}}{\partial t} & \textit{in } \mathcal{D}'(\mathbf{R}^3_T)^3, \\ \hat{\vec{B}} \in \partial\varphi_0(\hat{\vec{H}}) & \textit{a.e. in } \Omega_T, \\ \hat{\vec{B}} = \hat{\vec{H}} & \textit{a.e. in } \mathbf{R}^3_T \setminus \Omega_T, \\ \hat{\vec{J}} \in \partial\gamma_0(\hat{\vec{E}}, \hat{\vec{H}}) & \textit{a.e. in } \Omega_T, \\ \hat{\vec{J}} = \vec{0} & \textit{a.e. in } \mathbf{R}^3_T \setminus \Omega_T, \\ (1-\chi)\vec{E}(\cdot, 0) = (1-\chi)\vec{E}^0 & \textit{in } \mathcal{D}'(\mathbf{R}^3)^3, \\ \vec{B}(\cdot, 0) = \vec{B}^0 & \textit{in } \mathcal{D}'(\mathbf{R}^3)^3. \end{cases}$$

Homogenization of a Model of Phase Transitions

The above approach can also be applied to other models.

E.g., the homogenization of a doubly-nonlinear Stefan-type model of phase transitions for a nonhomogeneous material, in which the heat flux depends nonlinearly on the temperature gradient and on the temperature:

$$\begin{cases} \dfrac{\partial w_\varepsilon}{\partial t} + \nabla \cdot \vec{q}_\varepsilon = f & \text{in } \mathcal{D}'(\Omega_T), \\ w_\varepsilon \in \partial\varphi(u_\varepsilon, x/\varepsilon) & \text{a.e. in } \Omega_T, \\ \vec{q}_\varepsilon = -\vec{\alpha}(\nabla u_\varepsilon, u_\varepsilon, x/\varepsilon) & \text{a.e. in } \Omega_T, \\ w_\varepsilon(\cdot, 0) = w^0 & \text{a.e. in } \Omega, \\ u_\varepsilon = \tilde{u} & \text{on } (\partial\Omega)_T, \end{cases}$$

φ and $\vec{\alpha}$ being as above.

A. V.: *Homogenization of doubly-nonlinear equations.*
Rend. Lincei Mat. Appl. **17** (2006) 211-222

Homogenization of Visco-Elastic and Plastic Processes

Augusto Visintin - Trento

Analogical Models

A large class of mathematical models are built by coupling

- a (universal) balance law,

e.g., the dynamical equation, the Maxwell system, the energy balance, and so on,

- a set of constitutive relations (that characterize the specific material),
- appropriate initial- and boundary-conditions.

In continuum mechanics, electricity, magnetism, and so on
constitutive behaviours are often represented via so-called *analogical models*, namely

networks of elementary components arranged in series and / or in parallel.

If each element fulfils a constitutive law, a global law is then derived for each network.

Rheological Models

ε: deformation tensor, σ: stress tensor.

— For a discrete family of elements $\{A_j : j = 1, ..., M\}$

(i) *Combination in Series:* $\sigma = \sigma_1 = \sigma_2, \quad \varepsilon = \varepsilon_1 + \varepsilon_2;$

(ii) *Combination in Parallel:* $\varepsilon = \varepsilon_1 = \varepsilon_2, \quad \sigma = \sigma_1 + \sigma_2.$

E.g., for a parallel arrangement

$$\sigma_j = B_j : \varepsilon_j \quad \forall j, \quad \Rightarrow \quad \sigma = \sum_{j=1}^{M} B_j : \varepsilon.$$

— For a continuous distribution of elements $\{A(y) : y \in Y\}$ $(Y := [0, 1[^3)$:

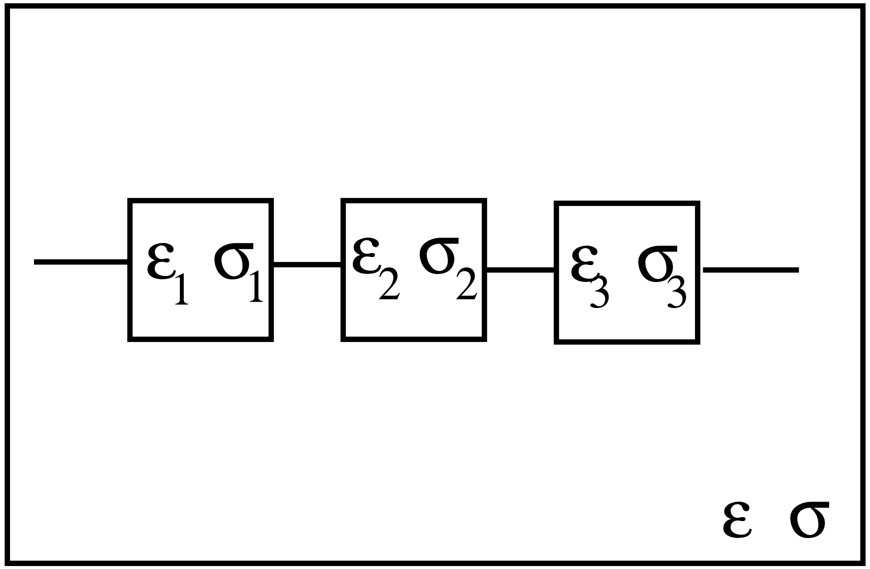
(i) *Combination in Series*: $\sigma(y) = \text{constant}, \qquad \varepsilon = \int_Y \varepsilon(y)\, dy;$

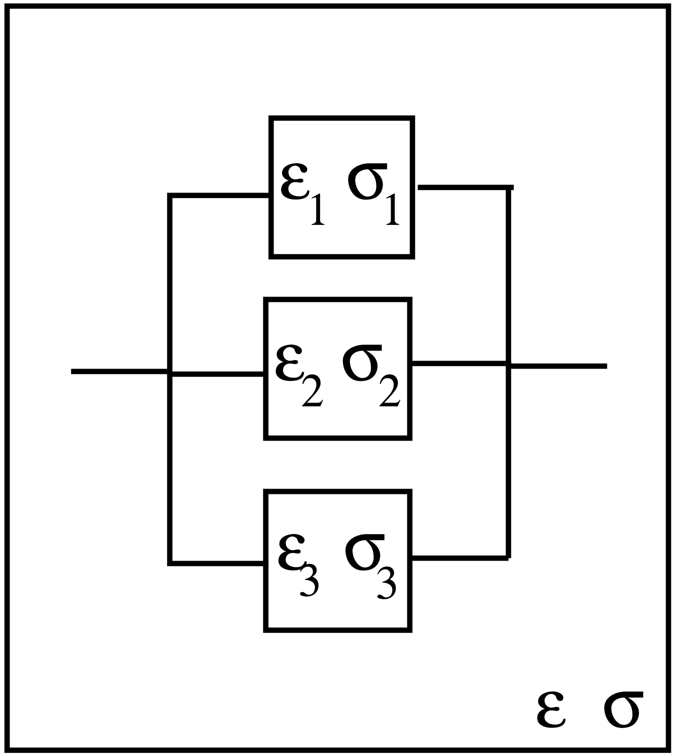
(ii) *Combination in Parallel*: $\varepsilon(y) = \text{constant}, \qquad \sigma = \int_Y \sigma(y)\, dy.$

E.g., for a parallel arrangement

$$\sigma(y) = B(y) : \varepsilon(y) \quad \text{for a.e. } y \quad \Rightarrow \quad \sigma = \int_Y B(y)\, dy : \varepsilon.$$

Schemes of Series and Parallel Arrangements





Series:
$$\begin{cases} \varepsilon = \sum_j \varepsilon_j \\ \sigma = \sigma_j \qquad \forall j. \end{cases}$$

Parallel:
$$\begin{cases} \sigma = \sum_j \sigma_j \\ \varepsilon = \varepsilon_j \qquad \forall j. \end{cases}$$

Examples of Basic Components

Classically linear elasticity is assumed for the spheric components: $\sigma_{(s)} = a\varepsilon_{(s)}$, whereas several relations are considered for the deviatoric components, e.g.:

(i) *Linear Elasticity:* $\sigma_{(d)} = A : \varepsilon_{(d)}$ $(A = A_{ijk\ell})$.

(ii) *Nonlinear Viscosity:* $\dot{\varepsilon}_{(d)} \in \partial\varphi(\sigma_{(d)})$, with φ l.s.c. and convex.

(iii) *Rigid Perfect Plasticity:* as above for $\varphi = I_K$, K being the yield criterion.

Examples of Composed Model

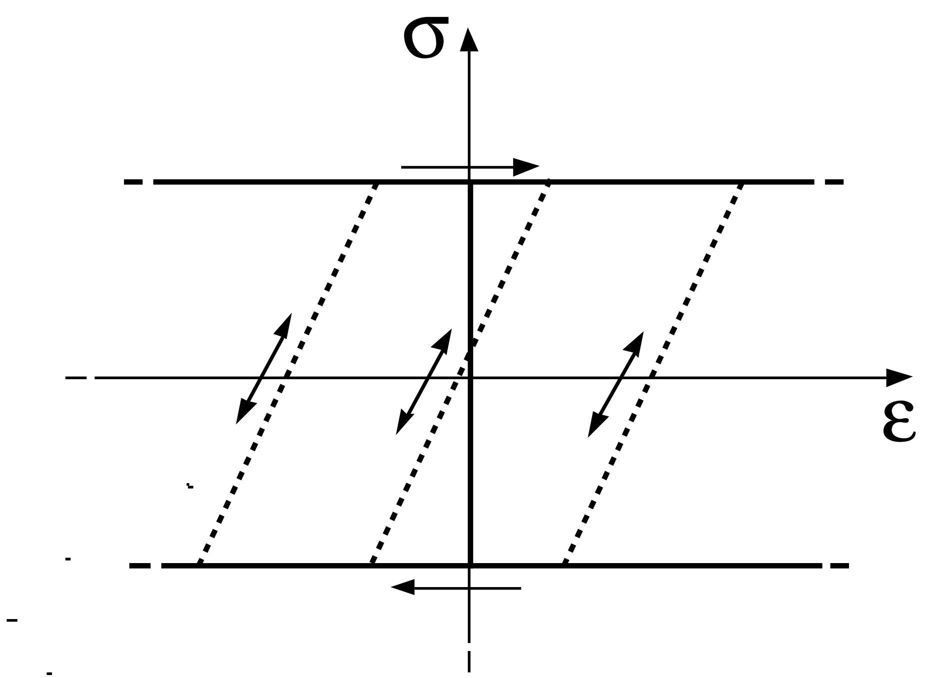
(i) *Maxwell model:* series arrangement of linear elasticity and nonlinear viscosity:

$$B : \dot{\sigma} + \partial\varphi(\sigma) \ni \dot{\varepsilon} \qquad \text{whence} \qquad \sigma = \mathcal{G}(\varepsilon).$$

(ii) *Generalized Maxwell model:* parallel arrangement of Maxwell models:

$$\sigma = \sum_j \mathcal{G}_j(\varepsilon) \qquad \text{or} \qquad \sigma = \int_Y \mathcal{G}(\varepsilon, y)\, dy.$$

Two Mechanical Models with Hysteresis



(iii) *Prandtl-Reuss Model* (or *Stop):* as in the Maxwell model, with $\varphi = I_K$:

$$\sigma = \mathcal{G}(\varepsilon) \qquad (\mathcal{G}: \text{ hysteresis operator}).$$

(iv) *Prandtl-Ishlinskiĭ Model of Stop-Type:* parallel arrangement of stops:

$$\sigma = \sum_j \mathcal{G}_j(\varepsilon) \qquad \text{or} \qquad \sigma = \int_Y \mathcal{G}(\varepsilon, y)\, dy.$$

Some References on Rheological Models

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B. Halphen, Nguyen Quoc Son: *Sur les matériaux standard généralisés.* J. de Méchanique **14** (1975), 39–63

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M.J. Leitman, G.M.C. Fisher: *The linear theory of viscoelasticity.* In: Handbuch der Physik (S. Flügge, ed.), vol. VIa/3. Springer, Berlin 1973, pp. 1–123

J. Lemaitre, J.-L. Chaboche: *Mechanics of Solid Materials.* Cambridge Univ. Press, Cambridge 1990

M. Reiner: *Rheology.* In: Handbuch der Physik (S. Flügge, ed.), vol. VI. Springer, Berlin 1958, pp. 434–550

What is the significance of analogical models ?

May networks of series / parallel arrangements represent composites?

May the corresponding constitutive relations be then retrieved via homogenization?

Which models do arise by assembling (either elementary or composite) models?

The answer depends upon the coupled PDEs and the space-dimension:

$$\varepsilon := \frac{\partial u}{\partial x}, \qquad \rho \frac{\partial^2 u}{\partial t^2} - \frac{\partial \sigma}{\partial x} = f \qquad \text{in 1 space-dimension,}$$

$$\varepsilon := \nabla^s \vec{u}, \qquad \rho \frac{\partial^2 \vec{u}}{\partial t^2} - \nabla \cdot \sigma = \vec{f} \qquad \text{in 3 space-dimensions.}$$

A Model of Elasto-Visco-Plasticity

σ: stress tensor, ε: linearized strain tensor,

$B(x)$: compliance tensor, $\varphi(\cdot, x) : \mathbf{R}^9_s \to \mathbf{R} \cup \{+\infty\}$ convex l.s.c.

$$\frac{\partial \varepsilon}{\partial t} - B(x) : \frac{\partial \sigma}{\partial t} \in \partial \varphi(\sigma, x), \qquad \text{namely} \tag{1}$$

$$\left(\frac{\partial \varepsilon}{\partial t} - B(x) : \frac{\partial \sigma}{\partial t} \right) : (\sigma - v) \geq \varphi(\sigma, x) - \varphi(v, x) \qquad \forall v \in \mathbf{R}^9_s. \tag{1$'$}$$

This relation accounts for elasto-visco-plasticity, including

the nonlinear Maxwell model, and

the Prandtl-Reuss model.

(The latter is a weak formulation of the evolution of the elasto-plastic interface...)

(1) is assumed pointwise and is coupled with the dynamical equation

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} - \nabla \cdot \sigma = \vec{f} \qquad \text{in } \Omega_T := \Omega \times]0, T[. \tag{2}$$

Program for Two- and Single-Scale Homogenization

1. Model of a Macroscopically Inhomogeneous Material. Here the fields only depend on the coarse-scale variable x (besides time). A single-scale initial- and boundary-value problem P_1 is formulated and solved.

2. Model of a Mesoscopically Inhomogeneous Material. The constitutive data B and φ are assumed to depend periodically on a fine-scale variable $y := x/\eta$ (η being a the ratio between the two space-scales). The problem P_1 is then relabelled as $P_{1\eta}$.

3. Two-Scale Homogenization. As $\eta \to 0$ a subsequence of solutions of $P_{1\eta}$ weakly *two-scale converges* to a solution of a two-scale problem, P_2, in which the fields depend on both the coarse- and fine-scale variables x and y (besides time).

4. Scale-Transformation of the Two-Scale Problem ("Upscaling"). A single-scale problem P_3 is derived from the two-scale problem P_2, by averaging the mesoscopic fields over the reference set $\mathcal{Y}$ and by homogenizing the constitutive relation.

5. Inversion of the Scale-Transformation ("Downscaling"). Conversely any solution of P_3 is represented as the $\mathcal{Y}$-average of a solution of problem P_2.

We may thus represent processes in our composite by means of <u>four different models:</u>

— (i) a single-scale model that can be represented via an analogical model, and rests on an (apparently unjustified) mean-field-type hypothesis;

— (ii) an approximate single-scale model, that is characterized by a *small* but finite parameter η; this might also be regarded as intermediate between a single-scale and a two-scale model;

— (iii) a detailed representation via a two-scale problem, in which the fields depend on both the coarse- and fine-scale variables x and y;

— (iv) a more synthetic but equivalent formulation, via a single-scale homogenized model in which the fields only depend on the coarse-scale variable x.

The models (iii) and (iv) contain the same amount of information, although this is fully displayed just in (iii).

In general the single-scale models (i) and (iv) need not be equivalent, for apparently there is no reason why either the stress or the strain should be mesoscopically uniform.

Two-Scale Convergence

After Nguetseng and Allaire, denoting by $\mathcal{Y}$ the N-dim. unit torus,

$$u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } L^2(\mathbf{R}^N \times \mathcal{Y}) \quad \underset{\text{def}}{\Longleftrightarrow} \quad \|u_\varepsilon\|_{L^2(\mathbf{R}^N)} \leq C \quad \text{and}$$

$$\int_{\mathbf{R}^N} u_\varepsilon(x)\, \psi\Big(x, \frac{x}{\varepsilon}\Big)\, dx \to \iint_{\mathbf{R}^N \times \mathcal{Y}} u(x,y)\, \psi(x,y)\, dx dy \quad \forall \psi \in \mathcal{D}(\mathbf{R}^N \times \mathcal{Y}).$$

Example. For any $\psi \in \mathcal{D}(\mathbf{R}^N \times \mathcal{Y})$, $\quad \psi(x, x/\varepsilon) \underset{2}{\rightharpoonup} \psi(x,y)$ in $L^2(]0,1[\times \mathcal{Y})$. E.g.:

$$x \sin(2\pi x/\varepsilon) \underset{2}{\rightharpoonup} x \sin(2\pi y) \qquad \text{in } L^2(]0,1[\times\mathcal{Y}).$$

G. Allaire: *Homogenization and two-scale convergence.* S.I.A.M. J. Math. Anal. **23** (1992) 1482–1518

G. Nguetseng: *A general convergence result for a functional related to the theory of homogenization.* S.I.A.M. J. Math. Anal. **20** (1989) 608–623

Theorem . *If*

$$u_\varepsilon \rightharpoonup u \qquad \text{in } H^1(\Omega), \tag{1}$$

then there exists $w \in L^2\big(\Omega; H^1(\mathcal{Y})\big)$ *such that* $\int_{\mathcal{Y}} w(\cdot, y)dy = 0$ *a.e. in* Ω*, and such that, as* $\varepsilon \to 0$ *along a suitable subsequence,*

$$\nabla u_\varepsilon \underset{2}{\rightharpoonup} \nabla u + \nabla_y w \qquad \text{in } L^2(\Omega \times \mathcal{Y})^3. \tag{2}$$

Example.

$$u_\varepsilon(x) := \varepsilon x \sin(2\pi x/\varepsilon) \rightharpoonup 0 =: u(x) \qquad \text{in } H^1(0,1), \tag{3}$$

$$\begin{aligned} D_x u_\varepsilon(x) &= \varepsilon \sin(2\pi x/\varepsilon) + 2\pi x \cos(2\pi x/\varepsilon) \\ &\underset{2}{\rightharpoonup} 2\pi x \cos(2\pi y) = D_x u(x) + D_y w(x,y) \quad \text{in } L^2(]0,1[\times \mathcal{Y}), \end{aligned} \tag{4}$$

where $w(x, y) = x \sin(2\pi y)$.

Some Basic References for Homogenization

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A. Braides, A. Defranceschi: *Homogenization of Multiple Integrals.* Oxford University Press, Oxford 1998

D. Cioranescu, P. Donato: *An Introduction to Homogenization.* Oxford Univ. Press, New York 1999

V.V. Jikov, S.M. Kozlov, O.A. Oleinik: *Homogenization of Differential Operators and Integral Functionals.* Springer, Berlin 1994

1. Model of a Macroscopically Inhomogeneous Material

Here the fields only depend on the coarse-scale variable x (besides time).

Problem 1. *Find* $(\vec{u}, \sigma)$ *such that, setting* $\varepsilon := \nabla^s \vec{u}$,

$$\vec{u} \in W^{2,\infty}\big(0,T; L^2(\Omega)^3\big) \cap W^{1,\infty}\big(0,T; W_0^{1,q}(\Omega)^3\big) \tag{1}$$

$$\sigma \in W^{1,\infty}\big(0,T; L^2(\Omega)_s^9\big), \qquad \nabla\cdot\sigma \in L^\infty\big(0,T; L^2(\Omega)_s^9\big) \tag{2}$$

$$\frac{\partial\varepsilon}{\partial t} - B(x) : \frac{\partial\sigma}{\partial t} \in \partial\varphi(\sigma, x) \qquad \textit{a.e. in } \Omega_T \tag{3}$$

$$\rho\frac{\partial^2\vec{u}}{\partial t^2} - \nabla\cdot\sigma = \vec{f} \qquad \textit{in } \mathcal{D}'(\Omega_T). \tag{4}$$

This problem is well-posed.

2. Model of a Mesoscopically Inhomogeneous Material

Just replace x by x/η, η being a (small) positive parameter.

3. Two-Scale Model

Problem 2. *Find* $\vec{u} = \vec{u}(x,t),\ \varepsilon = \varepsilon(x,y,t),\ \sigma = \sigma(x,y,t)$ *such that*

$$\vec{u} \in W^{2,\infty}\big(0,T;L^2(\Omega)^3\big) \cap W^{1,\infty}\big(0,T;W_0^{1,q}(\Omega)^3\big) \tag{1}$$

$$\sigma \in W^{1,\infty}\big(0,T;L^2(\Omega\times\mathcal{Y})_s^9\big), \quad \nabla\cdot\int_{\mathcal{Y}}\sigma\,dy \in L^\infty\big(0,T;L^2(\Omega)_s^9\big) \tag{2}$$

$$\exists\vec{u}_{(1)} \in L^q\big(\Omega_T;W^{1,q}(\mathcal{Y})^3\big): \qquad \varepsilon = \nabla^s\vec{u} + \nabla_y\vec{u}_{(1)} \quad \textit{a.e. in } \Omega_T\times\mathcal{Y} \tag{3}$$

$$\frac{\partial\varepsilon}{\partial t} - B(y):\frac{\partial\sigma}{\partial t} \in \partial\varphi(\sigma,y) \qquad \textit{a.e. in } \Omega_T\times\mathcal{Y} \tag{4}$$

$$\rho\frac{\partial^2\vec{u}}{\partial t^2} - \nabla\cdot\int_{\mathcal{Y}}\sigma\,dy = \vec{f} \qquad \textit{in } \mathcal{D}'(\Omega_T) \tag{5}$$

$$\nabla_y\cdot\sigma = \vec{0} \qquad \textit{in } \mathcal{D}'(\mathcal{Y})^3,\ \textit{a.e. in } \Omega_T. \tag{6}$$

This is retrieved by passing to the two-scale limit as $\eta \to 0$ in Problem 1_η.

4. Single-Scale Homogenization of the Constitutive Law

Basic scale decomposition: we define the average and fluctuating components:

$$\hat{v} := \int_{\mathcal{Y}} v(y)\,dy, \qquad \tilde{v} := v - \hat{v} \qquad \forall v \in L^1(\mathcal{Y}). \tag{1}$$

Henceforth we take $p = q = 2$. We define the spaces

$$W := \{\eta \in L^2(\mathcal{Y})^9 : \hat{\eta} = 0, \nabla\cdot\eta = \vec{0} \text{ in } \mathcal{D}'(\mathcal{Y})^3\} \tag{2}$$

$$Z := \{\zeta \in L^2(\mathcal{Y})^9 : \hat{\zeta} = 0, \zeta = \nabla^s\vec{v} \text{ a.e. in } \mathcal{Y}, \text{ for some } \vec{v} \in H^1(\mathcal{Y})^3\} \tag{3}$$

and notice the obvious orthogonality properties

$$\int_{\mathcal{Y}} \zeta(y):\eta(y)\,dy = 0 \qquad \forall\zeta \in Z, \forall\eta \in W \tag{4}$$

$$\int_{\mathcal{Y}} \hat{\zeta}:\tilde{\eta}(y)\,dy = 0 \qquad \forall\zeta,\eta \in L^2(\mathcal{Y})^9. \tag{5}$$

The Fenchel Properties

$$
\begin{aligned}
&\forall u, w, && F(u) + F^*(w) \geq w \cdot u && \text{(Fenchel inequality)} \\
&w \in \partial F(u) \;\Leftrightarrow && F(u) + F^*(w) = w \cdot u && \text{(Fenchel property – I).}
\end{aligned}
\tag{1}
$$

The latter statement then also reads

$$
w \in \partial F(u) \;\Leftrightarrow\; F(u) + F^*(w) \leq w \cdot u \qquad \text{(Fenchel property – II).} \tag{2}
$$

Trivial example : $F(v) = |v|^2/2$, whence $\partial F(u) = u$

$$
\begin{aligned}
&\forall u, w, && \frac{|u|^2}{2} + \frac{|w|^2}{2} \geq w \cdot u && \text{(Fenchel inequality)} \\
&w = u \;\Leftrightarrow && \frac{|u|^2}{2} + \frac{|w|^2}{2} = w \cdot u && \text{(Fenchel property – I)}
\end{aligned}
\tag{3}
$$

$$
w = u \;\Leftrightarrow\; \frac{|u|^2}{2} + \frac{|w|^2}{2} \leq w \cdot u \qquad \text{(Fenchel property – II).} \tag{4}
$$

By the Fenchel properties, $\frac{\partial\varepsilon}{\partial t} - B(x) : \frac{\partial\sigma}{\partial t} \in \partial\varphi(\sigma, x)$ a.e. in Ω_T is equivalent to

$$\varphi(\sigma, x) + \varphi^*\Big(\frac{\partial\varepsilon}{\partial t} - B(x) : \frac{\partial\sigma}{\partial t}, x\Big) = \sigma : \Big(\frac{\partial\varepsilon}{\partial t} - B(x) : \frac{\partial\sigma}{\partial t}\Big), \tag{1}$$

namely

$$\begin{aligned}&\iiint_{\Omega_\tau\times\mathcal{Y}}\Big[\varphi(\sigma, x) + \varphi^*\Big(\frac{\partial\varepsilon}{\partial t} - B(x) : \frac{\partial\sigma}{\partial t}, x\Big)\Big]\,dxdydt\\ &+ \frac{1}{2}\iint_{\Omega\times\mathcal{Y}}(\sigma : B(x) : \sigma)\Big|_{t=0}^{t=\tau}\,dxdy = \iiint_{\Omega_\tau\times\mathcal{Y}}\sigma : \frac{\partial\varepsilon}{\partial t}\,dxdydt \qquad \forall\tau\in\,]0, T].\end{aligned} \tag{2}$$

After a further integration in time and using the above orthogonality properties, we get an equation of the form

$$A(\sigma, \varepsilon) = \iiint_{\Omega_T\times\mathcal{Y}}(T - t)\sigma : \frac{\partial\varepsilon}{\partial t}\,dxdydt = \iint_{\Omega_T}(T - t)\hat{\sigma} : \frac{\partial\hat{\varepsilon}}{\partial t}\,dxdt. \tag{3}$$

Setting $\Lambda(\hat{\sigma}, \hat{\varepsilon}) := \inf\big\{A(\hat{\sigma} + \tilde{\sigma}, \hat{\varepsilon} + \tilde{\varepsilon}) : (\tilde{\sigma}, \tilde{\varepsilon}) \in L^2(\Omega_T; W \times Z)\big\}$, we then get (by the Fenchel properties...)

$$\Lambda(\hat{\sigma}, \hat{\varepsilon}) = \iint_{\Omega_T}(T - t)\hat{\sigma} : \frac{\partial\hat{\varepsilon}}{\partial t}\,dxdt. \tag{4}$$

4. Homogenized Single-Scale Model

Problem 3. *Find* $(\vec{u}, \overline{\varepsilon}, \overline{\sigma})$ *such that*

$$\vec{u} \in W^{2,\infty}\big(0,T;L^2(\Omega)^3\big) \cap W^{1,\infty}\big(0,T;H_0^1(\Omega)^3\big) \tag{1}$$

$$\overline{\sigma} \in W^{1,\infty}\big(0,T;L^2(\Omega)_s^9\big), \quad \nabla\cdot\overline{\sigma} \in L^{\infty}\big(0,T;L^2(\Omega)_s^9\big) \tag{2}$$

$$\Lambda(\overline{\sigma}, \overline{\varepsilon}) = \iint_{\Omega_T} (T-t)\overline{\sigma} : \frac{\partial\overline{\varepsilon}}{\partial t}\, dxdt \tag{3}$$

$$\rho\frac{\partial^2\vec{u}}{\partial t^2} - \nabla\cdot\overline{\sigma} = \vec{f} \qquad \textit{in } \mathcal{D}'(\Omega_T). \tag{4}$$

A.V. : *Homogenization of the nonlinear Kelvin-Voigt model of visco-elasticity and of the Prager model of plasticity.* Continuum Mech. Thermodyn. **18** (2006) 223-252

A.V. : *Homogenization of the nonlinear Maxwell model of visco-elasticity and of the Prandtl-Reuss model of elasto-plasticity.* (in preparation)