Structures of the Mechanics of Complex Bodies

Quantum Foundations of the Dynamics of Structured Bodies

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1 Some Philosophy

Newton: finite (perhaps countable???) systems of material points interacting via instantaneous, acting at a distance forces

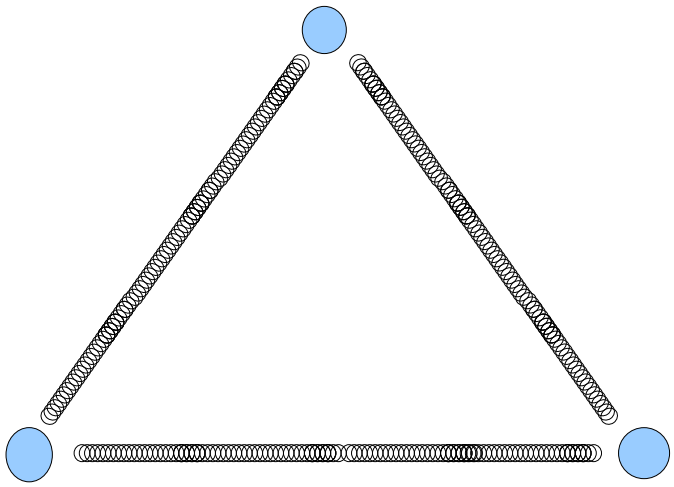

Fig. 1

Material points assumed structureless (atomic). Structure appeared as a feature of systems.
Configuration space of the N-point system

$$Q = E^N,$$

Cartesian product of N copies of the (3-dimensional) Euclidean space E.

Alternatively: $\varphi \in Q$ is a mapping (injective one)

$$\varphi : \{1, \dots, N\} \to E$$

Typical notation

$$y = (x_1, \dots, x_A, \dots, x_N) \in E^N, \qquad x_A = \varphi(A) \in E.$$

Continuum mechanics as developed by

Euler, Cauchy, Poisson, Lagrange:

$\{1, \dots, N\}$ replaced by B which in the modern language is a differential manifold, or manifold with boundary, of dimension 3 (2 in the theory of shells and plates, 1 in the theory of rods). B is parameterized by Lagrange coordinates.

Configuration space:

$$Q = E^B = \times_{_B} E$$

$\varphi \in Q$ is an injection subject to some smoothness conditions:

$$\varphi : B \to E, \qquad x^i = x^i\left(a^K\right),$$

the relationship between Lagrange and Euler coordinates,

$$x = \varphi(B)$$

is the current position of the B-th material point.

But later on some need appeared to consider granular media, when due to internal interactions the body becomes an "aggregate" of elements

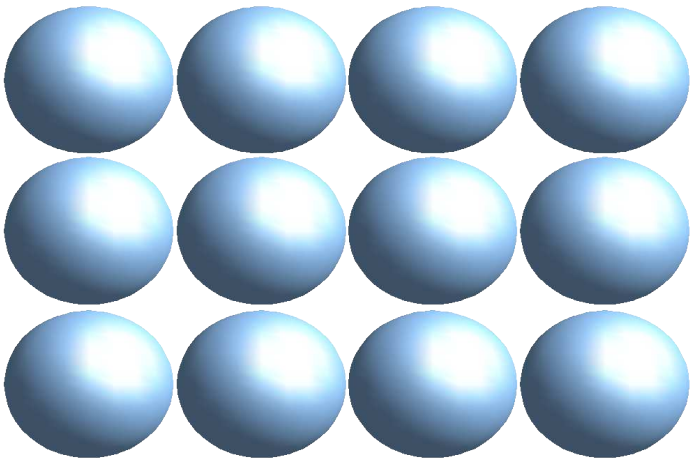

Fig. 2

which move translationally but have also internal, microrotational and macrodeformative motion.

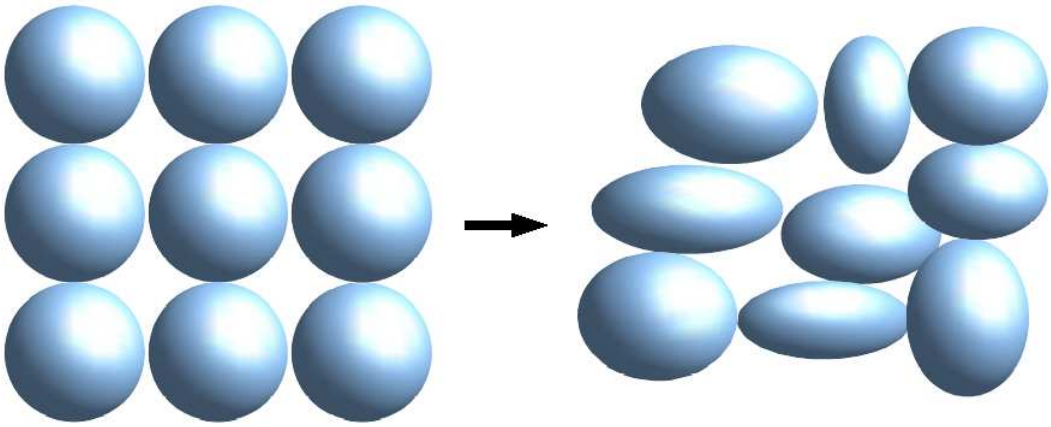

Fig. 3

Examples:

- micropolar Cosserat continua,
- micromorphic Eringen theory.

Those were still macroscopic ideas. But in modern atomistically-molecular picture of matter this became simply necessity — take molecules in molecular crystals — they rotate and deform. This leads to observable phenomena like the Raman scattering. But there is also a deeper motivation:

1) Why the material point should be a fundamental object? Why not just the point with an attached structure — a geometric object?

2) No doubt, some mathematical beauty does exist in fibre bundle models of internal degrees of freedom and we know for a long time that Physical World is organized mathematically and the mathematically elegant always finally becomes physically effective.

In micromorphic theory local states are described by linear bases attached to the material point.

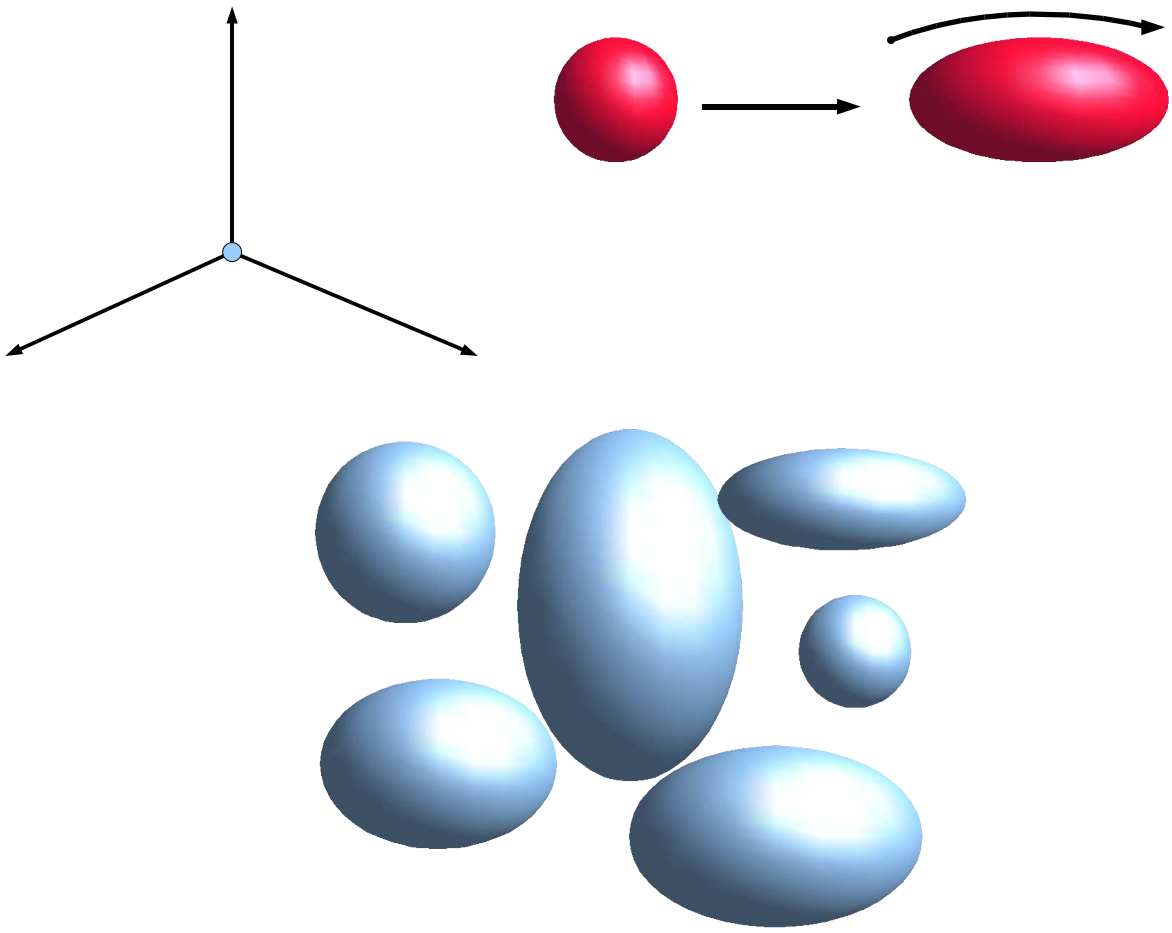

Fig. 4

After coordinate fixing they may be identified with elements of $\mathrm{GL}(n,\mathbb{R})$ or its subgroups like $\mathrm{SL}(n,\mathbb{R})$ (internal incompressibility) or $\mathrm{SO}(n,\mathbb{R})$ (internal rigidity, the Cosserat medium, orthonormal frames). Internal degrees of freedom are ruled by groups motivated by spatial geometry.

There are also other situations like the bubble-type or other void-type dynamics of objects having only dilatational internal modes [Gianfranco Capriz].

Other type of models — internal modes are not necessarily Lie groups but their more general homogeneous spaces or group-action spaces.

This is the case of nematic liquid crystals described as continua of rods, i.e., the internal space is the sphere:

$$Q = S^2(0, d) \simeq \mathrm{SO}(3, \mathbb{R})/\mathrm{SO}(2, \mathbb{R}).$$

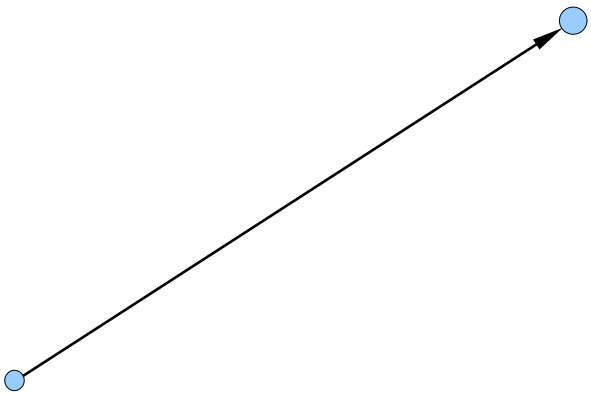

Fig. 5

These are continua of dipoles, strings or microphysically: two-atomic molecules, elongated molecules, etc.

If the poles are indistinguishable, Q is the so-called elliptic space:

$$Q = S^2(0, d)/\mathbb{Z}_2$$

(antipodal identification).

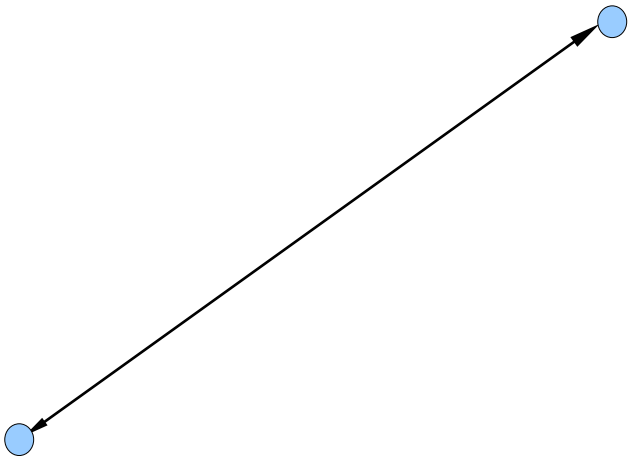

Fig. 6

The dual models are the continua or discrete aggregates of plates which may be "realistically" interpreted, e.g., as three-atomic molecules or flattened molecules:

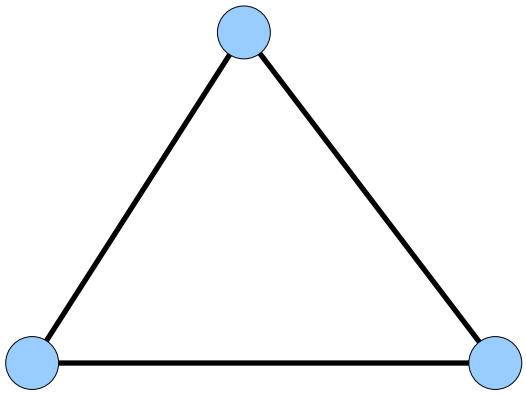

Fig. 7

Interesting: in some of those situations the internal configuration spaces like

$$\mathrm{GL}^{+}(3,\mathbb{R}), \qquad \mathrm{SL}(3,\mathbb{R}), \qquad \mathrm{SO}(3,\mathbb{R}), \qquad \mathrm{SO}(3,\mathbb{R})/\mathbb{Z}_2$$

are not simply-connected manifolds.

This brings about certain problems in the quantized version of theory (nano- and micro-level). In particular, multivalued wave functions may appear with all their paradoxes, like half-integer values of angular momentum even if there is no "true" spin.

Concentration below: the Eringen-like model when internal degrees of freedom are ruled by $\mathrm{GL}(3,\mathbb{R})$ (mathematically it is more convenient to start from $\mathbb{R}^n$ and $\mathrm{GL}(n,\mathbb{R})$), and only then to put $n = 3$ or $n = 2$). This is either granular medium or the system of molecules like ones in molecular crystals, just rotating and undergoing homogeneous deformations.

Other aspect of the model: the finite element method is becoming now not only the technical tool, but just the effective procedure of fundamental theoretical analysis, when the body is "triangulated" into the system of affinely-behaving objects [Miles Rubin]. And now it becomes "fashionable" to go with the size of these objects to the micro- and nano-scale. And then quantum mechanics of mutually interacting affine bodies must be used. This way of describing quantum phenomena may be competitive to traditional ones, based on the decomposition onto system of phonon modes.

2 Group-theoretical Model of Degrees of Freedom. Euler Tradition and Affine Models

G is a Lie group — usually linear:
$G \subset \mathrm{GL}(N,\mathbb{R})$ or $G \subset \mathrm{GL}(N,\mathbb{C})$ (but real, e.g., $\mathrm{U}(N)$)
$G' \subset \mathrm{L}(N,\mathbb{R})$ or $G' \subset \mathrm{L}(N,\mathbb{C})$ — Lie algebra
G'^* — Lie co-algebra
Typically: $G'^* \simeq G'$ via trace expression: $\langle f, x\rangle = \mathrm{Tr}(fx)$
$\mathbb{R} \ni t \to q(t) \in G$ — motion
Lie-algebraic velocities:

$$\Omega(t) = \dot{q}(t)q(t)^{-1}, \qquad \widehat{\Omega}(t) = q(t)^{-1}\dot{q}(t),$$
$$\Omega(t) = \mathrm{Ad}_{g(t)}\widehat{\Omega}(t), \qquad \mathrm{Ad}_g x = gxg^{-1}.$$

$\{\ldots, e_a, \ldots\}$ — basis in G,
$\{\ldots, e^a, \ldots\}$ — dual basis in G^*.

$$\Omega = \Omega^a(t)e_a, \qquad \widehat{\Omega} = \widehat{\Omega}^a(t)e_a, \qquad \Omega^a(t) = \left(\mathrm{Ad}_{g(t)}\right)^a{}_b\widehat{\Omega}^b(t)$$

Non-holonomic if G-non-Abelian

$$\Omega^a = \Omega^a{}_i(q)\dot{q}^i, \qquad \widehat{\Omega}^a = \widehat{\Omega}^a{}_i(q)\dot{q}^i$$

Structure constants $C^k{}_{lm}$,

$$[e_l, e_m] = C^k{}_{lm}e_k$$

γ — (pseudo)Euclidean metric on G',

$$\gamma \in G'^* \otimes G'^*, \qquad \gamma = \gamma_{ab}e^a \otimes e^b,$$

$\gamma_{ab} = \gamma_{ba}$ — constant.
Left-invariant kinetic energies, i.e., Riemannian metrics on G:

$$T = \frac{1}{2}\gamma_{ab}\widehat{\Omega}^a\widehat{\Omega}^b = \frac{1}{2}\gamma(\Omega,\Omega)$$

Tangent bundle TG-trivial: $TG \simeq G \times G'$
Cotangent language — phase space:
Σ, $\widehat{\Sigma}$ — non-holonomic momenta conjugate to quasivelocities Ω, $\widehat{\Omega}$

$$\Sigma = \Sigma_a e^a, \qquad \widehat{\Sigma} = \widehat{\Sigma}_a e^a$$

$T^*G \simeq G \times G'^*$ trivialization

$$\Sigma_a = \Sigma_a{}^i(q)p_i, \qquad \widehat{\Sigma}_a = \widehat{\Sigma}_a{}^i(q)p_i$$

$$\Sigma_a\Omega^a = \widehat{\Sigma}_a\widehat{\Omega}^a = p_i\dot{q}^i$$

Transformation properties under group translations:
Left $L_g : x \mapsto gx$

$$\Omega \mapsto g\Omega g^{-1} = \mathrm{Ad}_g\Omega, \qquad \widehat{\Omega} \mapsto \widehat{\Omega}$$
$$\Sigma \mapsto g\Sigma g^{-1} = \mathrm{Ad}^{*-1}_g\Sigma, \qquad \widehat{\Sigma} \mapsto \widehat{\Sigma}$$

Right $R_g : x \mapsto xg$

$$\Omega \mapsto \Omega, \qquad \widehat{\Omega} \mapsto g^{-1}\widehat{\Omega}g = \mathrm{Ad}_{g^{-1}}\widehat{\Omega}$$
$$\Sigma \mapsto \Sigma, \qquad \widehat{\Sigma} \mapsto g^{-1}\widehat{\Sigma}g = \mathrm{Ad}^{*}_g\widehat{\Sigma}$$

Poisson brackets:

$$\{\Sigma_i, \Sigma_j\} = C^k{}_{ij}\Sigma_k, \qquad \{\widehat{\Sigma}_i, \widehat{\Sigma}_j\} = -C^k{}_{ij}\widehat{\Sigma}_k, \qquad \{\Sigma_i, \widehat{\Sigma}_j\} = 0$$

$$\{\Sigma_a, f(q)\} = -\Sigma_a{}^i(q)\frac{\partial f}{\partial q^i}, \qquad \{\widehat{\Sigma}_a, f(q)\} = -\widehat{\Sigma}_a{}^i(q)\frac{\partial f}{\partial q^i}$$

Other Poisson brackets vanish.
Geometrically:
Σ_i — Hamiltonian generators of left regular translations (momentum mappings of L_G)
$\widehat{\Sigma}_i$ — Hamiltonian generators of right regular translations (momentum mappings of R_G)
Legendre transformation non-holonomically represented:

$$\Sigma_i = \frac{\partial T}{\partial \Omega^i}, \qquad \widehat{\Sigma}_i = \frac{\partial T}{\partial \widehat{\Omega}^i}$$

Kinetic energy term of the Hamiltonian:

$$\mathcal{T} = \frac{1}{2}\widetilde{\gamma}^{ab}\widehat{\Sigma}_a\widehat{\Sigma}_b \qquad - \text{ left invariant}$$

$$\mathcal{T} = \frac{1}{2}\widetilde{\gamma}^{ab}\Sigma_a\Sigma_b \qquad - \text{ right invariant}$$

$\widetilde{\gamma}^{ab}$, in short γ^{ab}: it is reciprocal contravariant metric:

$$\widetilde{\gamma}^{ac}\gamma_{cb} = \delta^a{}_b$$

Equations of motion in Poisson-bracket form:

$$\frac{df}{dt} = \{f, H\}$$

γ-Euler equations for left-invariant systems:

$$\frac{d\widehat{\Sigma}_a}{dt} = -\widetilde{\gamma}^{cd}C^b{}_{ac}\widehat{\Sigma}_d\widehat{\Sigma}_b + \widehat{N}_a$$

in $\widehat{\Omega}$-terms:

$$\gamma_{ab}\frac{d\widehat{\Omega}^b}{dt} = -\gamma_{bd}C^b{}_{ac}\widehat{\Omega}^c\widehat{\Omega}^d + \widehat{N}_a$$

or in mixed terms:

$$\frac{d\widehat{\Sigma}_a}{dt} = -C^b{}_{ac}\widehat{\Omega}^c\widehat{\Sigma}_b + \widehat{N}_a$$

If $\widehat{N}_a = 0$ (geodetic system), these equations are autonomously solvable with respect to $\widehat{\Sigma}$ or $\widehat{\Omega}$.

Then $q(t)$ is found by solving non-autonomous linear system:

$$\frac{dq}{dt} = q(t)\widehat{\Omega}$$

In general the last two equations form a closed mutually coupled system.
In geodetic models Σ_a are constants of motion:

$$\frac{d\Sigma_a}{dt} = 0$$

In non-geodetic case:

$$\frac{d\Sigma_a}{dt} = N_a = \{\Sigma_a, V\} = \Sigma_a{}^i \frac{dV}{dq^i}$$

In geodetic models the Casimir invariants built of $\widehat{\Sigma}_a$ are constants of motion and the problem reduces to orbits of the co-adjoint representations of G in G'^*. Poisson structures:
Right-invariant T-models when

$$T = \frac{1}{2}\gamma_{ab}\Omega^a\Omega^b$$

$$\frac{d\Sigma_a}{dt} = \widetilde{\gamma}^{cd}C^b{}_{ac}\Sigma_d\Sigma_b + N_a$$

Doubly-invariant T-models:
When γ is the **Killing tensor** on the semi-simple Lie group

$$\gamma_{ab} \simeq C^k{}_{la}C^l{}_{kb},$$

then T is doubly-invariant, C is totally γ-skew-symmetric

$$C^{ijk} = C^i{}_{ab}\widetilde{\gamma}^{aj}\widetilde{\gamma}^{bk} = -C^{jik} = -C^{kji} = -C^{ikj},$$

both Σ_a, $\widehat{\Sigma}_a$ are constants of motion and the non-dynamical Euler terms on the right-hand side of equations of motion do vanish.
Example: usual rigid body, $N = 3$, $G = \mathrm{SO}(3,\mathbb{R})$,

$$\Omega = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}, \qquad \widehat{\Omega} = \begin{bmatrix} 0 & -\widehat{\Omega}_3 & \widehat{\Omega}_2 \\ \widehat{\Omega}_3 & 0 & -\widehat{\Omega}_1 \\ -\widehat{\Omega}_2 & \widehat{\Omega}_1 & 0 \end{bmatrix}$$

$\mathrm{SO}(3,\mathbb{R})'$-skew-symmetric matrices identified with axial vectors

$$\Sigma = \begin{bmatrix} 0 & \Sigma_3 & -\Sigma_2 \\ -\Sigma_3 & 0 & \Sigma_1 \\ \Sigma_2 & -\Sigma_1 & 0 \end{bmatrix}, \qquad \widehat{\Sigma} = \begin{bmatrix} 0 & \widehat{\Sigma}_3 & -\widehat{\Sigma}_2 \\ -\widehat{\Sigma}_3 & 0 & \widehat{\Sigma}_1 \\ \widehat{\Sigma}_2 & -\widehat{\Sigma}_1 & 0 \end{bmatrix}$$

$$T = \sum_{a=1}^{3} \frac{I_a}{2}\widehat{\Omega}_a^2, \qquad \mathcal{T} = \sum_{a=1}^{3} \frac{1}{2I_a}\widehat{\Sigma}_a^2,$$

where I_a are the main co-moving moments of inertia.

$$\{\Sigma_a, \Sigma_b\} = \varepsilon_{abc}\Sigma_c, \qquad \{\widehat{\Sigma}_a, \widehat{\Sigma}_b\} = -\varepsilon_{abc}\widehat{\Sigma}_c, \qquad \{\Sigma_a, \widehat{\Sigma}_b\} = 0$$

Euler equations:

$$\frac{d\widehat{\Sigma}_1}{dt} = \left(\frac{1}{I_3} - \frac{1}{I_2}\right)\widehat{\Sigma}_2\widehat{\Sigma}_3 + \widehat{N}_1,$$
$$\frac{d\widehat{\Sigma}_2}{dt} = \left(\frac{1}{I_1} - \frac{1}{I_3}\right)\widehat{\Sigma}_1\widehat{\Sigma}_3 + \widehat{N}_2,$$
$$\frac{d\widehat{\Sigma}_3}{dt} = \left(\frac{1}{I_2} - \frac{1}{I_1}\right)\widehat{\Sigma}_1\widehat{\Sigma}_2 + \widehat{N}_3,$$

where $\widehat{\Sigma}_a = I_a\widehat{\Omega}_a$.
In velocity terms:

$$I_1\frac{d\widehat{\Omega}_1}{dt} = (I_2 - I_3)\,\widehat{\Omega}_2\widehat{\Omega}_3 + \widehat{N}_1,$$
$$I_2\frac{d\widehat{\Omega}_2}{dt} = (I_3 - I_1)\,\widehat{\Omega}_1\widehat{\Omega}_3 + \widehat{N}_2,$$
$$I_3\frac{d\widehat{\Omega}_3}{dt} = (I_1 - I_2)\,\widehat{\Omega}_1\widehat{\Omega}_2 + \widehat{N}_3$$

Doubly invariant T-spherical rigid body, Killing metric underlying T:

$$\frac{d\widehat{\Sigma}_a}{dt} = \widehat{N}_a, \qquad \frac{d\Sigma_a}{dt} = N_a$$

Geodetic dynamics — left invariant - isotropic in space, $L_{\mathrm{SO}(3,\mathbb{R})}$-invariant.
Euler equations for **ideal incompressible fluids**:

$$\frac{\partial v^a}{\partial t} + v^b\frac{\partial v^a}{\partial x^b} = -\frac{1}{\rho}\frac{\partial p}{\partial x^a},$$

v — Euler velocity field in space,
ρ — density,
p — pressure.
Both — achievements of Euler.
Apparently different.

Arnold: Euler fluid equations interpretable as a right-invariant geodetic system on the infinite-dimensional "Lie group" SDiff($\mathbb{R}^3$). More precisely:

- left-invariant under $\mathrm{SO}(3,\mathbb{R})$ — rotations in space (Euler coordinates)
- right-invariant under SDiff($\mathbb{R}^3$) — volume-preserving diffeomorphisms of the material space (Lagrange variables)
- geodetic system, no potential, $L = T$, $H = \mathcal{T}$,

$$T = \frac{\rho}{2}\int g_{ij}v^i(x)v^j(x)d_3\bar{x},$$

where v^i is the Euler velocity field:

$$v^i(t,x) = \frac{\partial x^i}{\partial t}(t, a(t,x))$$
$$\uparrow$$
only here $\dfrac{\partial}{\partial t}$ acts!

a are Lagrange coordinates.

We — something between — deformations, but finite dimensions.
Affinely-rigid body, homogeneously deformable gyroscope.
$G = \mathrm{GL}(3,\mathbb{R})$, more convenient to use $\mathrm{GL}(n,\mathbb{R})$ and later on to specify $n = 3, 2$.
Better — homogeneous space
$(N, U, \to, \eta)$ — material space
$(M, V, \to, g)$ — physical space

$$Q = M \times \mathrm{LI}(U,V),$$
$$\uparrow \qquad \uparrow$$
translational/internal motion

where $\mathrm{LI}(U,V)$ are linear isomorphisms of U onto V
If $M = N = U = V = \mathbb{R}^n$,

$$Q = \mathrm{GL}(n,\mathbb{R}) \times_s \mathbb{R}^n, \qquad \Phi \in Q:$$

$$\Phi(t,a)^i = \varphi^i{}_K(t) a^K + x^i(t)$$

Inertial objects: μ — mass distribution measure in N, it is positive and constant,

$$m = \int_N d\mu(a) \;-\; \text{total mass},$$

$$J^K = \int_N a^K d\mu(a) = 0 \;-\; a^K \text{ vanish at the material centre of mass},$$

$$J^{KL} = \int_N a^K a^L d\mu(a) \;-\; \text{inertial tensor, constant}$$
(Lagrangian) mass quadrupole.

Kinetic energy obtained in a usual way (summation over material points):

$$T = T_{\mathrm{tr}} + T_{\mathrm{int}} = \frac{m}{2} g_{ij} \frac{dx^i}{dt}\frac{dx^j}{dt} + \frac{1}{2} g_{ij} \frac{d\varphi^i{}_A}{dt}\frac{d\varphi^j{}_B}{dt} J^{AB}$$

Legendre transformation:

$$p_i = m g_{ij} \frac{dx^j}{dt}, \qquad p^A{}_i = g_{ij} \frac{d\varphi^j{}_B}{dt} J^{BA}$$

Kinetic part of the Hamiltonian:

$$\mathcal{T} = \frac{1}{2m} g^{ij} p_i p_j + \frac{1}{2} g^{ij} p^A{}_i p^B{}_j \widetilde{J}_{AB},$$

where $\widetilde{J}_{AC} J^{CB} = \delta_A{}^B$.

Non-doubtful range of **applications** of models

$$L = T - V(\varphi), \qquad H = \mathcal{T} + V(\varphi)$$

- macroscopic elasticity when the length of excited waves is comparable with the linear size of the body,
- micromorphic continua with internal degrees of freedom ruled by linear group [Eringen],
- molecular vibrations, molecular crystals,
- nuclear dynamics (collective droplet model of the atomic nuclei),
- astrophysical objects, vibrating stars, shape of Earth,
- integrable one-dimensional lattices and n-dimensional affinely-rigid body.

Drawbacks:

1. Geodetic models (without potentials — nonphysical, nonphysical - no vibrations, non-limited expansion and contraction)
2. No dynamical affine invariance — only kinematical one. Advantages of the group structure lost.

What would be affine models?
Do exist formally?
Are realistic?
Canonical objects, transformations, generators

$$p_i,\ p^A{}_i \qquad \text{conjugate to} \qquad x^i,\ \varphi^i{}_A$$

Legendre:

$$p_i = \frac{\partial T}{\partial v^i} = m g_{ij} v^j, \qquad p^A{}_i = \frac{\partial T}{\partial \dot{\varphi}^i{}_A} = g_{ij} \dot{\varphi}^j{}_B J^{BA}$$

Lie-algebraic objects:

$$\Omega = \dot{\varphi}\varphi^{-1}, \qquad \Omega^i{}_j = \dot{\varphi}^i{}_A \varphi^{-1A}{}_j,$$
$$\widehat{\Omega} = \varphi^{-1}\dot{\varphi}, \qquad \widehat{\Omega}^A{}_B = \varphi^{-1A}{}_i \dot{\varphi}^i{}_B.$$

Affine velocities. Eringen's "gyration".

Their g- and η-skew-symmetric parts — angular velocity. They are always skew-symmetric in rigid motion:

$$\omega^i{}_j = \Omega^i{}_j - \Omega_j{}^i, \qquad \widehat{\omega}^A{}_B = \widehat{\Omega}^A{}_B - \widehat{\Omega}_B{}^A$$

Their conjugate affine spins — Hamiltonian generators of

$$\begin{array}{rl} \varphi \mapsto A\varphi & \varphi \mapsto \varphi B \\ A \in \mathrm{GL}(V) & B \in \mathrm{GL}(U) \\ \Sigma = \varphi\pi & \Sigma = \pi\varphi \end{array}$$

Spin and vorticity:

$$S^i{}_j = \Sigma^i{}_j - \Sigma_j{}^i, \qquad V^A{}_B = \widehat{\Sigma}^A{}_B - \widehat{\Sigma}_B{}^A$$

(generators of spatial and material rotations).

Transformation rules:

$$\begin{array}{lll} A & : & \Sigma \mapsto A\Sigma A^{-1}, \qquad \widehat{\Sigma} \mapsto \widehat{\Sigma} \\ B & : & \Sigma \mapsto \Sigma, \qquad\qquad \widehat{\Sigma} \mapsto B^{-1}\widehat{\Sigma}B \\ A & : & \Omega \mapsto A\Omega A^{-1}, \qquad \widehat{\Omega} \mapsto \widehat{\Omega} \\ B & : & \Omega \mapsto \Omega, \qquad\qquad \widehat{\Omega} \mapsto B^{-1}\widehat{\Omega}B \end{array}$$

Co-moving translational objects:

$$\widehat{v}^A = \varphi^{-1A}{}_i v^i, \qquad \widehat{p}_A = p_i \varphi^i{}_A$$

Poisson brackets:

$$\begin{array}{rcl} \{\Sigma^i{}_j, \Sigma^k{}_l\} & = & \delta^i{}_l \Sigma^k{}_j - \delta^k{}_j \Sigma^i{}_l, \\ \{\widehat{\Sigma}^A{}_B, \widehat{\Sigma}^C{}_D\} & = & \delta^C{}_B \widehat{\Sigma}^A{}_D - \delta^A{}_D \widehat{\Sigma}^C{}_B, \\ \{\Sigma^i{}_j, \widehat{\Sigma}^A{}_B\} & = & 0, \\ \{\widehat{\Sigma}^A{}_B, \widehat{p}_C\} & = & \delta^A{}_C \widehat{p}_B, \\ \{I^i{}_j, p_k\} = \{\Lambda^i{}_j, p_k\} & = & \delta^i{}_k p_j, \end{array}$$

where

$$I(\mathcal{O})^i{}_j := \Lambda(\mathcal{O})^i{}_j + \Sigma^i{}_j, \qquad \Lambda(\mathcal{O})^i{}_j := x^i p_j$$

and x^i are Cartesian coordinates of the $\mathcal{O}$-radius vector of the current position of the centre of mass in M.

If F is any function depending only on the configurations variables, then, obviously,

$$\begin{array}{rcl} \{F, \Sigma^i{}_j\} & = & \varphi^i{}_A \dfrac{\partial F}{\partial \varphi^j{}_A}, \\ \{F, \Lambda^i{}_j\} & = & x^i \dfrac{\partial F}{\partial x^j}, \\ \{F, \widehat{\Sigma}^A{}_B\} & = & \varphi^i{}_B \dfrac{\partial F}{\partial \varphi^i{}_A}. \end{array}$$

Canonical affine spin:

$$\begin{array}{rcl} K^{ij} & = & \displaystyle\int \left(y^i - x^i\right)\left(\dot{y}^j - \dot{x}^j\right) d\mu_\varphi(y) = \int \left(y^i - x^i\right) \dot{\varphi}^j{}_K a^K d\mu(a) \\ & = & \varphi^i{}_A \dfrac{d\varphi^j{}_B}{dt} J^{AB} \end{array}$$

Dipole of distribution of linear momentum.

Affine moment of forces:

$$N^{ij} = \int \left(y^i - x^i\right) \mathcal{F}^j(y)\, d\mu(y),$$

where $\mathcal{F}^j$ is the force distribution.

Equations of motion:

$$\begin{array}{ccccc} m\dfrac{d^2x^i}{dt^2} & = & F^i & = & -g^{ij}\dfrac{dV}{dx^j}, \\ & & \text{(total force)} & & \text{(potential case)} \end{array}$$

$$\varphi^i{}_A \frac{d^2\varphi^j{}_B}{dt^2} J^{AB} = N^{ij} = -\varphi^i{}_A \frac{\partial V}{\partial \varphi^k{}_A} g^{kj}.$$

Balance form:

$$\begin{aligned}\frac{dk^i}{dt} &= F^i,\\ \frac{dK^{ij}}{dt} &= \frac{d\varphi^i{}_A}{dt}\frac{d\varphi^j{}_B}{dt}J^{AB}+N^{ij},\end{aligned}$$

where $k^i = g^{ij}p_j$ and $k^i = \varphi^i{}_A\widehat{k}^A$.

No really Euler form — the non-dynamical term does not vanish ever — affine symmetry of degrees of freedom broken to the orthogonal one:

$$\frac{d\varphi^i{}_A}{dt}\frac{d\varphi^j{}_B}{dt}J^{AB} = 2\frac{\partial T_{\rm int}}{\partial g_{ij}}.$$

Similarly:

$$\begin{aligned}\frac{d\widehat{k}^A}{dt} &= -\widehat{k}^B\widetilde{J}_{BC}\widehat{K}^{CA}+\widehat{F}^A,\\ \frac{d\widehat{K}^{AB}}{dt} &= -\widehat{K}^{AC}\widetilde{J}_{CD}\widehat{K}^{DB}+\widehat{N}^{AB},\end{aligned}$$

or, using non-holonomic velocities,

$$\begin{aligned}m\frac{d\widehat{v}^A}{dt} &= -m\widehat{\Omega}^A{}_B\widehat{v}^B+\widehat{F}^A,\\ J^{AC}\frac{d\widehat{\Omega}^B{}_C}{dt} &= -\widehat{\Omega}^B{}_D\widehat{\Omega}^D{}_C J^{CA}+\widehat{N}^{AB}.\end{aligned}$$

What would be affine models?

Left affinely invariant:

$$T_{\rm int} = \frac{1}{2}\mathcal{L}^B{}_A{}^D{}_C\widehat{\Omega}^A{}_B\widehat{\Omega}^C{}_D, \qquad \frac{d\Sigma^i{}_j}{dt} = N^i{}_j$$

Right affinely invariant:

$$T_{\rm int} = \frac{1}{2}\mathcal{R}^j{}_i{}^l{}_k\Omega^i{}_j\Omega^k{}_l, \qquad \frac{d\widehat{\Sigma}^A{}_B}{dt} = \widehat{N}^A{}_B$$

Doubly affinely invariant:

$$T_{\rm int} = \frac{A}{2}{\rm Tr}\left(\Omega^2\right)+\frac{B}{2}\left({\rm Tr}\ \Omega\right)^2 = \frac{A}{2}{\rm Tr}\left(\widehat{\Omega}^2\right)+\frac{B}{2}\left({\rm Tr}\ \widehat{\Omega}\right)^2$$

Comment to d'Alembert:

$$T_{\rm int} = \frac{1}{2}\mathcal{A}^K{}_i{}^L{}_j\frac{d\varphi^i{}_K}{dt}\frac{d\varphi^j{}_L}{dt},$$

where

$$\mathcal{A}^K{}_i{}^L{}_j = g_{ij}J^{KL}.$$

Translational motion:

$$T_{\rm tr} = \frac{m}{2}C_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = \frac{m}{2}\eta_{AB}\widehat{v}^A\widehat{v}^B$$

or

$$T_{\rm tr} = \frac{m}{2}g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = \frac{m}{2}G_{AB}\widehat{v}^A\widehat{v}^B$$

Equations of motion:

- $\mathcal{L}$-affine invariant:

$$\frac{dp_i}{dt} = Q_i, \qquad \frac{d\Sigma^i{}_j}{dt} = -\frac{1}{m}\widetilde{C}^{ik}p_k p_j + Q^i{}_j,$$

where

$$Q_i = -\frac{\partial V}{\partial x^i}, \qquad Q^i{}_j = -\varphi^i{}_A \frac{\partial V}{\partial \varphi^j{}_A},$$

or in the other form:

$$\frac{dp_i}{dt} = Q_i, \qquad \frac{dI(\mathcal{O})^i{}_j}{dt} = Q_{\rm tot}(\mathcal{O})^i{}_j,$$

where

$$\begin{aligned} I(\mathcal{O})^i{}_j &= \Lambda(\mathcal{O})^i{}_j + \Sigma^i{}_j = x^i p_j + \Sigma^i{}_j, \\ Q_{\rm tot}(\mathcal{O})^i{}_j &= Q_{\rm tr}(\mathcal{O})^i{}_j + Q^i{}_j = x^i Q_j + Q^i{}_j. \end{aligned}$$

- $\mathcal{R}$-affine invariant:

$$\frac{dp_i}{dt} = Q_i, \qquad \frac{d\widehat{\Sigma}^A{}_B}{dt} = \widehat{Q}^A{}_B,$$

where

$$\widehat{Q}^A{}_B = -\frac{\partial V}{\partial \varphi^i{}_A}\varphi^i{}_B = (\varphi^{-1})^A{}_i Q^i{}_j \varphi^j{}_B.$$

Left affine, right metrical:

$$T_{\rm int} = \frac{I}{2}\eta_{KL}\widehat{\Omega}^K{}_M\widehat{\Omega}^L{}_N\eta^{MN} + \frac{A}{2}\widehat{\Omega}^K{}_L\widehat{\Omega}^L{}_K + \frac{B}{2}\widehat{\Omega}^K{}_K\widehat{\Omega}^L{}_L$$

(drunk missile, effective mass).

Right affine, left metrical:

$$T_{\rm int} = \frac{I}{2}g_{ik}\Omega^i{}_j\Omega^k{}_l g^{jl} + \frac{A}{2}\Omega^i{}_j\Omega^j{}_i + \frac{B}{2}\Omega^i{}_i\Omega^j{}_j$$

(Arnold discretized?).

Two-polar decomposition:

$$\varphi = LDR^{-1},$$

where $L, R \in \mathrm{SO}(n, \mathbb{R})$ — diagonal, D — diagonal, deformation invariants.

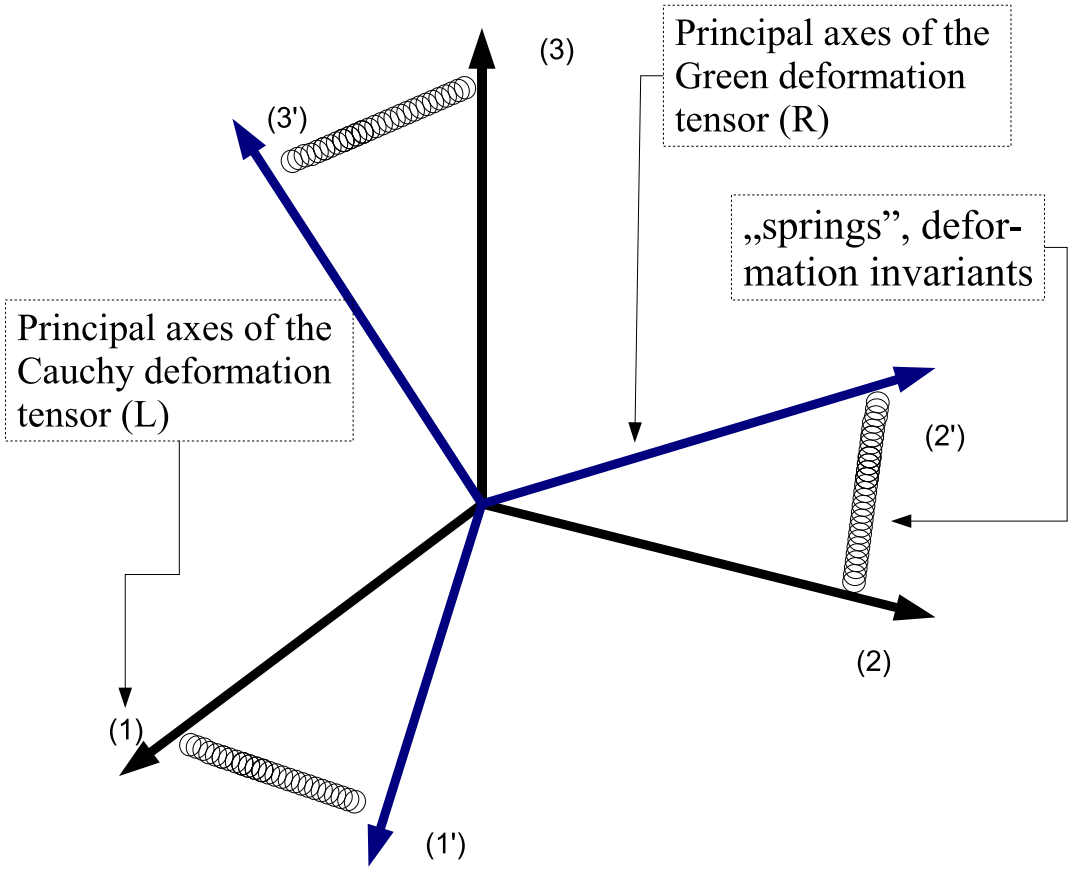

Fig. 8

Angular velocities and canonical momenta:

$$\widehat{\chi}^a{}_b = L^a{}_i \frac{dL^i{}_b}{dt}, \quad \text{its conjugate is} \quad \widehat{\rho}$$

$$\widehat{\vartheta}^a{}_b = R^a{}_K \frac{dR^K{}_b}{dt} \quad \text{its conjugate is} \quad \widehat{\tau}$$

$$M := -\widehat{\rho} - \widehat{\tau}, \qquad N := \widehat{\rho} - \widehat{\tau}$$

and then the second-order Casimir invariant has the form

$$C(2) = \sum_a p_a^2 + \frac{1}{16} \sum_{a,b} \frac{(M^a{}_b)^2}{\mathrm{sh}^2 \frac{q^a - q^b}{2}} - \frac{1}{16} \sum_{a,b} \frac{(N^a{}_b)^2}{\mathrm{ch}^2 \frac{q^a - q^b}{2}},$$

where the symbols are used: $Q^a = D^{aa}$, $q^a = \ln Q^a$.

Lattice structure:

$$\mathcal{T}_{\rm latt} = \frac{1}{2\alpha}\sum_a p_a^2 + \frac{1}{32\alpha}\sum_{a,b}\frac{(M^a{}_b)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}} - \frac{1}{32\alpha}\sum_{a,b}\frac{(N^a{}_b)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}}$$

- Hyperbolic Sutherland-like lattices:

$$\begin{aligned}
\mathcal{T}^{\rm aff}_{\rm int} &= \frac{1}{4(I+A)n}\sum_{a,b}(p_a-p_b)^2 + \frac{1}{32(I+A)}\sum_{a,b}\frac{(M^a{}_b)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}} \\
&- \frac{1}{32(I+A)}\sum_{a,b}\frac{(N^a{}_b)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}} + \frac{1}{2n(I+A+nB)}p^2, \\
\mathcal{T}^{\rm aff-metr}_{\rm int} &= \mathcal{T}^{\rm aff}_{\rm int} + \frac{I}{2(I^2-A^2)}\|V\|^2, \\
\mathcal{T}^{\rm metr-aff}_{\rm int} &= \mathcal{T}^{\rm aff}_{\rm int} + \frac{I}{2(I^2-A^2)}\|S\|^2.
\end{aligned}$$

- Calogero-Moser-like lattices:

$$\mathcal{T}_{\rm int} = \frac{1}{2I}\sum_a P_a^2 + \frac{1}{8I}\sum_{a,b}\frac{(M^a{}_b)^2}{(Q^a-Q^b)^2} + \frac{1}{8I}\sum_{a,b}\frac{(N^a{}_b)^2}{(Q^a+Q^b)^2}.$$

- Usual Sutherland-like lattices:

$$\begin{aligned}
\mathcal{T}_{\rm int} &= \frac{1}{2A}\sum_a p_a^2 - \frac{B}{2A(A+nB)}p^2 \\
&+ \frac{1}{32A}\sum_{a,b}\frac{(M^a{}_b)^2}{\sin^2\frac{q^a-q^b}{2}} + \frac{1}{32A}\sum_{a,b}\frac{(N^a{}_b)^2}{\cos^2\frac{q^a-q^b}{2}}.
\end{aligned}$$

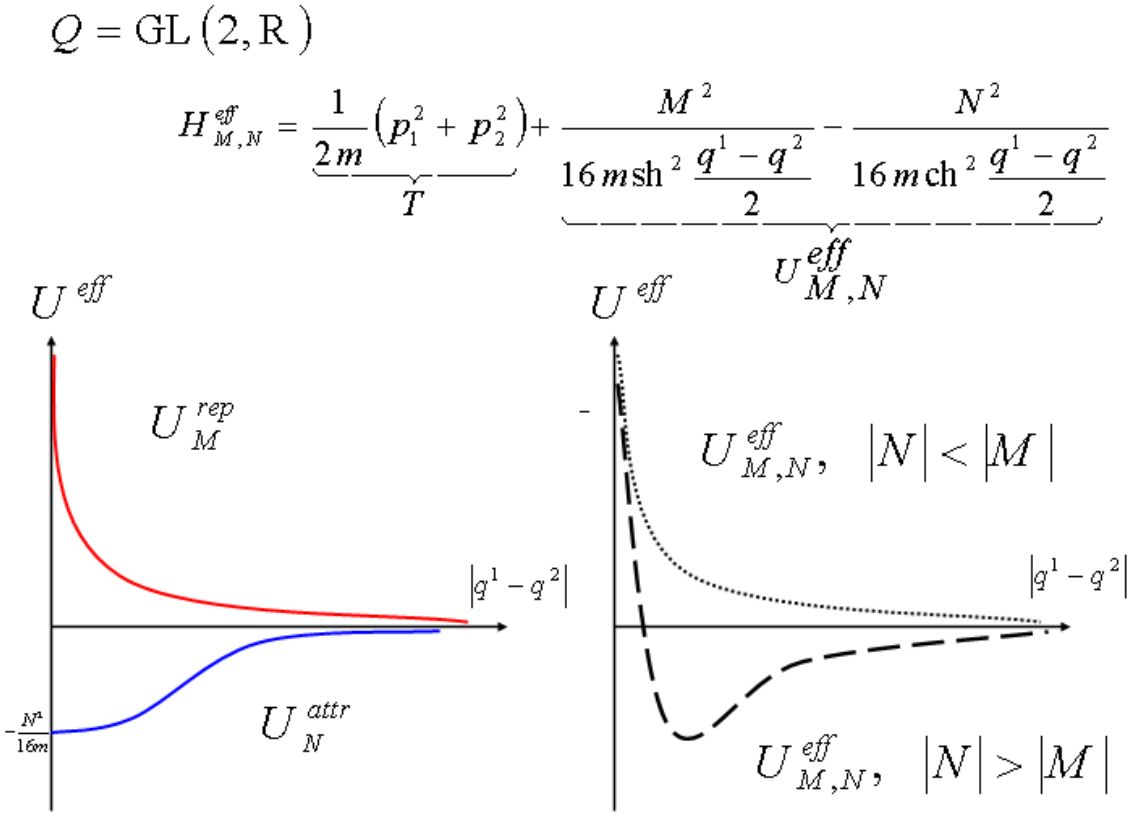

Fig. 9

3 Schrödinger Quantization — General and Affine Models

A fascinating feature of our models of affine collective dynamics is their extremely wide range of applications. It covers the nuclear and molecular dynamics, micromechanics of structured continua, perhaps nanostructure and defects phenomena, macroscopic elasticity and astrophysical phenomena like vibration of stars and clouds of cosmic dust. Obviously, microphysical applications must be based on the quantized version of the theory. And one is dealing then with a very curious convolution of quantum theory with mathematical methods of continuum mechanics. It is worth to mention that there were even attempts, mainly by Barut and Rączka, to describe the dynamics of strongly interacting elementary particles (hadrons) in terms of some peculiar, quantized continua. By the way, as French say, the extremes teach one another; it is not excluded that the dynamics of cosmic objects like neutron stars must be also described in quantum terms. They are though giant nuclei, very exotic ones, because composed exclusively of neutrons (enormous "mass numbers" and vanishing "atomic numbers").

Quantization of classical geodetic systems

Let us consider a classical geodetic system in a Riemannian manifold (Q, Γ), where Q denotes the configuration space, and Γ is the "metric" tensor field on Q underlying the kinetic energy form. In terms of generalized coordinates we have

$$T = \frac{1}{2}\Gamma_{\mu\nu}\frac{dq^\mu}{dt}\frac{dq^\nu}{dt},$$

or in Hamiltonian terms

$$\mathcal{T} = \frac{1}{2}\Gamma^{\mu\nu}p_\mu p_\nu,$$

where, obviously,

$$\Gamma^{\mu\alpha}\Gamma_{\alpha\nu} = \delta^\mu{}_\nu$$

and

$$p_\mu = \frac{\partial T}{\partial \dot{q}^\mu} = \Gamma_{\mu\nu}\frac{dq^\nu}{dt}.$$

As usual, the metric tensor Γ gives rise to the natural measure μ_Γ on Q,

$$d\mu_\Gamma(q) = \sqrt{|\det[\Gamma_{\mu\nu}]|}dq^1 \cdots dq^f,$$

where f denotes the number of degrees of freedom, i.e., $f = \dim Q$. For simplicity the square-root expression will be always denoted by $\sqrt{|\Gamma|}$. The mathematical framework of Schrödinger quantization is based on $\mathrm{L}^2(Q, \mu_\Gamma)$, i.e., the Hilbert space of complex-valued wave functions on Q, which are square-integrable in the μ_Γ-sense. Their scalar product is given by the usual formula:

$$< \Psi_1|\Psi_2 >= \int \overline{\Psi}_1(q)\Psi_2(q)d\mu_\Gamma(q).$$

The classical kinetic energy expression is replaced by the operator

$$\mathbf{T} = -\frac{\hbar^2}{2}\Delta(\Gamma),$$

where $\hbar$ denotes the ("crossed") Planck constant, and $\Delta(\Gamma)$ is the Laplace-Beltrami operator corresponding to Γ,

$$\Delta(\Gamma) = \frac{1}{\sqrt{|\Gamma|}}\sum_{\mu,\nu}\partial_\mu\sqrt{|\Gamma|}\Gamma^{\mu\nu}\partial_\nu = \Gamma^{\mu\nu}\nabla_\mu\nabla_\nu.$$

In the last expression ∇_μ denotes the Levi-Civita covariant differentiation in the Γ-sense. Therefore, the kinetic energy operator $\mathbf{T}$ is formally obtained from the corresponding classical expression $\mathcal{T}$ (kinetic Hamiltonian) by the substitution

$$p_\mu \mapsto \mathbf{p}_\mu = \frac{\hbar}{i}\nabla_\mu.$$

If the problem is non-geodetic and some potential $V(q)$ is admitted, the corresponding Hamilton (energy) operator is given by:

$$\mathbf{H} = \mathbf{T} + \mathbf{V},$$

where the operator $\mathbf{V}$ acts on wave functions simply multiplying them by V,

$$(\mathbf{V}\Psi)(q) = V(q)\Psi(q).$$

This is the reason why very often one does not distinguish graphically between $\mathbf{V}$ and V.

Stationary situation

We shall deal almost exclusively with stationary problems when the Hamilton operator $\mathbf{H}$ is time-independent, thus, the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \mathbf{H}\psi$$

will be replaced by its stationary form, i.e., by the eigenequation

$$\mathbf{H}\Psi = E\Psi,$$

where, obviously,

$$\psi = \exp\left(-\frac{i}{\hbar}Et\right)\Psi$$

and Ψ is a time-independent wave function on the configuration space.

Classical background for quantization

We concentrate on the collective modes ruled by the linear and affine groups. It is an important fact from the Lie group theory that Lie-algebraic objects $\Omega, \widehat{\Omega} \in G'$ give rise to some vector fields X, Y on G invariant, respectively, under right and left translations on G. Namely, for any fixed $\Omega, \widehat{\Omega} \in G'$, they are given by

$$X_g[\Omega] := \Omega g, \qquad Y_g[\widehat{\Omega}] := g\widehat{\Omega}.$$

Affine velocities are just the special case of these Lie-algebraic objects. The dual objects Σ, $\widehat{\Sigma}$, i.e., affine spin in two representations, can be introduced. These dual quantities exist also in the general case when G is an arbitrary Lie group. They are then elements of the dual space, i.e., Lie co-algebra, $\Sigma, \widehat{\Sigma} \in G'^*$. Their relationship with canonical momenta p and configurations g is given by the following formula involving evaluations of co-vectors on vectors:

$$< p, \dot{g} >=< \Sigma, \Omega >=< \widehat{\Sigma}, \widehat{\Omega} >,$$

where $\dot{g} \in T_gG$, $p \in T_g^*G$, and g, $\dot{g}$ are arbitrary. Denoting the adjoint transformation of Ad_g by the usual symbol Ad_g^*, we have that

$$\Sigma = \mathrm{Ad}_g^{*-1}\widehat{\Sigma},$$

the obvious generalization of the corresponding relationship between laboratory and co-moving representation of affine (or usual metrical) spin. And just as in this special case, the quantities Σ, $\widehat{\Sigma}$ are Hamiltonian generators of the groups of left and right regular translations L_G, R_G on G.

In applications we are usually dealing with some special Lie groups for which many important formulas and relationships may be written in a technically simple form avoiding the general abstract terms.

We are dealing almost exclusively with linear groups $G \in \mathrm{GL}(W) \subset \mathrm{L}(W)$, where W is a linear space, e.g., some $\mathbb{R}^n$ or $\mathbb{C}^n$.

All the mentioned simplifications follow from the obvious canonical isomorphism between $\mathrm{L}(W)$ and its dual $\mathrm{L}(W)^*$, based on the pairing

$$< C, D >= \mathrm{Tr}\,(CD)\,.$$

The Lie algebra G' is a linear subspace of $\mathrm{L}(W)$, therefore, its dual space G'^* may be canonically identified with the quotient space $\mathrm{L}(W)^*/\mathrm{An}G'$, where $\mathrm{An}G'$ denotes the subspace of linear functions vanishing on G'. But, according to the above identification between $\mathrm{L}(W)^*$ and $\mathrm{L}(W)$ itself, $\mathrm{An}G'$ may be identified with some linear subspace of $\mathrm{L}(W)$; we shall denote it by $G'^{\perp}$. Therefore, the Lie co-algebra G'^* is canonically isomorphic with the corresponding quotient, i.e.,

$$G'^* \simeq \mathrm{L}(W)/G'^{\perp}.$$

This is the general fact for linear groups and their Lie algebras. However, in some special cases, just ones of physical relevance, this quotient space admits a natural canonical isomorphism onto some distinguished linear subspace of $\mathrm{L}(W)$ consisting of natural representants of cosets, e.g., in the most practical cases G'^* is canonically isomorphic with G' itself. For example, it is so for $\mathrm{SO}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{R})$, where the Lie algebras $\mathrm{SO}(n,\mathbb{R})'$, $\mathrm{SL}(n,\mathbb{R})'$ may be identified with the duals $\mathrm{SO}(n,\mathbb{R})'^*$, $\mathrm{SL}(n,\mathbb{R})'^*$. By the way, for certain reasons it is more convenient to use the pairing

$$< A, B >= -\frac{1}{2}\mathrm{Tr}(AB)$$

for the orthogonal group $\mathrm{SO}(n,\mathbb{R})$.

Just as in the special case of affine objects, transformation rules for Σ, $\widehat{\Sigma}$ are analogous to those for Ω, $\widehat{\Omega}$; we mean transformations under regular translations:

$$\begin{aligned} L_k \;&:\quad \Sigma \mapsto \mathrm{Ad}_k^{*-1}\Sigma, \qquad \widehat{\Sigma} \mapsto \widehat{\Sigma}, \\ R_k \;&:\quad \Sigma \mapsto \Sigma, \qquad \widehat{\Sigma} \mapsto \mathrm{Ad}_k^{*}\widehat{\Sigma}. \end{aligned}$$

Using the identifications mentioned above (assuming that they work), we can write these rules in a form analogous to that for non-holonomic velocities,

$$\begin{aligned} L_k \;&:\quad \Sigma \mapsto k\Sigma k^{-1}, \qquad \widehat{\Sigma} \mapsto \widehat{\Sigma}, \\ R_k \;&:\quad \Sigma \mapsto \Sigma, \qquad \widehat{\Sigma} \mapsto k^{-1}\widehat{\Sigma}k, \end{aligned}$$

i.e., just as it is for the affine spin.

Geometrical meaning of Σ and $\widehat{\Sigma}$ is that of the momentum mappings induced, respectively, by the group of left and right regular translations. And the relationship between two versions of Σ-objects is given as follows:

$$\Sigma = g\widehat{\Sigma}g^{-1}.$$

The objects Σ and $\widehat{\Sigma}$ may be also interpreted in terms of right- and left-invariant differential forms (co-vector fields), i.e., Maurer-Cartan forms A, B on the group G. Assuming the afore-mentioned identification, we can express A, B for any fixed Σ, $\widehat{\Sigma}$ in the following forms:

$$A_g[\Sigma] = g^{-1}\Sigma, \qquad B_g[\widehat{\Sigma}] = \widehat{\Sigma}^{-1}g.$$

Just as in the special case of affine systems, Poisson bracket relations of Σ- and $\widehat{\Sigma}$-components are given by structure constants of G. Those for $\widehat{\Sigma}$ have opposite signs to those for Σ, and the mutual ones vanish (left regular translations commute with the right ones).

Hamiltonian systems on Lie group spaces

We assume that our configuration space Q is a Lie group G or, more precisely, its homogeneous space with trivial isotropy groups. Also in a more general situation when isotropy groups are nontrivial (even continuous) a large amount of analysis performed on group spaces remains useful.

Obviously, just as in the special case of affinely-rigid bodies, left- and right-invariant kinetic energies T are, respectively, quadratic forms of $\widehat{\Omega}$ and Ω with constant coefficients. Their underlying Riemannian structures on G are locally flat if and only if G is Abelian.

In both theoretical and practical problems the Hamilton language based on Poisson brackets is much more lucid and efficient than that based on Lagrange equations. If besides of geodetic inertia the system is influenced only by potential forces derivable from some potential energy term $V(q)$, then, obviously, the classical Hamiltonian is given by the following expression:

$$H = \mathcal{T} + V(q) = \frac{1}{2}\Gamma^{\mu\nu}(q)p_\mu p_\nu + V(q).$$

It is very convenient to express the Hamiltonian and all other essential quantities in terms of non-holonomic velocities and their conjugate non-holonomic (Poisson-non-commuting) momenta.

Let $\{E_\mu\}$ be some basis in the Lie algebra G' and q^μ be the corresponding canonical coordinates of the first kind on G, i.e.,

$$g(q) = \exp\left(q^\mu E_\mu\right).$$

Lie-algebraic objects $\Omega, \widehat{\Omega} \in G'$ will be, respectively, expanded as follows:

$$\Omega = \Omega^\mu E_\mu, \qquad \widehat{\Omega} = \widehat{\Omega}^\mu E_\mu.$$

Using the expansion coefficients Ω^μ, $\widehat{\Omega}^\mu$ one obtains the following simple expressions for the left- and right-invariant kinetic energies:

$$T_{\text{left}} = \frac{1}{2}\mathcal{L}_{\mu\nu}\widehat{\Omega}^\mu\widehat{\Omega}^\nu, \qquad T_{\text{right}} = \frac{1}{2}\mathcal{R}_{\mu\nu}\Omega^\mu\Omega^\nu,$$

where the matrices $\mathcal{L}$, $\mathcal{R}$ are constant, symmetric, and non-singular. The positive definiteness problem is a more delicate matter, and there are some hyperbolic-signature structures of some relevance both for physics and pure geometry.

For potential systems Legendre transformation may be easily described with the use of non-holonomic objects, respectively,

$$\widehat{\Sigma}_\mu = \frac{\partial T_{\text{left}}}{\partial \widehat{\Omega}^\mu} = \mathcal{L}_{\mu\nu}\widehat{\Omega}^\nu, \qquad \Sigma_\mu = \frac{\partial T_{\text{right}}}{\partial \Omega^\mu} = \mathcal{R}_{\mu\nu}\Omega^\nu,$$

where, obviously, $\widehat{\Sigma}_\mu$, Σ_μ are expansion coefficients of $\widehat{\Sigma}$, Σ with respect to the dual basis $\{E^\mu\}$ of the Lie co-algebra, i.e.,

$$\widehat{\Sigma} = \widehat{\Sigma}_\mu E^\mu, \qquad \Sigma = \Sigma_\mu E^\mu.$$

The resulting Hamiltonians have, respectively, the following forms:

$$H = \mathcal{T}_{\text{left}} + V(q) = \frac{1}{2}\mathcal{L}^{\mu\nu}\widehat{\Sigma}_\mu\widehat{\Sigma}_\nu + V(q),$$

$$H = \mathcal{T}_{\text{right}} + V(q) = \frac{1}{2}\mathcal{R}^{\mu\nu}\Sigma_\mu\Sigma_\nu + V(q),$$

where, obviously, the matrices $[\mathcal{L}^{\mu\nu}]$, $[\mathcal{R}^{\mu\nu}]$ are reciprocal to $[\mathcal{L}_{\mu\nu}]$, $[\mathcal{R}_{\mu\nu}]$.

If structure constants of G' with respect to the basis $\{E_\mu\}$ are defined according to the convention

$$[E_\mu, E_\nu] = E_\lambda C^\lambda{}_{\mu\nu},$$

then the Poisson brackets of Σ-objects are given as follows:

$$\{\Sigma_\mu, \Sigma_\nu\} = \Sigma_\lambda C^\lambda{}_{\mu\nu}, \qquad \{\widehat{\Sigma}_\mu, \widehat{\Sigma}_\nu\} = -\widehat{\Sigma}_\lambda C^\lambda{}_{\mu\nu}, \qquad \{\Sigma_\mu, \widehat{\Sigma}_\nu\} = 0.$$

Basic differential operators

Let us define basic differential operators generating left and right regular translations on G. We denote them respectively by $\mathbf{L}_\mu$ and $\mathbf{R}_\mu$. Their action on complex- or vector-valued functions F on G is defined as follows:

$$(\mathbf{L}_\mu F)(g) := \frac{\partial}{\partial q^\mu} F(k(q)g)\Big|_{q=0}, \qquad (\mathbf{R}_\mu F)(g) := \frac{\partial}{\partial q^\mu} F(gk(q))\Big|_{q=0}.$$

Their Lie-bracket (commutator) relations differ from the above Poisson rules for Σ-quantities by signs:

$$[\mathbf{L}_\mu, \mathbf{L}_\nu] = -\mathbf{L}_\lambda C^\lambda{}_{\mu\nu}, \qquad [\mathbf{R}_\mu, \mathbf{R}_\nu] = \mathbf{R}_\lambda C^\lambda{}_{\mu\nu}, \qquad [\mathbf{L}_\mu, \mathbf{R}_\nu] = 0.$$

Poisson brackets between Σ-objects and functions F depending only on coordinates q (pull-backs of functions defined on the configuration space $Q = G$) are given by

$$\{\Sigma_\mu, F\} = -\mathbf{L}_\mu F, \qquad \{\widehat{\Sigma}_\mu, F\} = -\mathbf{R}_\mu F.$$

The system of Poisson brackets quoted above is sufficient for calculating any other Poisson bracket with the help of well-known properties of this operation. Thus, e.g., for any pair of functions A, B depending in general on all phase-space variables we have the following expression:

$$\{A, B\} = \Sigma_\lambda C^\lambda{}_{\mu\nu} \frac{\partial A}{\partial \Sigma_\mu} \frac{\partial B}{\partial \Sigma_\nu} - \frac{\partial A}{\partial \Sigma_\mu} \mathbf{L}_\mu B + \frac{\partial B}{\partial \Sigma_\mu} \mathbf{L}_\mu A,$$

and, when the phase space is parameterized in terms of quantities q^μ, $\widehat{\Sigma}_\mu$, we have the similar expression:

$$\{A, B\} = \widehat{\Sigma}_\lambda C^\lambda{}_{\mu\nu} \frac{\partial A}{\partial \widehat{\Sigma}_\mu} \frac{\partial B}{\partial \widehat{\Sigma}_\nu} - \frac{\partial A}{\partial \widehat{\Sigma}_\mu} \mathbf{R}_\mu B + \frac{\partial B}{\partial \widehat{\Sigma}_\mu} \mathbf{R}_\mu A.$$

Obviously, the finite regular translations may be expressed in terms of the following exponential formulas:

$$F(k(q)g) = \exp(q^\mu \mathbf{L}_\mu) F, \qquad F(gk(q)) = \exp(q^\mu \mathbf{R}_\mu) F,$$

with all known provisos concerning exponentiation of differential operators.

Non-holonomic velocities Ω, $\widehat{\Omega}$ depend linearly on generalized velocities $\dot{q}$,

$$\Omega^\mu = \Omega^\mu{}_\nu(q)\dot{q}^\nu, \qquad \widehat{\Omega}^\mu = \widehat{\Omega}^\mu{}_\nu(q)\dot{q}^\nu.$$

Similarly, Σ and $\widehat{\Sigma}$ depend contragradiently on the conjugate momenta p,

$$\Sigma_\mu = p_\alpha \Sigma^\alpha{}_\mu(q), \qquad \widehat{\Sigma}_\mu = p_\alpha \widehat{\Sigma}^\alpha{}_\mu(q),$$

where, obviously,

$$\Sigma^\alpha{}_\mu \Omega^\mu{}_\beta = \delta^\alpha{}_\beta, \qquad \widehat{\Sigma}^\alpha{}_\mu \widehat{\Omega}^\mu{}_\beta = \delta^\alpha{}_\beta.$$

This leads to the following expressions for generators:

$$\mathbf{L}_\mu = \Sigma^\alpha{}_\mu \frac{\partial}{\partial q^\alpha}, \qquad \mathbf{R}_\mu = \widehat{\Sigma}^\alpha{}_\mu \frac{\partial}{\partial q^\alpha}.$$

Unitary transformations

It follows from the very nature of the Haar measure μ that on the level of wave functions the left and right regular translations are realized by unitary transformations on $\mathrm{L}^2(G,\mu)$. More precisely, let us define for any $k \in G$ the operators $\mathbf{L}(k)$, $\mathbf{R}(k)$ given by

$$\left(\mathbf{L}(k)\Psi\right)(g) := \Psi(kg), \qquad \left(\mathbf{R}(k)\Psi\right)(g) := \Psi(gk)$$

for any $g \in G$. It is clear that $\mathbf{L}(k)$, $\mathbf{R}(k)$ preserve the space $\mathrm{L}^2(G,\mu)$, moreover, they are unitary transformations,

$$\langle \mathbf{L}(k)\Psi_1 | \mathbf{L}(k)\Psi_2 \rangle = \langle \mathbf{R}(k)\Psi_1 | \mathbf{R}(k)\Psi_2 \rangle = \langle \Psi_1 | \Psi_2 \rangle .$$

The assignments $G \ni k \mapsto \mathbf{L}(k), \mathbf{R}(k)$ are, respectively, a unitary anti-representation and representation of G in $\mathrm{L}^2(G,\mu)$,

$$\mathbf{L}(k_1k_2) = \mathbf{L}(k_2)\mathbf{L}(k_1), \qquad \mathbf{R}(k_1k_2) = \mathbf{R}(k_1)\mathbf{R}(k_2).$$

To convert $\mathbf{L}$ into representation it is sufficient to replace $\Psi(kg)$ by $\Psi(k^{-1}g)$. Obviously, the difference is rather cosmetical and related to the conventions concerning the definition of the superposition of mappings. Nevertheless, any neglect may lead to the accumulation of sign errors and finally to numerically wrong results.

The operators $\mathbf{L}_\mu$, $\mathbf{R}_\mu$ generate the above representations, thus, we have

$$\mathbf{L}\left(\exp(q^\mu E_\mu)\right) = \exp(q^\mu \mathbf{L}_\mu), \qquad \mathbf{R}\left(\exp(q^\mu E_\mu)\right) = \exp(q^\mu \mathbf{R}_\mu),$$

with all known provisos concerning domains and exponents of evidently unbounded differential operators. It is important to remember that the left-hand sides are always well-defined bounded unitary operators acting on the whole $\mathrm{L}^2(G,\mu)$. Unlike this, $\mathbf{L}_\mu$, $\mathbf{R}_\mu$ act only on differentiable functions, they are unbounded, and the problems of domain and convergence appear on the right-hand sides of the above equations.

Unitarity of $\mathbf{L}$, $\mathbf{R}$ implies that their generators $\mathbf{L}_\mu$, $\mathbf{R}_\mu$ are formally anti-self-adjoint (physicists tell roughly: anti-Hermitian), i.e.,

$$\langle \mathbf{L}_\mu\Psi_1 | \Psi_2 \rangle = -\langle \Psi_1 | \mathbf{L}_\mu\Psi_2 \rangle , \qquad \langle \mathbf{R}_\mu\Psi_1 | \Psi_2 \rangle = -\langle \Psi_1 | \mathbf{R}_\mu\Psi_2 \rangle ,$$

assuming that the left- and right-hand sides are well-defined (this is the case, e.g., for differentiable compactly supported functions on G).

Now, let us introduce the following operators:

$$\mathbf{\Sigma}_\mu := \frac{\hbar}{i}\mathbf{L}_\mu, \qquad \widehat{\mathbf{\Sigma}}_\mu := \frac{\hbar}{i}\mathbf{R}_\mu.$$

They are formally self-adjoint, i.e., "Hermitian" in the rough language of quantum physicists:

$$\langle \mathbf{\Sigma}_\mu\Psi_1 | \Psi_2 \rangle = \langle \Psi_1 | \mathbf{\Sigma}_\mu\Psi_2 \rangle , \qquad \left\langle \widehat{\mathbf{\Sigma}}_\mu\Psi_1 | \Psi_2 \right\rangle = \left\langle \Psi_1 | \widehat{\mathbf{\Sigma}}_\mu\Psi_2 \right\rangle ,$$

with the same as previously provisos concerning the functions Ψ_1, Ψ_2. Obviously, $\hbar$ denotes the ("crossed") Planck constant.

The operators $\mathbf{\Sigma}_\mu$, $\widehat{\mathbf{\Sigma}}_\mu$ are quantized counterparts of classical physical quantities Σ_μ, $\widehat{\Sigma}_\mu$. They may be expressed as follows:

$$\mathbf{\Sigma}_\mu = \frac{\hbar}{i}\Sigma^\alpha{}_\mu(q)\frac{\partial}{\partial q^\alpha}, \qquad \widehat{\mathbf{\Sigma}}_\mu = \frac{\hbar}{i}\widehat{\Sigma}^\alpha{}_\mu(q)\frac{\partial}{\partial q^\alpha}.$$

There is no problem of ordering of q-variables and differential operators $\partial/\partial q^\alpha$. This ordering is exactly as above, just due to the interpretation of $\mathbf{L}_\mu$ and $\mathbf{R}_\mu$ as infinitesimal generators of one-parameter subgroups.

Quantum Poisson bracket

In virtue of the above group-theoretical arguments the quantum Poisson-bracket rules are analogous to the classical ones,

$$_Q\{\mathbf{\Sigma}_\mu, \mathbf{\Sigma}_\nu\} = \mathbf{\Sigma}_\lambda C^\lambda{}_{\mu\nu}, \qquad _Q\{\widehat{\mathbf{\Sigma}}_\mu, \widehat{\mathbf{\Sigma}}_\nu\} = -\widehat{\mathbf{\Sigma}}_\lambda C^\lambda{}_{\mu\nu}, \qquad _Q\{\mathbf{\Sigma}_\mu, \widehat{\mathbf{\Sigma}}_\nu\} = 0.$$

Let us remind that the quantum Poisson bracket of operators is defined as

$$_Q\{\mathbf{A}, \mathbf{B}\} := \frac{1}{i\hbar}[\mathbf{A}, \mathbf{B}] = \frac{1}{i\hbar}(\mathbf{AB} - \mathbf{BA}).$$

One can show that the kinetic energy operators for the left- and right-invariant models are given simply by the formerly quoted formulas with the classical generators Σ_μ, $\widehat{\Sigma}_\mu$ replaced by the corresponding operators $\mathbf{\Sigma}_\mu$, $\widehat{\mathbf{\Sigma}}_\mu$, i.e.,

$$\mathcal{T}_{\text{left}} = \frac{1}{2}\mathcal{R}^{\mu\nu}\widehat{\mathbf{\Sigma}}_\mu\widehat{\mathbf{\Sigma}}_\nu = -\frac{\hbar^2}{2}\mathcal{R}^{\mu\nu}\mathbf{R}_\mu\mathbf{R}_\nu,$$

$$\mathcal{T}_{\text{right}} = \frac{1}{2}\mathcal{L}^{\mu\nu}\mathbf{\Sigma}_\mu\mathbf{\Sigma}_\nu = -\frac{\hbar^2}{2}\mathcal{L}^{\mu\nu}\mathbf{L}_\mu\mathbf{L}_\nu.$$

As mentioned, the literal calculation of the Laplace-Beltrami operator in terms of local coordinates q^μ is usually very complicated and the resulting formula is, as a rule, quite obscure, non-readable, and because of this practically non-useful. Unlike this, the above block expression in terms of generators is geometrically lucid and well apt for solving procedure of the Schrödinger equation. In various problems it is sufficient to operate algebraically with quantum Poisson brackets. To complete the above system of brackets let us quote expressions involving generators and position-type variables. The latter ones are operators which multiply wave functions by other functions on the configuration space,

$$(\mathbf{F}\Psi)(q) := F(q)\Psi(q).$$

If there is no danger of misunderstanding, we will not distinguish graphically between $\mathbf{F}$ and F. Just as on the classical level we have

$$_Q\{\mathbf{\Sigma}_\mu, \mathbf{F}\} = -\mathbf{L}_\mu F, \qquad _Q\{\widehat{\mathbf{\Sigma}}_\mu, \mathbf{F}\} = -\mathbf{R}_\mu F.$$

Obviously, two position-type operators mutually commute.

Remark: Obviously, only for generators and position quantities the quantum and classical Poisson rules are identical. For other quantities it is no longer the case, moreover, there are problems with the very definition of quantum counterparts of other classical quantities. The very existence of the above distinguished family of physical quantities is due to the group-theoretical background of degrees of freedom.

Corresponding Haar measures

Let us now return to the main subject of our analysis, i.e., to the quantization of affine systems. For technical purposes we again fix some Cartesian coordinates x^i, a^K in M, N and identify analytically the configuration space $Q = \mathrm{LI}(U, V) \times M$ with the affine group $\mathrm{GAf}(n, \mathbb{R}) \simeq \mathrm{GL}(n, \mathbb{R}) \times_s \mathbb{R}^n$. Similarly, the internal configuration space $Q_{\text{int}} = \mathrm{LI}(U, V)$ is identified with $\mathrm{GL}(n, \mathbb{R})$. The corresponding Haar measures will be denoted respectively by α, λ, i.e.,

$$\begin{aligned} d\alpha(\varphi, x) &= (\det\varphi)^{-n-1} dx^1 \cdots dx^n d\varphi^1{}_1 \cdots d\varphi^n{}_n \\ &= (\det\varphi)^{-1} d\lambda(\varphi) dx^1 \cdots dx^n, \\ d\lambda(\varphi) &= (\det\varphi)^{-n} d\varphi^1{}_1 \cdots d\varphi^n{}_n. \end{aligned}$$

In terms of the binary decomposition we have the following expression:

$$d\lambda(\varphi) = d\lambda(l; q; r) = \prod_{i \neq j} \left|\mathrm{sh}(q^i - q^j)\right| d\mu(l) d\mu(r) dq^1 \cdots dq^n,$$

where μ denotes the Haar measure on $\mathrm{SO}(n,\mathbb{R})$. Due to the compactness of $\mathrm{SO}(n,\mathbb{R})$ we can, but of course need not, normalize μ to unity, $\mu\left(\mathrm{SO}(n,\mathbb{R})\right)=1$.

The Haar measure on $\mathrm{SL}(n,\mathbb{R})$ used in quantum mechanics of incompressible objects may be symbolically written with the use of Dirac distribution as follows:

$$d\lambda_{SL}(\varphi)=\prod_{i\neq j}\left|\mathrm{sh}(q^i-q^j)\right|\,d\mu(l)d\mu(r)\delta(q^1+\cdots+q^n)dq^1\cdots dq^n.$$

Kinetic energy operators for affine models

Affine spin and its co-moving representation are, respectively, given by the following formally self-adjoint operators:

$$\mathbf{\Sigma}^a{}_b:=\frac{\hbar}{i}\mathbf{L}^a{}_b=\frac{\hbar}{i}\varphi^a{}_K\frac{\partial}{\partial\varphi^b{}_K},\qquad \widehat{\mathbf{\Sigma}}^A{}_B:=\frac{\hbar}{i}\mathbf{R}^A{}_B=\frac{\hbar}{i}\varphi^m{}_B\frac{\partial}{\partial\varphi^m{}_A}.$$

The usual spin and vorticity operators are respectively given by

$$\mathbf{S}^a{}_b:=\mathbf{\Sigma}^a{}_b-g^{ac}g_{bd}\mathbf{\Sigma}^d{}_c,\qquad \mathbf{V}^A{}_B:=\widehat{\mathbf{\Sigma}}^A{}_B-\eta^{AC}\eta_{BD}\widehat{\mathbf{\Sigma}}^D{}_C.$$

Kinetic energy operators corresponding to the formerly described classical models of internal kinetic energies are simply obtained by replacing the classical quantities $\Sigma^a{}_b$, $\widehat{\Sigma}^A{}_B$ by the above operators $\mathbf{\Sigma}^a{}_b$, $\widehat{\mathbf{\Sigma}}^A{}_B$ without any attention to be paid to the ordering problem (just because of the group-theoretic interpretation of these quantities).

Thus, for the affine-affine model (affine both in space and in the material) we have

$$\begin{aligned}\mathbf{T}^{\mathrm{aff-aff}}_{\mathrm{int}} &= \frac{1}{2A}\mathbf{\Sigma}^i{}_j\mathbf{\Sigma}^j{}_i-\frac{B}{2A(A+nB)}\mathbf{\Sigma}^i{}_i\mathbf{\Sigma}^j{}_j\\ &= \frac{1}{2A}\widehat{\mathbf{\Sigma}}^A{}_B\widehat{\mathbf{\Sigma}}^B{}_A-\frac{B}{2A(A+nB)}\widehat{\mathbf{\Sigma}}^A{}_A\widehat{\mathbf{\Sigma}}^B{}_B.\end{aligned}$$

Similarly, for models with the mixed metrical-affine and affine-metrical invariance we have, respectively,

$$\begin{aligned}\mathbf{T}^{\mathrm{met-aff}}_{\mathrm{int}} &= \frac{1}{2\tilde{I}}g_{ik}g^{jl}\mathbf{\Sigma}^i{}_j\mathbf{\Sigma}^k{}_l+\frac{1}{2\tilde{A}}\mathbf{\Sigma}^i{}_j\mathbf{\Sigma}^j{}_i+\frac{1}{2\tilde{B}}\mathbf{\Sigma}^i{}_i\mathbf{\Sigma}^j{}_j,\\ \mathbf{T}^{\mathrm{aff-met}}_{\mathrm{int}} &= \frac{1}{2\tilde{I}}\eta_{AB}\eta^{CD}\widehat{\mathbf{\Sigma}}^A{}_C\widehat{\mathbf{\Sigma}}^B{}_D+\frac{1}{2\tilde{A}}\widehat{\mathbf{\Sigma}}^A{}_B\widehat{\mathbf{\Sigma}}^B{}_A+\frac{1}{2\tilde{B}}\widehat{\mathbf{\Sigma}}^A{}_A\widehat{\mathbf{\Sigma}}^B{}_B\end{aligned}$$

with the same as previously meaning of symbols $\tilde{I}$, $\tilde{A}$, $\tilde{B}$.

Similarly, the corresponding expressions for $\mathcal{T}_{\mathrm{tr}}$ have the following forms:

$$\begin{aligned}\mathbf{T}^{\mathrm{met-aff}}_{\mathrm{tr}} &= \frac{m}{2}g^{ij}\mathbf{P}_i\mathbf{P}_j=\frac{m}{2}\widetilde{G}^{AB}\widehat{\mathbf{P}}_A\widehat{\mathbf{P}}_B,\\ \mathbf{T}^{\mathrm{aff-met}}_{\mathrm{tr}} &= \frac{m}{2}\widetilde{C}^{ij}\mathbf{P}_i\mathbf{P}_j=\frac{m}{2}\eta^{AB}\widehat{\mathbf{P}}_A\widehat{\mathbf{P}}_B,\end{aligned}$$

where $\mathbf{P}_i$, $\widehat{\mathbf{P}}_A$ are linear momentum operators respectively in laboratory and co-moving representations,

$$\mathbf{P}_a=\frac{\hbar}{i}\frac{\partial}{\partial x^a},\qquad \widehat{\mathbf{P}}_K=\varphi^a{}_K\mathbf{P}_a=\frac{\hbar}{i}\varphi^a{}_K\frac{\partial}{\partial x^a}.$$

Just as previously, $\widetilde{C}$, $\widetilde{G}$ are contravariant reciprocals of deformation tensors,

$$\widetilde{C}^{ik}C_{kj}=\delta^i{}_j,\qquad \widetilde{G}^{AC}G_{CB}=\delta^A{}_B.$$

As mentioned, there are no affine-affine models of $\mathbf{T}_{\mathrm{tr}}$, and therefore, no affine-affine models of $\mathbf{T}$. The corresponding "metric tensors" on $\mathrm{GAf}(n,\mathbb{R})$ would have to be singular.

Another important physical quantity is the canonical momentum conjugate to the dilatational coordinate q. On the quantum level it is represented by the formally self-adjoint operator

$$\mathbf{p} = \frac{\hbar}{i}\frac{\partial}{\partial q}.$$

It is also convenient to use the deviatoric (shear) parts of the affine spin,

$$\mathbf{s}^a{}_b := \mathbf{\Sigma}^a{}_b - \frac{\mathbf{p}}{n}\delta^a{}_b, \qquad \widehat{\mathbf{s}}^A{}_B := \widehat{\mathbf{\Sigma}}^A{}_B - \frac{\mathbf{p}}{n}\delta^A{}_B;$$

obviously,

$$\mathbf{p} = \mathbf{\Sigma}^a{}_a = \widehat{\mathbf{\Sigma}}^A{}_A.$$

Due to the group-theoretical structure of the above objects as generators, the classical splitting of $\mathbf{T}$ into incompressible (shear-rotational) and dilatational parts remains literally valid, namely, we have the following expressions:

$$\begin{aligned}
\mathbf{T}^{\mathrm{aff-aff}}_{\mathrm{int}} &= \frac{1}{2A}\mathbf{C}_{\mathrm{SL}(n)}(2) + \frac{1}{2n(A+nB)}\mathbf{p}^2,\\
\mathbf{T}^{\mathrm{met-aff}}_{\mathrm{int}} &= \frac{1}{2(I+A)}\mathbf{C}_{\mathrm{SL}(n)}(2)\\
&+ \frac{1}{2n(I+A+nB)}\mathbf{p}^2 + \frac{I}{2(I^2-A^2)}\|\mathbf{S}\|^2,\\
\mathbf{T}^{\mathrm{aff-met}}_{\mathrm{int}} &= \frac{1}{2(I+A)}\mathbf{C}_{\mathrm{SL}(n)}(2)\\
&+ \frac{1}{2n(I+A+nB)}\mathbf{p}^2 + \frac{I}{2(I^2-A^2)}\|\mathbf{V}\|^2,
\end{aligned}$$

where, obviously,

$$\mathbf{C}_{\mathrm{SL}(n)}(k) := \mathbf{s}^a{}_b\mathbf{s}^b{}_c\cdots\mathbf{s}^r{}_s\mathbf{s}^s{}_a = \widehat{\mathbf{s}}^A{}_B\widehat{\mathbf{s}}^B{}_C\cdots\widehat{\mathbf{s}}^R{}_S\widehat{\mathbf{s}}^S{}_A,$$

k terms in these expressions, and

$$\|\mathbf{S}\|^2 = -\frac{1}{2}\mathbf{S}^a{}_b\mathbf{S}^b{}_a, \qquad \|\mathbf{V}\|^2 = -\frac{1}{2}\mathbf{V}^A{}_B\mathbf{V}^B{}_A.$$

As mentioned, the $\mathrm{SL}(n,\mathbb{R})$-part of $\mathbf{T}$ has both discrete and continuous spectrum and predicts the bounded oscillatory solutions even if no extra potential on $\mathrm{SL}(n,\mathbb{R})$ is used (classically this is the geodetic model with an open subset of bounded trajectories in the complete solution). In particular, there is an open range of inertial parameters $(A,B,C)\in\mathbb{R}^3$ for which the spectrum is positive or at least bounded from below.

One can hope that on the basis of commutation relations for the Lie algebra $\mathrm{SL}(n,\mathbb{R})'$ some information concerning spectra and wave functions may be perhaps obtained without the explicit solving of differential equations.

There are $\mathrm{GL}(n,\mathbb{R})$-problems where the separation of isochoric $\mathrm{SL}(n,\mathbb{R})$-terms is not necessary, sometimes it is even undesirable. Then it is more convenient to use the quantized version, i.e.,

$$\begin{aligned}
\mathbf{T}^{\mathrm{aff-aff}}_{\mathrm{int}} &= \frac{1}{2A}\mathbf{C}(2) - \frac{B}{2A(A+nB)}\mathbf{p}^2,\\
\mathbf{T}^{\mathrm{met-aff}}_{\mathrm{int}} &= \frac{1}{2\alpha}\mathbf{C}(2) + \frac{1}{2\beta}\mathbf{p}^2 + \frac{1}{2\mu}\|\mathbf{S}\|^2,\\
\mathbf{T}^{\mathrm{aff-met}}_{\mathrm{int}} &= \frac{1}{2\alpha}\mathbf{C}(2) + \frac{1}{2\beta}\mathbf{p}^2 + \frac{1}{2\mu}\|\mathbf{V}\|^2,
\end{aligned}$$

where α, β, μ are previously introduced constants and $\mathbf{C}(k)$ are operators of the full $\mathrm{GL}(n,\mathbb{R})$-Casimirs,

$$\mathbf{C}(k) := \mathbf{\Sigma}^a{}_b\mathbf{\Sigma}^b{}_c\cdots\mathbf{\Sigma}^r{}_s\mathbf{\Sigma}^s{}_a = \widehat{\mathbf{\Sigma}}^A{}_B\widehat{\mathbf{\Sigma}}^B{}_C\cdots\widehat{\mathbf{\Sigma}}^R{}_S\widehat{\mathbf{\Sigma}}^S{}_A;$$

the above contracted products contain k terms. In particular,

$$\mathbf{C}(2) := \mathbf{\Sigma}^a{}_b\mathbf{\Sigma}^b{}_a = \widehat{\mathbf{\Sigma}}^A{}_B\widehat{\mathbf{\Sigma}}^B{}_A, \qquad \mathbf{C}(1) := \mathbf{\Sigma}^a{}_a = \widehat{\mathbf{\Sigma}}^A{}_A.$$

In particular, if the inertial constant B vanishes, then the model $\mathbf{T}^{\mathrm{aff}-\mathrm{aff}}_{\mathrm{int}}$ may be interpreted in terms of one-dimensional multi-body problems in the sense of Calogero, Moser, Sutherland, etc., quite independently of our primary motivation, i.e., n-dimensional affine systems.

As mentioned, on $\mathrm{GL}(n,\mathbb{R})$, i.e., for compressible objects with dilatations, some dilatation-stabilizing potential $V(q)$ must be introduced if the system has to possess bound states. For more general doubly isotropic potentials $V(q^1,\ldots,q^n)$ depending only on deformation invariants, there is no possibility of avoiding differential equations (with the help of ladder procedures). Nevertheless, the problem is then still remarkably simplified in comparison with the general case, because the quantum dynamics of deformation invariants is autonomous (in this respect the quantum problem is in a sense simpler than the classical one). The procedure is based then on the two-polar decomposition, which by the way is also very convenient on the level of purely geodetic models. In certain problems, e.g., spatially isotropic but materially anisotropic ones, the polar decomposition is also convenient.

Two-polar decomposition for quantum case

On the quantum level the classical quantities $\rho = S$, $\tau = -V$ become the operators of spin and minus vorticity $\mathbf{S}$, $-\mathbf{V}$, i.e., Hermitian generators of the unitary groups of spatial and material rotations acting argument-wise on wave functions. Classical quantities $\widehat{\rho}$, $\widehat{\tau}$ were co-moving representants of tensors $\rho = S$, $\tau = -V$, i.e., their projections onto principal axes of the Cauchy and Green deformation tensors. Their quantum counterparts, i.e., operators $\widehat{\mathbf{r}}$, $\widehat{\mathbf{t}}$ are also co-moving representants of $\mathbf{r} = \mathbf{S}$, $\mathbf{t} = -\mathbf{V}$, i.e.,

$$\widehat{\mathbf{r}}^a{}_b = L^a{}_i L^j{}_b \mathbf{S}^i{}_j, \qquad \widehat{\mathbf{t}}^a{}_b = -R^A{}_b R^a{}_B \mathbf{V}^B{}_A.$$

They are Hermitian generators of the argument-wise right-hand side action of $\mathrm{SO}(n,\mathbb{R})$ on the wave functions. Just as in classical theory, it is convenient to introduce operators

$$\mathbf{M}^a{}_b := -\widehat{\mathbf{r}}^a{}_b - \widehat{\mathbf{t}}^a{}_b, \qquad \mathbf{N}^a{}_b := \widehat{\mathbf{r}}^a{}_b - \widehat{\mathbf{t}}^a{}_b.$$

Commutation relations for operators $\mathbf{S}$, $\mathbf{V}$, $\widehat{\mathbf{r}}$, $\widehat{\mathbf{t}}$, $\mathbf{M}$, $\mathbf{N}$ are directly isomorphic with those for the generators of $\mathrm{SO}(n,\mathbb{R})$ and are expressed in a straightforward way in terms of $\mathrm{SO}(n,\mathbb{R})$-structure constants.

Now we are ready to write down explicitly our kinetic energy and Hamiltonian operators in terms of the two-polar splitting. We begin with the traditional integer spin models, and later on we show how half-integer angular momentum of extended bodies may appear in a natural way.

Quantum operators $\mathbf{S}^i{}_j$, $-\mathbf{V}^A{}_B$ have the following form:

$$\mathbf{S}^i{}_j = \frac{\hbar}{i}\mathbf{\Lambda}^i{}_j(L), \qquad -\mathbf{V}^A{}_B = \frac{\hbar}{i}\mathbf{\Lambda}^A{}_B(R),$$

where $\mathbf{\Lambda}^i{}_j(L)$ and $\mathbf{\Lambda}^A{}_B(R)$ are real first-order differential operators generating left regular translations on $\mathrm{SO}(n,\mathbb{R})$, or, more precisely, on the isometric factors $L:\mathbb{R}^n\to V$, $R:\mathbb{R}^n\to U$ of the two-polar splitting, i.e.,

$$\begin{aligned} F\left(W(\omega)L\right) &= \left(\exp\left(\frac{1}{2}\omega^j{}_i\mathbf{\Lambda}^i{}_j\right)F\right)(L), \\ F\left(W(\omega)R\right) &= \left(\exp\left(\frac{1}{2}\omega^B{}_A\mathbf{\Lambda}^A{}_B\right)F\right)(R). \end{aligned}$$

In the formulas above, F are functions on the manifolds of isometries from $(\mathbb{R}^n,\delta)$ to (V,g) and from $(\mathbb{R}^n,\delta)$ to (U,η). Analytically, in Cartesian coordinates they are simply functions on $\mathrm{SO}(n,\mathbb{R})$. Matrices $[\omega^a{}_b]$, $[\omega^A{}_B]$ are respectively g- and η-antisymmetric:

$$\omega^a{}_b=-g^{ac}g_{bd}\omega^d{}_c,\qquad \omega^A{}_B=-\eta^{AC}\eta_{BD}\omega^D{}_C.$$

Their independent components are canonical coordinates of the first kind on $\mathrm{SO}(V,g)$, $\mathrm{SO}(U,\eta)$ (roughly, on $\mathrm{SO}(n,\mathbb{R})$),

$$W(\omega)=\exp\left(\frac{1}{2}\omega^b{}_aE^a{}_b\right),\qquad W(\omega)=\exp\left(\frac{1}{2}\omega^B{}_AE^A{}_B\right),$$

where $E^a{}_b\in\mathrm{SO}(V,g)'$, $E^A{}_B\in\mathrm{SO}(U,\eta)'$ are basic elements corresponding to some (arbitrary) choice of bases in V, U, i.e.,

$$(E^a{}_b)^i{}_j=\delta^a{}_j\delta^i{}_b-g^{ai}g_{bj},\qquad (E^A{}_B)^C{}_D=\delta^A{}_D\delta^C{}_B-\eta^{AC}\eta_{BD}.$$

One could reproach against our permanent changing between the simplified analytical description based on $\mathbb{R}^n$, $\mathrm{GL}^+(n,\mathbb{R})$, $\mathrm{SO}(n,\mathbb{R})$ and the careful geometric distinguishing between the material and physical spaces U, V and the manifolds $\mathrm{LI}(U,V)$, $\mathrm{O}^+(\mathbb{R}^n,\delta;V,g)$, $\mathrm{O}^+(\mathbb{R}^n,\delta;U,\eta)$; the latter two denoting the manifolds of orientation-preserving isometries between indicated Euclidean spaces (equivalently, manifolds of positively oriented orthonormal frames $\mathrm{F}^+(V,g)$, $\mathrm{F}^+(U,\eta)$). However, this "monkey" way of changing branches has some advantages, provided that done carefully. There are relationships easily representable for computational purposes in matrix terms, however, in certain fundamental formulas this may be misleading and risky.

And now, at some final stage of our discussion there appear some expressions where the calculus on $\mathbb{R}^n$ as such (not on $\mathbb{R}^n$ base-identified with U, V) becomes not only temporarily admissible but just mathematically proper one. Namely, it is just the matrix group $\mathrm{SO}(n,\mathbb{R})$ that acts on the right on the objects $L\in\mathrm{O}^+(\mathbb{R}^n,\delta;V,g)$ and $R\in\mathrm{O}^+(\mathbb{R}^n,\delta;U,\eta)$. As said above, on the classical level the corresponding Hamiltonian generators, i.e., momentum mappings, are given by $[\widehat{\rho}^a{}_b]$, $[\widehat{\tau}^a{}_b]$. In quantized theory the same role is played by the formally self-adjoint differential operators $\widehat{\mathbf{r}}^a{}_b$, $\widehat{\mathbf{t}}^a{}_b$,

$$\begin{aligned}F\left(LW(\omega)\right)&=\left(\exp\left(\frac{1}{2}\omega^b{}_a\boldsymbol{\Upsilon}^a{}_b\right)F\right)(L)=\left(\exp\left(\frac{i}{2\hbar}\omega^b{}_a\widehat{\mathbf{r}}^a{}_b\right)F\right)(L),\\ F\left(RW(\omega)\right)&=\left(\exp\left(\frac{1}{2}\omega^b{}_a\boldsymbol{\Upsilon}^a{}_b\right)F\right)(R)=\left(\exp\left(\frac{i}{2\hbar}\omega^b{}_a\widehat{\mathbf{t}}^a{}_b\right)F\right)(R).\end{aligned}$$

Here the skew-symmetry of $[\omega^a{}_b]$ is meant in the literal Kronecker-delta sense; nothing like g and η is implicitly assumed:

$$\omega^a{}_b=-\omega_b{}^a=-\delta^{ac}\delta_{bd}\omega^d{}_c.$$

Just $\mathbb{R}^n$ as such with its numerical metric is used here. In the physical three-dimensional case one uses the duality between skew-symmetric tensors and axial vectors, thus, on the quantum operator level we use the quantities $\widehat{\mathbf{r}}_a$, $\widehat{\mathbf{t}}_a$, $\boldsymbol{\Upsilon}(L)_a$, $\boldsymbol{\Upsilon}(R)_a$, where

$$\widehat{\mathbf{r}}^a{}_b=\epsilon^a{}_b{}^c\widehat{\mathbf{r}}_c,\qquad \widehat{\mathbf{r}}_a=\frac{1}{2}\epsilon_{ab}{}^c\widehat{\mathbf{r}}^b{}_c,$$

$$\widehat{\mathbf{t}}^a{}_b=\epsilon^a{}_b{}^c\widehat{\mathbf{t}}_c,\qquad \widehat{\mathbf{t}}_a=\frac{1}{2}\epsilon_{ab}{}^c\widehat{\mathbf{t}}^b{}_c,$$

$$\boldsymbol{\Upsilon}^a{}_b=\epsilon^a{}_b{}^c\boldsymbol{\Upsilon}_c,\qquad \boldsymbol{\Upsilon}_a=\frac{1}{2}\epsilon_{ab}{}^c\boldsymbol{\Upsilon}^b{}_c.$$

Obviously, the expressions $\boldsymbol{\Upsilon}^a{}_b$, $\boldsymbol{\Upsilon}_a$ are meant in two versions, as acting on the L,R-variables, thus, puristically we should have used the symbols $\boldsymbol{\Upsilon}^a{}_b(L)$, $\boldsymbol{\Upsilon}_a(L)$, $\boldsymbol{\Upsilon}^a{}_b(R)$, $\boldsymbol{\Upsilon}_a(R)$, however, when non-necessary, we prefer to avoid the crowd of symbols. Commutation relations are in both cases:

$$[\boldsymbol{\Upsilon}_a, \boldsymbol{\Upsilon}_b] = \epsilon_{ab}{}^c \boldsymbol{\Upsilon}_c,$$

i.e., in terms of quantum Poisson brackets:

$$\frac{1}{i\hbar}[\widehat{\mathbf{r}}_a, \widehat{\mathbf{r}}_b] = -\epsilon_{ab}{}^c \widehat{\mathbf{r}}_c, \qquad \frac{1}{i\hbar}[\widehat{\mathbf{t}}_a, \widehat{\mathbf{t}}_b] = -\epsilon_{ab}{}^c \widehat{\mathbf{t}}_c.$$

It is clear that

$$[\widehat{\mathbf{r}}_a, \widehat{\mathbf{t}}_b] = 0, \qquad [\boldsymbol{\Upsilon}_a(L), \boldsymbol{\Upsilon}_b(R)] = 0.$$

Obviously, the raising and lowering of indices is meant here in the trivial Kronecker-delta sense, so it is written only for cosmetic reasons, e.g.,

$$\epsilon^a{}_b{}^c = \delta^{ak}\delta^{cl}\epsilon_{kbl},$$

etc. What concerns the V- and U-space objects like $\mathbf{S}^i{}_j = \mathbf{r}^i{}_j$, $\mathbf{V}^A{}_B = -\mathbf{t}^A{}_B$, analogous expressions are true when one uses orthonormal coordinates, i.e., when $g_{ij} =_* \delta_{ij}$, $\eta_{AB} =_* \delta_{AB}$. When more general rectilinear coordinates are used, the formulas become more complicated because various expressions involving $\det[g_{ij}]$, $\det[\eta_{AB}]$ appear; there is, however, no practical need to use this representation.

In orthonormal coordinates in V and U spaces we have again the following expressions in terms of axial vectors:

$$\mathbf{r}^i{}_j = \mathbf{S}^i{}_j = \epsilon^i{}_j{}^k \mathbf{r}_k = \epsilon^i{}_j{}^k \mathbf{S}_k,$$
$$\mathbf{t}^A{}_B = -\mathbf{V}^A{}_B = \epsilon^A{}_B{}^C \mathbf{t}_C = -\epsilon^A{}_B{}^C \mathbf{V}_C.$$

These quantities are expressed through differential operators $\boldsymbol{\Lambda}^i{}_j(L)$ and $\boldsymbol{\Lambda}^A{}_B(R)$, i.e.,

$$\boldsymbol{\Lambda}^i{}_j(L) = \epsilon^i{}_j{}^k \boldsymbol{\Lambda}_k(L), \qquad \boldsymbol{\Lambda}_k(L) = \frac{1}{2}\epsilon_{ij}{}^k \boldsymbol{\Lambda}^j{}_k(L),$$

$$\boldsymbol{\Lambda}^A{}_B(R) = \epsilon^A{}_B{}^C \boldsymbol{\Lambda}_C(R), \qquad \boldsymbol{\Lambda}_A(R) = \frac{1}{2}\epsilon_{AB}{}^C \boldsymbol{\Lambda}^B{}_C(R).$$

When using the convention of "small" and "capital" indices, one can omit the L- and R-labels at $\boldsymbol{\Lambda}$-symbols. Obviously, we have:

$$\mathbf{S}_i = \mathbf{r}_i = \frac{\hbar}{i}\boldsymbol{\Lambda}_i, \qquad \mathbf{V}_A = -\mathbf{t}_A = -\frac{\hbar}{i}\boldsymbol{\Lambda}_A.$$

One should be careful with some subtle sign problems in commutation relations,

$$[\boldsymbol{\Lambda}_i, \boldsymbol{\Lambda}_j] = -\epsilon_{ij}{}^k \boldsymbol{\Lambda}_k, \qquad [\boldsymbol{\Lambda}_A, \boldsymbol{\Lambda}_B] = -\epsilon_{AB}{}^C \boldsymbol{\Lambda}_C, \qquad [\boldsymbol{\Lambda}_i, \boldsymbol{\Lambda}_A] = 0,$$

therefore,

$$\frac{1}{i\hbar}[\mathbf{S}_i, \mathbf{S}_j] = \epsilon_{ij}{}^k \mathbf{S}_k, \qquad \frac{1}{i\hbar}[\mathbf{V}_A, \mathbf{V}_B] = -\epsilon_{AB}{}^C \mathbf{V}_C, \qquad [\mathbf{S}_i, \mathbf{V}_A] = 0.$$

Let us also notice that

$$[\boldsymbol{\Lambda}_i, \boldsymbol{\Upsilon}_a(L)] = 0, \qquad [\boldsymbol{\Lambda}_i, \boldsymbol{\Lambda}_A] = 0, \qquad [\boldsymbol{\Lambda}_i, \boldsymbol{\Upsilon}_a(R)] = 0,$$

$$[\boldsymbol{\Lambda}_A, \boldsymbol{\Upsilon}_a(L)] = 0, \qquad [\boldsymbol{\Upsilon}_a(L), \boldsymbol{\Upsilon}_a(R)] = 0, \qquad [\boldsymbol{\Lambda}_A, \boldsymbol{\Upsilon}_a(R)] = 0.$$

"Rotation vector" space language

Obviously, "coordinates" $\omega^a{}_b$ on $\mathrm{SO}(n,\mathbb{R})$ are redundant, unless we restrict ourselves to $\omega_{ab} = \delta_{ac}\omega^c{}_b$, $a < b$ (or conversely). If $n = 3$, one uses so-called "rotation vector" k^a, where

$$\omega^a{}_b = -\epsilon^a{}_{bc}k^c, \qquad k^a = -\frac{1}{2}\epsilon^a{}_b{}^c\omega^b{}_c.$$

It is convenient to use the "magnitude" $k = \sqrt{(k^1)^2 + (k^2)^2 + (k^3)^2}$. In this parameterization, $\mathrm{SO}(3,\mathbb{R})$ is covered by the ball $k \leq \pi$ with the proviso that antipodal points on the sphere $k = \pi$ describe the same half-rotation, i.e., rotation by π about a given axis. For $k < \pi$ the representation is unique. The magnitude k equals the angle of rotation, whereas the versor $\overline{n} := \overline{k}/k$ represents the oriented rotation axis in the right screw sense (for $k = \pi$ it does not matter right or left ones; they coincide). In certain expressions it is convenient to use the spherical coordinates k, ϑ, φ in the $\overline{k}$-space, thus,

$$k^1 = k\sin\vartheta\cos\varphi, \qquad k^2 = k\sin\vartheta\sin\varphi, \qquad k^3 = k\cos\vartheta.$$

For the completeness, let us quote some important three-dimensional formulas.

The "basic" matrices $E^a{}_b \in \mathrm{SO}(3,\mathbb{R})'$ are represented dually by the actually basic system of E_a, where

$$E^a{}_b = \epsilon^a{}_b{}^c E_c, \qquad E_a = \frac{1}{2}\epsilon_{ab}{}^c E^b{}_c, \qquad (E_a)^b{}_c = -\epsilon_a{}^b{}_c.$$

The structure constants are then given simply by "epsilons":

$$[E_a, E_b] = \epsilon_{ab}{}^c E_c.$$

For any rotation vector $\overline{k} \in \mathbb{R}^3$, corresponding matrices $W\left(\overline{k}\right) \in \mathrm{SO}(3,\mathbb{R})$ act on vectors $\overline{u} \in \mathbb{R}^3$ as follows:

$$W\left(\overline{k}\right)\cdot\overline{u} = \cos k\overline{u} + \frac{(1-\cos k)}{k^2}\left(\overline{k}\cdot\overline{u}\right)\overline{k} + \frac{\sin k}{k}\overline{k}\times\overline{u};$$

obviously, the scalar and vector product are meant in the standard $\mathbb{R}^3$-sense. The components of $\overline{k}$ are canonical coordinates of the first kind on $\mathrm{SO}(3,\mathbb{R})$,

$$W\left(\overline{k}\right) = \exp\left(k^a E_a\right) = \sum_{m=0}^{\infty}\frac{1}{m!}\left(k^a E_a\right)^m.$$

One can show that

$$\begin{aligned} W\left(\overline{k}\right)\cdot\overline{u} &= \overline{u} + \overline{k}\times\overline{u} + \frac{1}{2}\overline{k}\times\left(\overline{k}\times\overline{u}\right) + \cdots \\ &+ \frac{1}{n!}\overline{k}\times\left(\overline{k}\times\left(\overline{k}\times\cdots\left(\overline{k}\times\overline{u}\right)\cdots\right)\right) + \cdots \end{aligned}$$

This infinite series is an alternative representation of the exponential formula. The term with multiplicator $1/n!$ contains the n-fold vector multiplication of $\overline{u}$ by $\overline{k}$. Explicitly the matrix of $W\left(\overline{k}\right)$ is given by

$$W\left(\overline{k}\right)^a{}_b = \cos k\ \delta^a{}_b + (1-\cos k)\frac{k^a k_b}{k^2} + \sin k\ \epsilon^a{}_{bc}\frac{k^c}{k};$$

obviously, the raising and lowering of indices is meant here in the trivial (purely cosmetic) delta-sense.

One can show that generators of right regular translations on $\mathrm{SO}(3,\mathbb{R})$ are given by the following expression:

$$\boldsymbol{\Upsilon}_a = \frac{k}{2}\mathrm{ctg}\frac{k}{2}\frac{\partial}{\partial k^a} + \left(1 - \frac{k}{2}\mathrm{ctg}\frac{k}{2}\right)\frac{k_a k^b}{k^2}\frac{\partial}{\partial k^b} - \frac{1}{2}\epsilon_{ab}{}^c k^b\frac{\partial}{\partial k^c}.$$

This is a common formula for $\boldsymbol{\Upsilon}_a(L)$, $\boldsymbol{\Upsilon}_a(R)$, and now for simplicity we again use the analytical matrix representation, when U and V are identified with $\mathbb{R}^3$ and the L,R-terms of the two-polar

decomposition are identified with elements of $\mathrm{SO}(3,\mathbb{R})$. To specify this formula to $\boldsymbol{\Upsilon}_a(L)$, $\boldsymbol{\Upsilon}_a(R)$ one must replace the general symbol of the rotation vector $\overline{k}$ on $\mathrm{SO}(3,\mathbb{R})$ by the rotation vectors $\overline{l}$, $\overline{r}$ parameterizing the L,R-terms:

$$L\left(\overline{l}\right)=\exp\left(l^aE_a\right),\qquad R\left(\overline{r}\right)=\exp\left(r^aE_a\right).$$

Generators of the left regular translations on $\mathrm{SO}(3,\mathbb{R})$ are as follows:

$$\boldsymbol{\Lambda}_a=\frac{k}{2}\mathrm{ctg}\frac{k}{2}\frac{\partial}{\partial k^a}+\left(1-\frac{k}{2}\mathrm{ctg}\frac{k}{2}\right)\frac{k_ak^b}{k^2}\frac{\partial}{\partial k^b}+\frac{1}{2}\epsilon_{ab}{}^ck^b\frac{\partial}{\partial k^c}.$$

And this again specifies to $\boldsymbol{\Lambda}_a(L)$, $\boldsymbol{\Lambda}_a(R)$ when instead of $\overline{k}$ we substitute respectively $\overline{l}$, $\overline{r}$, i.e., rotation vectors parameterizing the manifolds of L,R-factors in the two-polar decomposition.

Let us observe that

$$\boldsymbol{\Lambda}_a-\boldsymbol{\Upsilon}_a=\mathbf{D}_a=\epsilon_{ab}{}^ck^b\frac{\partial}{\partial k^c},$$

and these differential operators generate the group of inner automorphisms of $\mathrm{SO}(3,\mathbb{R})$:

$$W\left(\overline{k}\right)\mapsto UW\left(\overline{k}\right)U^{-1}=W\left(U\overline{k}\right),$$

where U runs over $\mathrm{SO}(3,\mathbb{R})$. Roughly speaking, these transformations result in rotations of the rotation vectors. And, just as previously, substituting here $\overline{l}$ and $\overline{r}$ in place of $\overline{k}$ we obtain the corresponding transformations of the manifolds of $L\left(\overline{l}\right)$- and $R\left(\overline{r}\right)$-terms of the two-polar decompositions. One can show that the generators of the left and right regular translations on $\mathrm{SO}(3,\mathbb{R})$ may be expressed in terms of operators $\partial/\partial k$ and $\mathbf{D}_a$ acting, respectively, along the radius and tangently to spheres in the representative spaces $\mathbb{R}^3$ of the rotation vector $\overline{k}$, i.e.,

$$\begin{aligned}\boldsymbol{\Lambda}_a&=\frac{k_a}{k}\frac{\partial}{\partial k}-\frac{1}{2}\mathrm{ctg}\frac{k}{2}\epsilon_{ab}{}^ck^b\mathbf{D}_c+\frac{1}{2}\mathbf{D}_a,\\ \boldsymbol{\Upsilon}_a&=\frac{k_a}{k}\frac{\partial}{\partial k}-\frac{1}{2}\mathrm{ctg}\frac{k}{2}\epsilon_{ab}{}^ck^b\mathbf{D}_c-\frac{1}{2}\mathbf{D}_a.\end{aligned}$$

Obviously,

$$[\mathbf{D}_a,\mathbf{D}_b]=-\epsilon_{ab}{}^c\mathbf{D}_c.$$

In many formulas we need orthogonal invariants like $\|\mathbf{S}\|^2$, $\|\mathbf{V}\|^2$. They are based on the Casimir invariants $C_{\mathrm{SO}(n,\mathbb{R})}(2)$ built of generators $\boldsymbol{\Lambda}_a$, $\boldsymbol{\Upsilon}_a$ of the left and right regular translations on $\mathrm{SO}(n,\mathbb{R})$. If $n=3$, these Casimirs have the following form:

$$\boldsymbol{\Lambda}^2=\boldsymbol{\Upsilon}^2=\boldsymbol{\Lambda}_1^2+\boldsymbol{\Lambda}_2^2+\boldsymbol{\Lambda}_3^2=\boldsymbol{\Upsilon}_1^2+\boldsymbol{\Upsilon}_2^2+\boldsymbol{\Upsilon}_3^2,$$

and one can show that analytically

$$\mathbf{C}_{\mathrm{SO}(3,\mathbb{R})}(2)=\boldsymbol{\Lambda}^2=\boldsymbol{\Upsilon}^2=\left(\frac{\partial^2}{\partial k^2}+\mathrm{ctg}\frac{k}{2}\frac{\partial}{\partial k}\right)+\frac{1}{4\sin^2\frac{k}{2}}\mathbf{D}^2,$$

where

$$\mathbf{D}^2=\mathbf{D}_1^2+\mathbf{D}_2^2+\mathbf{D}_3^2.$$

Obviously,

$$\|\mathbf{S}\|^2=-\hbar^2\mathbf{C}_{\mathrm{SO}(3,\mathbb{R})}\left(L\left(\overline{l}\right)\right),\qquad\|\mathbf{V}\|^2=-\hbar^2\mathbf{C}_{\mathrm{SO}(3,\mathbb{R})}\left(R\left(\overline{r}\right)\right),$$

where the last two terms multiplied by $-\hbar^2$ are obtained from the previous $\mathbf{C}_{\mathrm{SO}(3,\mathbb{R})}$ by substituting the $\overline{l}$- and $\overline{r}$-variables in place of $\overline{k}$.

Remark: Obviously, the equality $\boldsymbol{\Lambda}^2=\boldsymbol{\Upsilon}^2$ holds only when $\boldsymbol{\Lambda}_a$ and $\boldsymbol{\Upsilon}_a$ involve the same kind of independent variables, e.g., $\overline{k}$ on the abstract $\mathrm{SO}(3,\mathbb{R})$ as generators of the left or right regular

translations, $\bar{l}$ when both operating on the left two-polar factor $L(\bar{l})$, or $\bar{r}$ when both acting on the right two-polar factor. But of course $\|\mathbf{S}\|^2$ and $\|\mathbf{V}\|^2$ are different for any dimension n, although, of course, the following always holds:

$$\|\mathbf{S}\|^2 = \|\widehat{\mathbf{r}}\|^2, \qquad \|\mathbf{V}\|^2 = \|\widehat{\mathbf{t}}\|^2.$$

Expansion of wave functions

When we use the two-polar decomposition $\varphi = LDR^{-1}$, then, according to the Peter-Weyl theorem, the wave functions on $\mathrm{GL}^+(n,\mathbb{R})$ may be expanded in L, R-variables with respect to matrix elements of irreducible representations of the compact group $\mathrm{SO}(n,\mathbb{R})$. Obviously, the expansion coefficients depend on deformation invariants, i.e., on the diagonal factor D (equivalently, on the variables Q^a or $q^a = \ln Q^a$). In general, we have that

$$\Psi(\varphi) = \Psi(L,D,R) = \sum_{\alpha,\beta\in\Omega} \sum_{m,n=1}^{N(\alpha)} \sum_{k,l=1}^{N(\beta)} \mathcal{D}^{\alpha}_{mn}(L) f^{\alpha\beta}_{\substack{nk\\ml}}(D) \mathcal{D}^{\beta}_{kl}(R^{-1}),$$

where the meaning of symbols is as follows: Ω is the set of equivalence classes of unitary irreducible representations of $\mathrm{SO}(n,\mathbb{R})$, $N(\alpha)$ is the dimension of the α-th representation class. It is finite because $\mathrm{SO}(n,\mathbb{R})$ is compact, $\mathcal{D}^{\alpha}$ is the α-th representation matrix. For many classical groups $\mathcal{D}^{\alpha}$ are explicitly known (at least in terms of some well-investigated special functions).

Analytically $\mathcal{D}^{\alpha}(L)$, $\mathcal{D}^{\beta}(R^{-1})$ are matrices depending on the group coordinates ${\omega_L}^a{}_b$, ${\omega_R}^a{}_b$ of L, R, e.g., rotation vectors $\bar{l}$, $\bar{r}$ if $n = 3$. The argument D of f is the system of q-variables $q^1, \ldots, q^n$. According to the mentioned multi-valuedness of the two-polar decomposition, the reduced amplitudes $f^{\alpha\beta}(q^1, \ldots, q^n)$ must obey some conditions, because Ψ must not distinguish triplets (L, D, R) corresponding to the same configuration $\varphi = LDR^{-1}$.

Therefore, on the submanifold $M^{(n)} \subset \mathrm{SO}(n,\mathbb{R}) \times \mathbb{R}^n \times \mathrm{SO}(n,\mathbb{R})$ with non-degenerate systems of $(q^1, \ldots, q^n)$ (no coincidences) we must have that

$$f^{\alpha\beta}_{\substack{nk\\ml}}(q^{\pi_W(1)}, \ldots, q^n) = \sum_{r=1}^{N(\alpha)} \sum_{s=1}^{N(\beta)} \mathcal{D}^{\alpha}_{nr}(W^{-1}) f^{\alpha\beta}_{\substack{rs\\ml}}(q^1, \ldots, q^n) \mathcal{D}^{\beta}_{sk}(W)$$

for any $W \in K^+$. The same holds on the subsets $M^{(k;p_1,\ldots,p_k)} \subset \mathrm{SO}(n,\mathbb{R}) \times \mathbb{R}^n \times \mathrm{SO}(n,\mathbb{R})$ with degenerate systems $(q^1, \ldots, q^n)$ (coincidences of some q's). The difference is that in degenerate cases W runs over the continuous subgroups of $\mathrm{SO}(n,\mathbb{R})$ generated by K^+ and the subgroups $H^{(k;p_1,\ldots,p_k)}$ described above. The special case of the total degeneracy is extreme and, because of this, very simple one. Indeed, then in the two-polar decomposition it is only LR^{-1} that is meaningful whereas L, R separately are not well-defined. Therefore, if $D = cI_n$, i.e., $q^1 = \cdots = q^n = q$, then the reduced amplitude obeys very severe restrictions, i.e.,

$$\begin{aligned} f^{\alpha\beta}(cI_n) &= 0 \quad \text{if} \quad \alpha \neq \beta, \\ f^{\alpha\alpha}_{\substack{rs\\ml}}(cI_n) &= g_{ml}\delta_{rs}. \end{aligned}$$

The non-uniqueness is extreme here, namely, for any $Z \in \mathrm{SO}(n,\mathbb{R})$ the triplets (L, cI_n, R) and (LZ, cI_n, RZ) represent the same classical configuration, thus, the wave functions do not distinguish them.

It is seen that if $q^1, \ldots, q^n$ are interpreted as coordinates of some fictitious material points on the real axis $\mathbb{R}$, one is dealing with a very peculiar system of identical para-statistical particles.

It is clear that in geodetic models or in models with doubly isotropic potentials (ones depending only on deformation invariants; dilatation-stabilizing potentials $V(q)$ provide the simplest example), m and l in the Peter-Weyl expansion are "good" quantum numbers. In other words, the spin and vorticity operators ${\mathbf{S}^i}_j$, ${\mathbf{V}^A}_B$ do commute with the Hamilton operator $\mathbf{H}$. The same concerns representation

labels $\alpha, \beta \in \Omega$, i.e., finally, the systems of eigenvalues for the Casimir operators of the groups $\mathrm{SO}(V,g)$, $\mathrm{SO}(U,\eta)$ acting argument-wise on wave functions. Let us remind that these Casimirs are given by

$$\begin{aligned}\mathbf{C}_{\mathrm{SO}(V,g)}(p) &\simeq \mathbf{S}^i{}_k\mathbf{S}^k{}_m\cdots\mathbf{S}^r{}_z\mathbf{S}^z{}_i,\\ \mathbf{C}_{\mathrm{SO}(U,\eta)}(p) &\simeq \mathbf{V}^A{}_K\mathbf{V}^K{}_M\cdots\mathbf{V}^R{}_Z\mathbf{V}^Z{}_A,\end{aligned}$$

p operator multipliers in every expression; $p \leq n$ and even.

In such situation it is convenient to keep α, β, m, l fixed and use the following reduced amplitudes:

$$\Psi(\varphi) = \Psi^{\alpha\beta}_{ml}(L,D,R) = \sum_{n=1}^{N(\alpha)}\sum_{k=1}^{N(\beta)} \mathcal{D}^{\alpha}_{mn}(L) f^{\alpha\beta}_{nk}(D) \mathcal{D}^{\beta}_{kl}(R^{-1}),$$

with the same as previously provisos concerning the one-valuedness of Ψ as a function of φ.

In the physical case $n = 3$, we have obviously the following standard form of $\mathrm{SO}(3,\mathbb{R})$-Casimirs:

$$\mathbf{C}_{\mathrm{SO}(V,g)}(2) = \mathbf{S}_1^2 + \mathbf{S}_2^2 + \mathbf{S}_3^2 = \widehat{\mathbf{r}}_1^2 + \widehat{\mathbf{r}}_2^2 + \widehat{\mathbf{r}}_3^2 = \mathbf{C}_{\mathrm{SO}(3,\mathbb{R})}(2),$$

$$\mathbf{C}_{\mathrm{SO}(U,\eta)}(2) = \mathbf{V}_1^2 + \mathbf{V}_2^2 + \mathbf{V}_3^2 = \widehat{\mathbf{t}}_1^2 + \widehat{\mathbf{t}}_2^2 + \widehat{\mathbf{t}}_3^2 = \mathbf{C}_{\mathrm{SO}(3,\mathbb{R})}(2).$$

Our expansions for wave functions are then described in terms of well-known expressions found by Wigner, and, of course, the family of rotational Casimirs begins and terminates on $p = 2$.

Obviously, for $n = 3$, Ω is the set of non-negative integer, α, β are traditionally denoted by symbols like $s, j = 0, 1, 2, \ldots$, etc., $N(s) = 2s+1$, $N(j) = 2j+1$, and the indices (m,n), (k,l) are considered as jumping by 1, respectively, from $-s$ to s and from $-j$ to j; here the tradition is too strong to respect the formal logical conventions. Thus, according to the mentioned conventions:

$$\Psi(\varphi) = \Psi(L,D,R) = \sum_{s,j=0}^{\infty}\sum_{m,n=-s}^{s}\sum_{k,l=-j}^{j} \mathcal{D}^{s}_{mn}(L) f^{sj}_{\substack{nk\\ml}}(D) \mathcal{D}^{j}_{kl}(R^{-1}).$$

Similarly, the reduced amplitudes are written as:

$$\Psi(\varphi) = \Psi^{sj}_{ml}(L,D,R) = \sum_{n=-s}^{s}\sum_{k=-j}^{j} \mathcal{D}^{s}_{mn}(L) f^{sj}_{nk}(D) \mathcal{D}^{j}_{kl}(R^{-1}).$$

Here $\mathcal{D}^s$ are celebrated Wigner matrices of $(2s+1)$-dimensional irreducible representations of the three-dimensional rotation group. They are well-known special functions of mathematical physics and may be assumed to be something in principle standard and well-know.

Obviously, the amplitudes Ψ^{sj}_{ml} are eigenfunctions of rotational Casimir invariants, i.e., essentially angular momentum and vorticity:

$$\|\mathbf{S}\|^2\Psi^{sj}_{ml} = \|\widehat{\mathbf{r}}\|^2\Psi^{sj}_{ml} = \hbar^2 s(s+1)\Psi^{sj}_{ml},$$

$$\|\mathbf{V}\|^2\Psi^{sj}_{ml} = \|\widehat{\mathbf{t}}\|^2\Psi^{sj}_{ml} = \hbar^2 j(j+1)\Psi^{sj}_{ml},$$

where, let us remind, in three dimensions we have the following expressions:

$$\|\mathbf{S}\|^2 = \mathbf{S}_1^2 + \mathbf{S}_2^2 + \mathbf{S}_3^2, \qquad \|\mathbf{V}\|^2 = \mathbf{V}_1^2 + \mathbf{V}_2^2 + \mathbf{V}_3^2,$$

and similarly for $\widehat{\mathbf{r}}$, $\widehat{\mathbf{t}}$. According to tradition, one uses such a basis that Ψ^{sj}_{ml} are also eigenfunctions of the third components of rotational generators,

$$\mathbf{S}_3\Psi^{sj}_{ml} = \hbar m\Psi^{sj}_{ml}, \qquad \mathbf{V}_3\Psi^{sj}_{ml} = \hbar l\Psi^{sj}_{ml}.$$

And, obviously, when the values n, k in the superposition are kept fixed and we retain only the corresponding single term, for the resulting Ψ we have

$$\widehat{\mathbf{r}}_3\Psi^{sj}_{\substack{ml\\nk}} = \hbar n\Psi^{sj}_{\substack{ml\\nk}}, \qquad \widehat{\mathbf{t}}_3\Psi^{sj}_{\substack{ml\\nk}} = \hbar k\Psi^{sj}_{\substack{ml\\nk}}.$$

Representation matrices

In this way one is dealing with quantum states of well-definite values of magnitudes and third components of the angular momentum and vorticity. For the general n, the amplitudes $\Psi^{\alpha\beta}_{ml}$ have, of course, the well-definite values $(\hbar/i)^p C(\alpha, p)$, $(\hbar/i)^p C(\beta, p)$ of the Casimirs. And now it will be convenient to return for a while (at least in a formal way) to the general case of dimension n.

Let us again use the exponential formulas for the elements of $W(\omega) \in \mathrm{SO}(V, g)$, $W(\omega) \in \mathrm{SO}(U, \eta)$, and just their simply numerical counterparts in $\mathrm{SO}(n, \mathbb{R})$,

$$W(\omega) = \exp\left(\frac{1}{2}\omega^a{}_b E^b{}_a\right),$$

where the basic matrices $E^b{}_a$ are simply given by

$$(E^b{}_a)^c{}_d = \delta^b{}_d\delta^c{}_a - \delta^{bc}\delta_{ad}$$

(just showing that one works just in $\mathbb{R}^n$ and $\mathrm{SO}(n, \mathbb{R})'$ not in V, U, $\mathrm{SO}(V, g)$, $\mathrm{SO}(U, \eta)$ basis-identified with the previous ones). And from now on let us again decide to work in purely analytical matrix form using orthonormal coordinates in V, U and identifying them with $\mathbb{R}^n$. Representation matrices $\mathcal{D}^\alpha$ are given by the following expresion:

$$\mathcal{D}^\alpha(\omega) = \exp\left(\frac{1}{2}\omega^a{}_b M^{\alpha b}{}_a\right),$$

where the $N(\alpha) \times N(\alpha)$ anti-hermitian matrices $M^{\alpha b}{}_a$ form irreducible representations of the Lie algebra $\mathrm{SO}(n, \mathbb{R})'$, thus, their commutation rules are identical with those for $E^b{}_a$.

Remark: For any $\alpha \in \Omega$ and for any pair of indices b, a, $M^{\alpha b}{}_a$ are just matrices not (b, a)-matrix elements of some M^α; let us notice in this connection that $a, b = \overline{1, n}$, whereas any $M^{\alpha b}{}_a$ is an $N(\alpha) \times N(\alpha)$-matrix. Obviously, when dealing with matrices $\mathcal{D}^\alpha(L)$, $\mathcal{D}^\beta(R)$, we must specialize the redundant "coordinates" $\omega^a{}_b$ to the ones parameterizing respectively the L- and R-terms of the two-polar splitting, writing, e.g.,

$$\mathcal{D}^\alpha\left(L(l)\right) = \exp\left(\frac{1}{2}l^a{}_b M^{\alpha b}{}_a\right), \qquad \mathcal{D}^\beta\left(R(r)\right) = \exp\left(\frac{1}{2}r^a{}_b M^{\beta b}{}_a\right).$$

For example, in three dimensions, where the pseudovector $\overline{k}$ may be used instead of the tensor $\omega^b{}_a$, i.e., $\mathcal{D}^s\left(W\left(\overline{k}\right)\right) = \exp\left(k^a M^s{}_a\right)$, we should write that

$$\mathcal{D}^s\left(L\left(\overline{l}\right)\right) = \exp\left(l^a M^s{}_a\right), \qquad \mathcal{D}^j\left(R\left(\overline{r}\right)\right) = \exp\left(r^a M^j{}_a\right),$$

where $M^s{}_a$ (s being non-negative integers and $a = 1, 2, 3$) are basic $(2s+1) \times (2s+1)$, thus, odd-dimensional, anti-hermitian matrices representing in an irreducible way the Lie algebra $\mathrm{SO}(3, \mathbb{R})'$. Therefore,

$$[M^s{}_a, M^s{}_b] = -\epsilon_{ab}{}^c M^s{}_c,$$

and it is impossible to reduce simultaneously all $M^s{}_a$ to the block form. The apparently impossible even dimension $(2s+1)$ of $M^s{}_a$, thus, positive half-integer s will be an important point of our further analysis because $\mathrm{SO}(3, \mathbb{R})'$ (just as any $\mathrm{SO}(n, \mathbb{R})'$, $n \geq 3$) admits even-dimensional representations corresponding to the half-integer angular momentum, both for rigid and homogeneously deformable bodies.

Algebraic form of equations

Let us introduce Hermitian matrices

$$S^{\alpha a}{}_b = \frac{\hbar}{i} M^{\alpha a}{}_b,$$

thus, for $n = 3$,

$$S^j{}_a = \frac{\hbar}{i} M^j{}_a,$$

and

$$\frac{1}{i\hbar}[S^j{}_a, S^j{}_b] = \epsilon_{ab}{}^c S^j{}_c.$$

These are standard well-known matrices, possible to be determined in purely algebraic terms, basing only on the commutation relations. And it was just a surprise that there exist even-dimensional irreducible representations, experimentally compatible with the half-integer internal angular momentum spin. The $(2j+1) \times (2j+1)$ matrices S^j provide the quantum description of the angular momentum with the quantized magnitude $\hbar^2 j(j+1)$; j being a non-negative integer, or also a positive half-integer in the theory of fermionic objects.

The representation property of $\mathcal{D}^\alpha$, i.e.,

$$\mathcal{D}^\alpha(R_1 R_2) = \mathcal{D}^\alpha(R_1)\mathcal{D}^\alpha(R_2),$$

together with the definition of generators imply that certain obvious relationships which enable one to replace some differential operations and equations by algebraic ones. Namely, it is clear from the above formulas that

$$\begin{aligned}
\frac{\hbar}{i}\mathbf{\Lambda}^i{}_j(L)\mathcal{D}^\alpha(L) &= S^{\alpha i}{}_j \mathcal{D}^\alpha(L), \\
\frac{\hbar}{i}\mathbf{\Lambda}^A{}_B(R)\mathcal{D}^\beta(R) &= \mathcal{D}^\beta(R) S^{\beta A}{}_B, \\
\frac{\hbar}{i}\mathbf{\Upsilon}^a{}_b(L)\mathcal{D}^\alpha(L) &= \mathcal{D}^\alpha(L) S^{\alpha a}{}_b, \\
\frac{\hbar}{i}\mathbf{\Upsilon}^a{}_b(R)\mathcal{D}^\beta(R) &= S^{\beta a}{}_b \mathcal{D}^\beta(R);
\end{aligned}$$

expressions on the right-hand side meant, obviously, in the sense of the matrix multiplication.

In other words,

$$\mathbf{S}^i{}_j \Psi^{\alpha\beta} = S^{\alpha i}{}_j \Psi^{\alpha\beta}, \qquad \mathbf{V}^A{}_B \Psi^{\alpha\beta} = \Psi^{\alpha\beta} S^{\beta A}{}_B,$$

where $\Psi^{\alpha\beta}$ is an abbreviation for the $N(\alpha) \times N(\beta)$ matrices $\left[\Psi^{\alpha\beta}_{ml}\right]$ $\left(m = \overline{1, N(\alpha)},\ l = \overline{1, N(\beta)}\right)$. Obviously, everything is formally correct because $S^{\alpha i}{}_j$, $S^{\beta A}{}_B$ are, respectively, $N(\alpha) \times N(\alpha)$- and $N(\beta) \times N(\beta)$-matrices. Let us stress once again that the indices (i, j), (A, B) label basic matrices within their sets; they do not refer to matrix elements.

From now on it will be convenient to write

$$\Psi^{\alpha\beta}(L, D, R) = \mathcal{D}^\alpha(L) f^{\alpha\beta}(D) \mathcal{D}^\beta(R^{-1});$$

obviously, the reduced amplitude $f^{\alpha\beta}(D)$ is an $N(\alpha) \times N(\beta)$-matrix depending only on deformation invariants $D_{aa} = Q^a = \exp(q^a)$.

Similarly, $\widehat{\mathbf{r}}^a{}_b$ and $\widehat{\mathbf{t}}^a{}_b$ act on $\Psi^{\alpha\beta}$ as follows:

$$\begin{aligned}
\widehat{\mathbf{r}}^a{}_b \Psi^{\alpha\beta} &= \mathcal{D}^\alpha(L) S^{\alpha a}{}_b f^{\alpha\beta}(D) \mathcal{D}^\beta(R^{-1}), \\
\widehat{\mathbf{t}}^a{}_b \Psi^{\alpha\beta} &= \mathcal{D}^\alpha(L) f^{\alpha\beta}(D) S^{\beta a}{}_b \mathcal{D}^\beta(R^{-1}).
\end{aligned}$$

Therefore, this action reduces simply to the action on the reduced amplitude $f^{\alpha\beta}$ only. It will be convenient to denote it as follows:

$$\overrightarrow{S^{\alpha a}}_{b} f^{\alpha\beta} := S^{\alpha a}{}_{b} f^{\alpha\beta}, \qquad \overleftarrow{S^{\beta a}}_{b} f^{\alpha\beta} := f^{\alpha\beta}(D) S^{\beta a}{}_{b}.$$

By assumption, the representations $\mathcal{D}^{\alpha}$ of $\mathrm{SO}(n,\mathbb{R})$ are irreducible, therefore, the matrices

$$C^{\alpha}(p) = \underbrace{S^{\alpha a}{}_{b} S^{\alpha b}{}_{c} \cdots S^{\alpha u}{}_{w} S^{\alpha w}{}_{a}}_{p \text{ factors}}$$

are proportional to the $N(\alpha) \times N(\alpha)$ identity matrices,

$$C^{\alpha}(p) = \left(\frac{\hbar}{i}\right)^{p} C(\alpha, p) I_{N(\alpha)},$$

where the numbers $C(\alpha, p)$ are eigenvalues of the corresponding Casimir operators built of the generators of the left and right regular translations on $\mathrm{SO}(n,\mathbb{R})$, e.g.,

$$\mathbf{C}_{\mathrm{SO}(n,\mathbb{R})}(p) = \underbrace{\mathbf{\Lambda}^{a}{}_{b} \mathbf{\Lambda}^{b}{}_{c} \cdots \mathbf{\Lambda}^{u}{}_{w} \mathbf{\Lambda}^{w}{}_{a}}_{p \text{ factors}}.$$

So, finally, let us summarize the corresponding formulas for the physical case $n = 3$,

$$\|\mathbf{S}\|^2 \Psi^{sj} = \|\widehat{\mathbf{r}}\|^2 \Psi^{sj} = \hbar^2 s(s+1) \Psi^{sj},$$

$$\|\mathbf{V}\|^2 \Psi^{sj} = \|\widehat{\mathbf{t}}\|^2 \Psi^{sj} = \hbar^2 j(j+1) \Psi^{sj},$$

$$\mathbf{S}_a \Psi^{sj} = S^{s}{}_{a} \Psi^{sj}, \qquad \mathbf{V}_a \Psi^{sj} = \Psi^{sj} S^{j}{}_{a},$$

in particular, in the standard representation,

$$\mathbf{S}_3 \Psi^{sj}_{ml} = \hbar m \Psi^{sj}_{ml}, \qquad \mathbf{V}_3 \Psi^{sj}_{ml} = \hbar l \Psi^{sj}_{ml}.$$

And just as for the general dimension value n, a little more complicated action of $\widehat{\mathbf{r}}_a$, $\widehat{\mathbf{t}}_a$ resulting in affecting the reduced $f(D)$-amplitudes,

$$\begin{aligned} \widehat{\mathbf{r}}_a &: \quad f^{sj} \mapsto S^{s}{}_{a} f^{sj} = \overrightarrow{S^{s}}_{a} f^{sj}, \\ \widehat{\mathbf{t}}_a &: \quad f^{sj} \mapsto f^{sj} S^{j}{}_{a} = \overleftarrow{S^{j}}_{a} f^{sj}. \end{aligned}$$

In particular, again in the standard representation,

$$\begin{aligned} \widehat{\mathbf{r}}_3 &: \quad \left[f^{sj}_{ml}\right] \mapsto \left[\hbar m f^{sj}_{ml}\right], \\ \widehat{\mathbf{t}}_3 &: \quad \left[f^{sj}_{ml}\right] \mapsto \left[\hbar l f^{sj}_{ml}\right]. \end{aligned}$$

Reduction to Cartan subgroup

Matrix elements of irreducible representations have important well-investigated properties which enable one to algebraize a good deal of differential equations problems and to perform an effective reduction of the quantum dynamics. Roughly speaking, this is reduction to the Cartan subgroup of $\mathrm{GL}(n,\mathbb{R})$, i.e., to its maximal Abelian subgroup. This is just the group of diagonal matrices, i.e., degrees of freedom parameterized by deformation invariants $q^1, \ldots, q^n$. This reduction from n^2 to n degrees of freedom is possible for geodetic problems, for dilatationally-stabilized problems (i.e., essentially for geodetic problems on $\mathrm{SL}(n,\mathbb{R})$) and, more generally, for doubly isotropic models when the potential energy is non-trivial but depends only on the deformation invariants, i.e., it has the form $V(q^1, \ldots, q^n)$. Let us remind that in this sense quantum mechanics of affine bodies is "simpler" than

the classical one where for $n > 2$ there is no simple way of reducing equations of motion to the Cartan subgroup.

It is convenient to start again with the general n, and later on to restrict ourselves to the special cases $n = 2, 3$. Due to the standard orthogonality properties of ${\mathcal{D}^{\alpha}}_{mn}$, the scalar product of wave functions Ψ may be reduced to one for the amplitudes $f^{\alpha\beta}$ depending only on deformation invariants, i.e.,

$$< \Psi_1 | \Psi_2 >= \sum_{\alpha,\beta \in \Omega} \frac{1}{N(\alpha)N(\beta)} \int \sum_{n,m=1}^{N(\alpha)} \sum_{k,l=1}^{N(\beta)} \overline{f_1}^{\alpha\beta}_{\substack{nk \\ ml}} f_{2\substack{nk \\ ml}}^{\alpha\beta} P dq^1 \cdots dq^n,$$

where, let us remind, the weight P is given by the following expression:

$$P\left(q^1, \ldots, q^n\right) = \prod_{i \neq j} \left| \mathrm{sh}(q^i - q^j) \right|.$$

If we fix the labels α, β, m, l ("good" quantum numbers for doubly-isotropic problems) and consider the simplified $N(\alpha) \times N(\beta)$-matrix amplitudes,

$$\Psi^{\alpha\beta}\left(L; q^1, \ldots, q^n; R\right) = \mathcal{D}^{\alpha}(L) f^{\alpha\beta}\left(q^1, \ldots, q^n\right) \mathcal{D}^{\beta}\left(R^{-1}\right),$$

then the scalar product reduces to

$$< \Psi_1^{\alpha\beta} | \Psi_2^{\alpha\beta} >= \frac{1}{N(\alpha)N(\beta)}$$

$$\int \mathrm{Tr}\left(f_1^{\alpha\beta +}\left(q^1, \ldots, q^n\right) f_2^{\alpha\beta}\left(q^1, \ldots, q^n\right) \right) P\left(q^1, \ldots, q^n\right) dq^1 \cdots dq^n,$$

where, obviously, $f_1^{\alpha\beta +}$ denotes the Hermitian conjugate of the matrix $f_1^{\alpha\beta}$.

Obviously, for the general expansion the corresponding formula involves the summation over α, β, and the multiplication of reduced amplitudes and trace operation meant in the sense of two-matrices with the entries labelled by two-indices $f_{\substack{nk \\ ml}}^{\alpha\beta}$, i.e.,

$$< \Psi_1 | \Psi_2 >= \sum_{\alpha,\beta \in \Omega} \frac{1}{N(\alpha)N(\beta)}$$

$$\int \mathrm{Tr}\left(f_1^{\alpha\beta +}\left(q^1, \ldots, q^n\right) f_2^{\alpha\beta}\left(q^1, \ldots, q^n\right) \right) P\left(q^1, \ldots, q^n\right) dq^1 \cdots dq^n,$$

For the sake of completeness, let us write explicitly

$$\mathrm{Tr}\left(f_1^{\alpha\beta +} f_2^{\alpha\beta} \right) = \sum_{n,m=1}^{N(\alpha)} \sum_{k,l=1}^{N(\beta)} \overline{f_1}^{\alpha\beta}_{\substack{nk \\ ml}} f_{2\substack{nk \\ ml}}^{\alpha\beta}.$$

When we consider the class of problems with α, β, m, l fixed once for all, then one can avoid the divisor $N(\alpha)N(\beta)$, with the proviso of being careful with the normalization of amplitudes so as not to violate the statistical interpretation.

In certain problems it may be convenient to avoid the weight factor P in the above expressions for the scalar product. To achieve this one should introduced rescaled amplitudes given by the following matrices:

$$g^{\alpha\beta} := \sqrt{P} f^{\alpha\beta}.$$

Then the factor P disappears from the above formulas, $f^{\alpha\beta}$ becomes replaced by $g^{\alpha\beta}$, and everything else remains as previously.

Metric tensors and arc elements

Essentially everything said above remains valid when discussing the half-integer angular momentum. Orthogonal groups $\mathrm{SO}(n,\mathbb{R})$ in the two-polar decomposition are then replaced by their coverings $\mathrm{Spin}(n)$, but it does not change anything in local analytical expressions. Technically, the only change is that the range of group parameters changes. And where for different parameter values the corresponding elements of $\mathrm{SO}(n,\mathbb{R})$ were identical, in $\mathrm{Spin}(n)$ they are different. It was described above in some details for $\mathrm{SO}(3,\mathbb{R})$ and its covering $\mathrm{Spin}(n)=\mathrm{SU}(2)$, where the main analytical novelty was replacing the range $[0,\pi]$ for the rotation vector magnitude k with $[0,2\pi]$. All analytical formulas remain formally the same, e.g., those for the generators of left and right regular translations $\mathbf{\Lambda}_a$, $\mathbf{\Upsilon}_a$. The metric Killing tensors on $\mathrm{SO}(3,\mathbb{R})$ and $\mathrm{SU}(2)$ normalized to be δ_{ij} in $\overline{k}$-coordinates at the group identity (thus, differing by the minus one-half factor in comparison with the general Lie-algebraic definition), i.e.,

$$\Gamma(a,b)=-\frac{1}{2}\mathrm{Tr}(ab),\qquad \Gamma(a,b)=-2\mathrm{Tr}(ab)$$

respectively, on $\mathrm{SO}(3,\mathbb{R})$ and $\mathrm{SU}(2)$, in both cases they are analytically given by the same formula:

$$\Gamma_{ab}=\frac{4}{k^2}\sin^2\frac{k}{2}\delta_{ab}+\left(1-\frac{4}{k^2}\sin^2\frac{k}{2}\right)\frac{k_a k_b}{k^2}.$$

In other words, the corresponding arc element is as follows:

$$ds^2=\Gamma_{ab}dk^a dk^b=dk^2+4\sin^2\frac{k}{2}\left(d\vartheta^2+\sin^2\vartheta d\varphi^2\right).$$

Obviously, this metric is conformally flat, e.g., defining new coordinates

$$\overline{r}=(a/k)\mathrm{tg}(k/4)\overline{k},\qquad a>0,$$

we obtain that

$$ds^2=\frac{16a^2}{a^2+r^2}\left(dr^2+r^2\left[d\vartheta^2+\sin^2\vartheta d\varphi^2\right]\right),$$

where the second factor is just the arc element in Euclidean $\mathbb{R}^3$ expressed in terms of spherical coordinates. This is the conformal mapping of $\mathrm{SU}(2)$ onto $\mathbb{R}^3$ if we consider the total range $r\in[0,\infty]$. It is interesting that $r\in[0,a]$ on $\mathrm{SO}(3,\mathbb{R})$. This is also some kind of arguments that $\mathrm{SO}(3,\mathbb{R})$ is somehow "imperfect" in comparison with its universal covering $\mathrm{SU}(2)$.

The Haar measure μ in both cases is given by

$$d\mu\left(\overline{k}\right)=\frac{4}{k^2}\sin^2\frac{k}{2}d_3\overline{k}=4\sin^2\frac{k}{2}\sin\vartheta dk d\vartheta d\varphi$$

if we wish its weight function to be equal one in $\overline{k}$-coordinates at the unit element $\left(\overline{k}=0\right)$. But if we wish, as we often do, to normalize the total measure of the compact group to unity, then both cases will differ by a constant factor.

Quantizing affine-affine kinetic energies

One can show after some calculations that the operator $\mathbf{T}_{\mathrm{int}}^{\mathrm{aff-aff}}$ of kinetic energy invariant under both spatial and material affine transformation is as follows:

$$\begin{aligned}\mathbf{T}_{\mathrm{int}}^{\mathrm{aff-aff}} &= -\frac{\hbar^2}{2A}\mathbf{D}+\frac{\hbar^2 B}{2A(A+nB)}\frac{\partial^2}{\partial q^2}\\ &+ \frac{1}{32A}\sum_{a,b}\frac{(\mathbf{M}^a{}_b)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}}-\frac{1}{32A}\sum_{a,b}\frac{(\mathbf{N}^a{}_b)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}},\end{aligned}$$

where A, B are constants as previously in classical formulas,

$$\mathbf{M}^a{}_b=-\widehat{\mathbf{r}}^a{}_b-\widehat{\mathbf{t}}^a{}_b,\qquad \mathbf{N}^a{}_b=\widehat{\mathbf{r}}^a{}_b-\widehat{\mathbf{t}}^a{}_b,$$

and

$$\mathbf{D} = \frac{1}{P}\sum_a \frac{\partial}{\partial q^a} P \frac{\partial}{\partial q^a} = \sum_a \frac{\partial^2}{\partial (q^a)^2} + \sum_a \frac{\partial \ln P}{\partial q^a}\frac{\partial}{\partial q^a}$$

(every differentiation operator acts on everything on the right of it), P is the previously introduced weight factor.

It is seen that this is almost the previously used classical formula with classical canonical quantities, e.g., $\widehat{\rho}^a{}_b$, $\widehat{\tau}^a{}_b$ replaced by the corresponding operators $\widehat{\mathbf{r}}^a{}_b$, $\widehat{\mathbf{t}}^a{}_b$. There is, however, some difference and possibility of an easy mistake in the sector of (q^a, p_a)-variables. Namely, the term involving differentiation with respect to q^a is not, as it might be expected, the usual $\mathbb{R}^n$-Laplace operator in q^a variables, although it contains such a term. Let us observe that in the $\varphi = LDR^{-1}$-representation the $\partial/\partial q^a$ operators act only on the $f^{\alpha\beta}$ amplitude, whereas $\widehat{\mathbf{r}}^a{}_b$, $\widehat{\mathbf{t}}^a{}_b$ act only, respectively, on the L- and R-variables. Therefore, there is no problem of ordering of operators in $\mathbf{T}^{\mathrm{aff}-\mathrm{aff}}_{\mathrm{int}}$. One could get rid off the first derivatives of Ψ with respect to q^a by the substitution which was already used within a slightly different context, namely,

$$\varphi = \sqrt{P}\Psi.$$

The action of the last three terms in $\mathbf{T}^{\mathrm{aff}-\mathrm{aff}}_{\mathrm{int}}$ on φ is exactly as that on Ψ because $\partial/\partial q^a$, $\mathbf{M}^a{}_b$, $\mathbf{N}^a{}_b$ do not act on $(q^a - q^b)$-quantities of which P is built; roughly speaking, the $\sqrt{P}$ is "transparent" for these operators. It is no longer the case with the $\mathbf{D}$-term, both in the good and in the bad senses. Namely, the action of $-(\hbar^2/2A)\mathbf{D}$ on Ψ is represented by the action of the following operator $-(\hbar^2/2A)\widetilde{\mathbf{D}}$ on φ:

$$-\frac{\hbar^2}{2A}\widetilde{\mathbf{D}} = -\frac{\hbar^2}{2A}\sum_a \frac{\partial^2}{\partial (q^a)^2} + \widetilde{\mathbf{V}},$$

where $\widetilde{\mathbf{V}}$ is the following artificial potential term:

$$\widetilde{\mathbf{V}} = -\frac{\hbar}{2A}\frac{1}{P^2} + \frac{\hbar^2}{4A}\frac{1}{P}\sum_a \left(\frac{\partial P}{\partial q^a}\right)^2.$$

In other words,

$$\widetilde{\mathbf{D}}\varphi = \sqrt{P}\mathbf{D}\Psi.$$

There are no first derivatives of φ with respect to q^a, and the differential action is given by the usual $\mathbb{R}^n$-Laplace operator, just as in mechanics of n q^a-particles on $\mathbb{R}$. But this simplification is only seeming one because, if $n > 2$, it is completely destroyed by the "potential" $\widetilde{\mathbf{V}}$. Obviously, in realistic problems concerning deformable objects Hamiltonian should also contain dilatation-stabilizing potential, i.e.,

$$\mathbf{H} = \mathbf{T}^{\mathrm{aff}-\mathrm{aff}}_{\mathrm{int}} + \mathbf{V}(q).$$

Although such simple $\mathrm{SL}(n, \mathbb{R})$-geodetic models may successfully describe elastic vibrations, some more general isotropic potentials $V(q^1, \ldots, q^n)$ are also acceptable and compatible with the above description.

Metric-affine and affine-metric models

Quantizing metric-affine and affine-metric kinetic energies we obtain, respectively, the following operators:

$$\begin{aligned}
\mathbf{T}^{\mathrm{met-aff}}_{\mathrm{int}} &= -\frac{\hbar^2}{2\alpha}\mathbf{D} - \frac{\hbar^2}{2\beta}\frac{\partial^2}{\partial q^2} \\
&+ \frac{1}{32\alpha}\sum_{a,b}\frac{(\mathbf{M}^a{}_b)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}} - \frac{1}{32\alpha}\sum_{a,b}\frac{(\mathbf{N}^a{}_b)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}} + \frac{1}{2\mu}\|\mathbf{S}\|^2, \\
\mathbf{T}^{\mathrm{aff-met}}_{\mathrm{int}} &= -\frac{\hbar^2}{2\alpha}\mathbf{D} - \frac{\hbar^2}{2\beta}\frac{\partial^2}{\partial q^2} \\
&+ \frac{1}{32\alpha}\sum_{a,b}\frac{(\mathbf{M}^a{}_b)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}} - \frac{1}{32\alpha}\sum_{a,b}\frac{(\mathbf{N}^a{}_b)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}} + \frac{1}{2\mu}\|\mathbf{V}\|^2,
\end{aligned}$$

with the same meaning of operator symbols as above and the same relationship between inertial constants (α, β, μ) and the primary ones (I, A, B) as previously.

Potential case

As mentioned above, for Hamiltonians $\mathbf{H} = \mathbf{T} + \mathbf{V}$ with some dilatation-stabilizing potentials $V(q)$, or more generally, with some doubly-isotropic potentials $V(q^1, \ldots, q^n)$, the action of operators $\mathbf{M}^a{}_b$ and $\mathbf{N}^a{}_b$ become algebraic and standard, and the stationary Schrödinger equation, i.e., energy eigenproblem

$$\mathbf{H}\Psi = E\Psi,$$

splits into family of eigenproblems for the amplitudes $f^{\alpha\beta}$; they are partial differential equations involving q^a-variables only:

$$\mathbf{H}^{\alpha\beta}f^{\alpha\beta} = E^{\alpha\beta}f^{\alpha\beta},$$

where $f^{\alpha\beta}$ for any $\alpha, \beta \in \Omega$ is an $N(\alpha) \times N(\beta)$ matrix depending on $q^1, \ldots, q^n$. In a consequence of the double (spatial and material) isotropy, this problem is $N(\alpha) \times N(\beta)$-fold degenerate, i.e., for every component of $f^{\alpha\beta}$ there exists an $N(\alpha) \times N(\beta)$-dimensional subspace of solutions. Let us remind that in the primary symbols $f^{\alpha\beta}_{\substack{nk\\ml}}$ the indices m, l just label the degeneracy of solutions for every $f^{\alpha\beta}_{nk}$. $\mathbf{H}^{\alpha\beta}$ is an $N(\alpha) \times N(\beta)$-matrix of second-order differential operators,

$$\mathbf{H}^{\alpha\beta} = \mathbf{T}^{\alpha\beta} + \mathbf{V},$$

where $\mathbf{V}$ denotes a dilatation-stabilizing or general doubly-isotropic potential, and $\mathbf{T}^{\alpha\beta}$ denotes the kinetic energy operator. It is one of the previous ones restricted to the corresponding (α, β)-subspace.

Therefore, for the affine-affine, metric-affine, and affine-metric models we have, respectively,

$$\begin{aligned}
\mathbf{T}^{\alpha\beta}f^{\alpha\beta} &= -\frac{\hbar^2}{2A}\mathbf{D}f^{\alpha\beta} + \frac{1}{32A}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta a}}_b - \overrightarrow{S^{\alpha a}}_b\right)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}}f^{\alpha\beta} \\
&- \frac{1}{32A}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta a}}_b + \overrightarrow{S^{\alpha a}}_b\right)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}}f^{\alpha\beta} + \frac{\hbar^2 B}{2A(A+nB)}\frac{\partial^2}{\partial q^2}f^{\alpha\beta}, \\
\mathbf{T}^{\alpha\beta}f^{\alpha\beta} &= -\frac{\hbar^2}{2\alpha}\mathbf{D}f^{\alpha\beta} - \frac{\hbar^2}{2\mu}C(\alpha,2)f^{\alpha\beta} + \frac{1}{32\alpha}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta a}}_b - \overrightarrow{S^{\alpha a}}_b\right)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}}f^{\alpha\beta} \\
&- \frac{1}{32\alpha}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta a}}_b + \overrightarrow{S^{\alpha a}}_b\right)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}}f^{\alpha\beta} - \frac{\hbar^2}{2\beta}\frac{\partial^2}{\partial q^2}f^{\alpha\beta}, \\
\mathbf{T}^{\alpha\beta}f^{\alpha\beta} &= -\frac{\hbar^2}{2\alpha}\mathbf{D}f^{\alpha\beta} - \frac{\hbar^2}{2\mu}C(\beta,2)f^{\alpha\beta} + \frac{1}{32\alpha}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta a}}_b - \overrightarrow{S^{\alpha a}}_b\right)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}}f^{\alpha\beta} \\
&- \frac{1}{32\alpha}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta a}}_b + \overrightarrow{S^{\alpha a}}_b\right)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}}f^{\alpha\beta} - \frac{\hbar^2}{2\beta}\frac{\partial^2}{\partial q^2}f^{\alpha\beta}.
\end{aligned}$$

The constants α, β, μ are exactly as previously; do not confuse them with labels α, β at $f^{\alpha\beta}$. In the physical case $n=3$, $\alpha = s = 0, 1/2, 1, \ldots \in \mathbb{N}/2 \cup \{0\}$ and similarly $\beta = j = 0, 1/2, 1, \ldots \in \mathbb{N}/2 \cup \{0\}$ assuming that the half-integer values of angular momentum and vorticity are admitted. Otherwise we would have $s, j \in \mathbb{N} \cup \{0\}$. Obviously, in this case $C(s,2) = -s(s+1)$, $C(j,2) = -j(j+1)$, and the additional constants in the last two formulas are simply $(\hbar^2/2\mu)s(s+1)$, $(\hbar^2/2\mu)j(j+1)$, expressions close to the heart of any physicist. Let us stress that, even if half-integers are admitted, there is a restriction that $(j-s)$ must be integer, i.e., j and s have the same "half-ness". In any case, it must be so if wave functions are to be well-defined on $\overline{\mathrm{GL}(3,\mathbb{R})}$ not only on the "artificial" configuration space $\mathrm{SU}(2)\times\mathbb{R}^3\times\mathrm{SU}(2)$. If they are to be statistically interpretable in $\mathrm{GL}(3,\mathbb{R})$ itself, then only the terms with half-integer (s,j) or integer (s,j) may be separately superposed, no mutual superposition admissible (although some blasphemic doubts may be raised against this superselection, i.e., against statistical interpretation in $\mathrm{GL}(3,\mathbb{R})$).

In three dimensions the above-mentioned additional terms $(\hbar^2/2\mu)s(s+1)$, $(\hbar^2/2\mu)j(j+1)$ seem to be physically interesting and, at least qualitatively, compatible with some experimental data. It is so as if the doubly affine background (affine invariance in space and in the body) was responsible for some fundamental part of the spectra, which later on, the more the μ is smaller, splits due to some internal rotations. The term $(\hbar^2/2\mu)s(s+1)$ is physically intuitive and classically corresponds to the situation when in the system some regime of rigid rotations was established after time of transition processes. But, perhaps, $(\hbar^2/2\mu)j(j+1)$ appearing in the affine-metrical model is even more interesting. Being a formal analogue of certain aspects of angular momentum, it is not angular momentum and may be perhaps semiclassically related to the isotopic spin or similar internal quantities ruled by SU(2) and appearing in nuclear and elementary particle physics.

Remark: just as previously, the terms with the first-order derivatives of $f^{\alpha\beta}$ with respect to q^a may be avoided by the substitution

$$g^{\alpha\beta} := \sqrt{P}f^{\alpha\beta},$$

which was also used for simplifying the scalar product. But then again the artificial potential $\mathbf{V}$ appears in all reduced Schrödinger equations.

Combine models and their possible use

By the way, one can have both things, i.e., the terms $(\hbar^2/2\mu)s(s+1)$ and $(\hbar^2/2\mu)j(j+1)$ terms. For this purpose we would have to use the kinetic energy consisting of four terms:

$$T_{\rm int} = \frac{I_1}{2} g_{ik} g^{jl} \Omega^i{}_j \Omega^k{}_l + \frac{I_2}{2} \eta_{KL} \eta^{MN} \widehat{\Omega}^K{}_M \widehat{\Omega}^L{}_N + \frac{A}{2} \widehat{\Omega}^K{}_L \widehat{\Omega}^L{}_K + \frac{B}{2} \widehat{\Omega}^K{}_K \widehat{\Omega}^L{}_L,$$

where, obviously, the last two terms might be as well written as follows:

$$\frac{A}{2} \Omega^i{}_j \Omega^j{}_i + \frac{B}{2} \Omega^i{}_i \Omega^j{}_j.$$

In matrix language, using Cartesian coordinates $g_{ik} =_* \delta_{ik}$, $\eta_{AB} =_* \delta_{AB}$, we would simply write that

$$\begin{aligned} T_{\rm int} &= \frac{I_1}{2} {\rm Tr}(\Omega^T \Omega) + \frac{I_2}{2} {\rm Tr}(\widehat{\Omega}^T \widehat{\Omega}) + \frac{A}{2} {\rm Tr}(\widehat{\Omega}^2) + \frac{B}{2} ({\rm Tr}\widehat{\Omega})^2 \\ &= \frac{I_1}{2} {\rm Tr}(\Omega^T \Omega) + \frac{I_2}{2} {\rm Tr}(\widehat{\Omega}^T \widehat{\Omega}) + \frac{A}{2} {\rm Tr}(\Omega^2) + \frac{B}{2} ({\rm Tr}\Omega)^2. \end{aligned}$$

But now some reproach might be raised that, doing as above, we forget our primary motivation concerning the dynamical ${\rm GL}(n,\mathbb{R})$-invariance and return to models which are only orthogonally invariant (geometrically speaking, ${\rm O}(V,g)$- and ${\rm O}(U,\eta)$-invariant), and it is again only pure kinematics that is ruled by affine group. This would be true, and we indeed do not insist on the above model. Let us notice, however, that this model, having still high dynamical symmetry, may also work as a purely geodetic model encoding a kind of elastic bounded vibrations without any extra introduced potential. Moreover, due to the lack of dilatational invariance, it is not excluded (we are not yet sure; this is a conjecture) that even dilatation-stabilizing potentials would not be necessary.

Doubly isotropic d'Alembert models

The above remarks again put our attention on the doubly isotropic "d'Alembert" models of classical kinetic energy with $J^{KL} = I\eta^{KL}$. This time, as a measure particularly convenient for quantization, the usual Lebesgue measure l on ${\rm L}(n)$ should be used,

$$dl(\varphi) = d\varphi^1{}_1 \cdots \varphi^n{}_n.$$

In terms of the two-polar splitting,

$$dl(L,D,R) = P_l(Q) d\mu(L) d\mu(R) dQ^1 \cdots dQ^n,$$

where μ, as previously, is the Haar measure on ${\rm SO}(n,\mathbb{R})$, and the weight factor P_l is now given by the following expression:

$$P_l = \prod_{a \neq b} \left| (Q^a)^2 - (Q^b)^2 \right| = \prod_{a \neq b} \left| (Q^a + Q^b)(Q^a - Q^b) \right|.$$

Everything concerning quantization looks in a similar way like previously for affinely-invariant models. For example, expansion of wave functions Ψ with respect to $\mathcal{D}^\alpha(L)$, $\mathcal{D}^\beta(R)$ with $f^{\alpha\beta}(D)$-reduced amplitudes is exactly the same. The difference appears in details concerning the integration procedure, just the weight factor P_l is substituted instead of P. Also, in spite of formal similarities, the particular form of the kinetic energy operator is different,

$$\mathbf{T}^{\rm d.A}_{\rm int} = -\frac{\hbar^2}{2I} \mathbf{D}_l + \frac{1}{8I} \sum_{a,b} \frac{(\mathbf{M}^a{}_b)^2}{(Q^a - Q^b)^2} + \frac{1}{8I} \sum_{a,b} \frac{(\mathbf{N}^a{}_b)^2}{(Q^a + Q^b)^2},$$

where now

$$\mathbf{D}_l = \frac{1}{P_l} \sum_a \frac{\partial}{\partial Q^a} P_l \frac{\partial}{\partial Q^a} = \sum_a \frac{\partial^2}{\partial (Q^a)^2} + \sum_a \frac{\partial \ln P_l}{\partial Q^a} \frac{\partial}{\partial Q^a}.$$

Just as previously, the weight factor P_l in the scalar product and first-order differentiations $\partial/\partial Q^a$ may be avoided by rescaling

$$\varphi = \sqrt{P_l}\Psi,$$

but in the resulting differential operator acting on φ also some rather unpleasant potential term appears, i.e.,

$$\widetilde{\mathbf{V}}_l = -\frac{\hbar}{2I}\frac{1}{P_l^2} + \frac{\hbar^2}{4I}\frac{1}{P_l}\sum_a \left(\frac{\partial P_l}{\partial Q^i}\right)^2.$$

It is obvious that without an appropriate potential term $\mathbf{V}$ the geodetic Hamiltonian $\mathbf{T}^{\mathrm{d.A}}$ cannot work in theory of deformable objects because just as on the classical level it describes only purely scattering, non-bounded motions. Indeed, the above operator

$$\mathbf{T}^{\mathrm{d.A}} = -\frac{\hbar^2}{2I}\Delta^{n^2} = -\frac{\hbar^2}{2I}\sum_{i,A}\frac{\partial^2}{\partial(\varphi^i{}_A)^2}$$

is simply proportional to the usual Laplace operator in $\mathbb{R}^{n^2}$ written in non-typical coordinates.

Therefore, the only realistic applications of the above $\mathbf{T}$ are those as a term of some doubly isotropic Hamiltonian

$$\mathbf{H} = \mathbf{T}^{\mathrm{d.A}} + \mathbf{V}\left(Q^1, \ldots, Q^n\right).$$

Just as previously, due to the double isotropy of the model, the resulting stationary Schrödinger equation

$$\mathbf{H}\Psi = E\Psi$$

splits into the family of equations for partial amplitudes $f^{\alpha\beta}$ depending only on q^a-variables,

$$\mathbf{H}^{\alpha\beta}f^{\alpha\beta} = E^{\alpha\beta}f^{\alpha\beta},$$

where

$$\begin{aligned}\mathbf{H}^{\alpha\beta}f^{\alpha\beta} &= -\frac{\hbar^2}{2I}\mathbf{D}_l f^{\alpha\beta} + \frac{1}{8I}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta}}{}^a{}_b - \overrightarrow{S^{\alpha}}{}^a{}_b\right)^2}{(Q^a - Q^b)^2}f^{\alpha\beta}\\ &+ \frac{1}{8I}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta}}{}^a{}_b + \overrightarrow{S^{\alpha}}{}^a{}_b\right)^2}{(Q^a + Q^b)^2}f^{\alpha\beta} + V\left(Q^1, \ldots, Q^n\right)f^{\alpha\beta}.\end{aligned}$$

For d'Alembert models, the problem of coverings and multi-valued wave functions looks exactly like in affine theories. Simply $\mathrm{SO}(n,\mathbb{R})$-groups in the two-polar decomposition must be replaced by the coverings $\mathrm{Spin}(n)$. In particular, for $n = 3$ when $\alpha, \beta = s, j = 0, 1/2, 1, \ldots$, everything said above remains true, and $S^{sa}{}_b$, $S^{ja}{}_b$ are replaced by the standard Wigner matrices of angular momentum, $S^s{}_a$, $S^j{}_a$.

Usual Wigner matrices $S^j{}_a$

In three dimensions those terms of the affine-affine reduced operator $\mathbf{T}^{\alpha\beta}$ which contain the factor $1/32A$ may be written in the following form involving the usual Wigner matrices $S^j{}_a$:

$$\begin{aligned}&\frac{1}{16A}\sum_{a=1}^{3}\frac{(S^s{}_a)^2 f^{sj} - 2S^s{}_a f^{sj}S^j{}_a + f^{sj}(S^j{}_a)^2}{\mathrm{sh}^2\frac{q^b - q^c}{2}}\\ -&\frac{1}{16A}\sum_{a=1}^{3}\frac{(S^s{}_a)^2 f^{sj} + 2S^s{}_a f^{sj}S^j{}_a + f^{sj}(S^j{}_a)^2}{\mathrm{ch}^2\frac{q^b - q^c}{2}},\end{aligned}$$

where in any a-th term of both summations we have obviously $b \neq a$, $c \neq a$, $b \neq c$ (it is clear that it does not matter what is the sequence of b, c).

The same holds for the metric-affine and affine-metric models with the proviso that the inertial factor A is replaced by α. As mentioned, the last constant-multiplicator terms are respectively

$$\frac{\hbar^2}{2\mu}s(s+1)f^{sj}, \qquad \frac{\hbar^2}{2\mu}j(j+1)f^{sj}.$$

Similarly, in reduced d'Alembert expressions the terms with the $1/8I$-factor become for $n=3$:

$$\frac{1}{4I}\sum_{a=1}^{3}\frac{(S^s{}_a)^2f^{sj}-2S^s{}_af^{sj}S^j{}_a+f^{sj}(S^j{}_a)^2}{(Q^b-Q^c)^2}$$
$$+\frac{1}{4I}\sum_{a=1}^{3}\frac{(S^s{}_a)^2f^{sj}+2S^s{}_af^{sj}S^j{}_a+f^{sj}(S^j{}_a)^2}{(Q^b+Q^c)^2},$$

with the same as previously convention concerning indices a, b, c.

For affinely-invariant geodetic models the bounded state L^2-solutions appear for particular relationships between s and j (α and β) in n dimensions). For the d'Alembert models of kinetic energy this is impossible, an appropriate potential $V(Q^1,\ldots,Q^n)$ must be always used.

Both the affine and d'Alembert expressions become particularly simple for the lowest possible values of rotational quantum numbers s, j, and then there exists some hope for rigorous or at least numerical solutions. Thus, for $s=j=0$ the corresponding expressions vanish at all, and the resulting Schrödinger equations for f^{00} are purely scalar. For $s=j=1/2$ we obtain the spinor-spinor state, which is also relatively simple because then $S^{1/2}{}_a=(\hbar/2)\sigma_a$, $\left(S^{1/2}{}_a\right)^2=(\hbar^2/4)I_2$, where, obviously, σ_a are Pauli matrices, and I_2 is the unit 2×2 matrix.

Two-dimensional case on classical level

In some physical problems also the two-dimensional case $n=2$ may be physically interesting. And in any case it is mathematically exceptionally simple. This is, so to speak, "pathological" simplicity following from the commutativity of $\mathrm{SO}(2,\mathbb{R})$. Although this exceptional simplicity is rather "exotic" from the point of view of the general n, it may suggest some guiding hints for analysis of this general situation.

The main two-dimensional peculiarity is that

$$\widehat{\rho}=\rho=S, \qquad \widehat{\tau}=\tau=-V.$$

This is exactly due to the commutativity of $\mathrm{SO}(2,\mathbb{R})$. Because of this, the convenient quantities $\widehat{\rho}$, $\widehat{\tau}$ are constants of motion for geodetic models and models with doubly-invariant potentials. It was not the case for $n>2$, where only S, V are constants of motion (for invariant geodetic models and, more generally, for doubly-isotropic models). But it is just the use of $\widehat{\rho}$ and $\widehat{\tau}$, or equivalently M and N, that simplifies the problem and enables one to perform a partial separation of variables, especially effective on the quantum level. If $n=2$, the two things coincide, and the problem may be effectively reduced to the Cartan subgroup of diagonal matrices (deformation invariants) even on the classical level.

Let us begin with the classical description. In the two-polar decomposition $\varphi=LDR^{-1}$ we shall use the following parameterization:

$$L=\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}, \qquad R=\begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix},$$

$$D=\begin{bmatrix}Q^1 & 0\\ 0 & Q^2\end{bmatrix}=\begin{bmatrix}\exp q^1 & 0\\ 0 & \exp q^2\end{bmatrix}.$$

The splitting $\mathrm{GL}^+(2,\mathbb{R})=\mathbb{R}^+\mathrm{SL}(2,\mathbb{R})$ is well-suited to coordinates

$$q=\frac{1}{2}\left(q^1+q^2\right), \qquad x=q^2-q^1,$$

and their conjugate canonical momenta, respectively,

$$p = p_1 + p_2, \qquad p_x = \frac{1}{2}\left(p_2 - p_1\right).$$

Before using these convenient coordinates, let us express classical kinetic energies in terms of primary variables. First of all, let us notice the obvious fact that the angular velocities of L- and R-rotators are given, respectively, by

$$\chi = \frac{dL}{dt}L^{-1} = L^{-1}\frac{dL}{dt} = \widehat{\chi} = \frac{d\alpha}{dt}\left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right],$$

$$\vartheta = \frac{dR}{dt}R^{-1} = R^{-1}\frac{dR}{dt} = \widehat{\vartheta} = \frac{d\beta}{dt}\left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right].$$

The corresponding spin and vorticity quantities are given (in canonical representation) by the following expressions:

$$S = \rho = \widehat{\rho} = p_\alpha \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right], \qquad V = -\tau = -\widehat{\tau} = p_\beta \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right],$$

where p_α, p_β are, respectively, canonical momenta conjugate to α, β. The corresponding duality pairings are as follows:

$$p_\alpha \frac{d\alpha}{dt} = \frac{1}{2}\mathrm{Tr}(S\chi) = \frac{1}{2}\mathrm{Tr}(\rho\chi) = \frac{1}{2}\mathrm{Tr}(\widehat{\rho}\widehat{\chi}),$$
$$p_\beta \frac{d\beta}{dt} = \frac{1}{2}\mathrm{Tr}(V\vartheta) = -\frac{1}{2}\mathrm{Tr}(\tau\vartheta) = -\frac{1}{2}\mathrm{Tr}(\widehat{\tau}\widehat{\vartheta}),$$

where $d\alpha/dt$, $d\beta/dt$ are arbitrary virtual velocities of the variables α, β.

The corresponding classical quantities

$$M = -\widehat{\rho} - \widehat{\tau}, \qquad N = \widehat{\rho} - \widehat{\tau}$$

are respectively given by the following expressions:

$$M = \mathsf{m}\left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right] = (p_\beta - p_\alpha)\left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right],$$

$$N = \mathsf{n}\left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right] = (p_\beta + p_\alpha)\left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right],$$

where

$$\mathsf{m} := p_\beta - p_\alpha, \qquad \mathsf{n} := p_\beta + p_\alpha$$

may be interpreted as canonical momenta conjugate to the corresponding "mixtures" of angles β, α:

$$\gamma := \frac{1}{2}(\beta - \alpha), \qquad \delta := \frac{1}{2}(\beta + \alpha),$$

i.e.,

$$\alpha = \delta - \gamma, \qquad \beta = \delta + \gamma.$$

In fact, one can easily show that

$$\mathsf{m}\dot{\gamma} + \mathsf{n}\dot{\delta} = p_\alpha\dot{\alpha} + p_\beta\dot{\beta}$$

for arbitrary virtual velocities occurring in these formulas, thus,

$$\mathsf{m} = p_\gamma = p_\beta - p_\alpha, \qquad \mathsf{n} = p_\delta = p_\beta + p_\alpha,$$

and conversely,

$$p_\alpha = \frac{1}{2}(\mathsf{n} - \mathsf{m}), \qquad p_\beta = \frac{1}{2}(\mathsf{n} + \mathsf{m}).$$

The previously used magnitudes of S, V become:

$$\|S\| = |p_\alpha| = \frac{1}{2}|\mathsf{n} - \mathsf{m}|, \qquad \|V\| = |p_\beta| = \frac{1}{2}|\mathsf{n} + \mathsf{m}|.$$

For the classical affine-affine kinetic energy in Hamiltonian representation we obtain the following expression:

$$\begin{aligned} \mathcal{T}_{\rm int}^{\rm aff-aff} &= \frac{1}{2A}\left(p_1^2 + p_2^2\right) - \frac{B}{2A(A+2B)}p^2 \\ &+ \frac{1}{16A}\frac{\mathsf{m}^2}{\mathrm{sh}^2\frac{q^2-q^1}{2}} - \frac{1}{16A}\frac{\mathsf{n}^2}{\mathrm{ch}^2\frac{q^2-q^1}{2}}; \end{aligned}$$

the meaning of symbols A, B is like previously, and $n = 2$ is substituted to constant factors.

Similarly, for the metrical-affine and affine-metrical models we obtain, respectively,

$$\begin{aligned} \mathcal{T}_{\rm int}^{\rm met-aff} &= \frac{1}{2\alpha}(p_1^2 + p_2^2) + \frac{1}{2\beta}p^2 \\ &+ \frac{1}{16\alpha}\frac{\mathsf{m}^2}{\mathrm{sh}^2\frac{q^2-q^1}{2}} - \frac{1}{16\alpha}\frac{\mathsf{n}^2}{\mathrm{ch}^2\frac{q^2-q^1}{2}} + \frac{1}{8\mu}(\mathsf{n}-\mathsf{m})^2, \\ \mathcal{T}_{\rm int}^{\rm aff-met} &= \frac{1}{2\alpha}(p_1^2 + p_2^2) + \frac{1}{2\beta}p^2 \\ &+ \frac{1}{16\alpha}\frac{\mathsf{m}^2}{\mathrm{sh}^2\frac{q^2-q^1}{2}} - \frac{1}{16\alpha}\frac{\mathsf{n}^2}{\mathrm{ch}^2\frac{q^2-q^1}{2}} + \frac{1}{8\mu}(\mathsf{n}+\mathsf{m})^2, \end{aligned}$$

where the meaning of constants α, β, μ is like previously, but with $n = 2$ substituted, thus,

$$\alpha = I + A, \qquad \beta = -\frac{(I+A)(I+A+2B)}{B}, \qquad \mu = \frac{(I^2 - A^2)}{I}.$$

As m and n, or equivalently p_α and p_β, are now constants of motion, it is seen that for geodetic problems and for problems with doubly-isotropic potentials $V(q^1, q^2)$, e.g., with dilatation-stabilizing ones $V(q)$, everything reduces to the two-dimensional dynamics in variables q^1, q^2 ruled by the effective Hamiltonian obtained by the formal substitution of fixed values p_α, p_β (or m, n) to the above expressions. Moreover, for $\mathrm{SL}(2,\mathbb{R})$-geodetic problems, or for $\mathrm{GL}(2,\mathbb{R})$-problems with separated variables potentials $V(q,x) = V_{\rm dil}(q) + V_{\rm sh}(x)$, everything reduces trivially to independent one-dimensional motions. In the above geodetic models it is only the relationship between constant values of m, n that decides whether the motion is oscillatory or unbounded. The first case happens, obviously, when $|\mathsf{n}| > |\mathsf{m}|$; then at large "distances" $|q^2 - q^1|$ the attractive ch^{-2}-term prevails. On the contrary, if $|\mathsf{n}| < |\mathsf{m}|$, one deals with the repulsive case, i.e., with the decaying motion of invariants q^1, q^2. This is the simplest example of the fact mentioned above that affinely-invariant geodetic models admit an open family of bounded (vibrating) and an open family of non-bounded (decaying) motions. Obviously, for general $n > 2$ the situation is more complicated because then $M^a{}_b$, $N^a{}_b$ fail to be constants of motion and perform oscillations somehow coupled with those of q^a. Using new variables q, x, p, p_x,

we can rewrite the above models of $\mathcal{T}$ in the following forms:

$$\begin{aligned}
\mathcal{T}_{\text{int}}^{\text{aff}-\text{aff}} &= \frac{p^2}{4(A+2B)} + \frac{p_x^2}{A} + \frac{(p_\alpha - p_\beta)^2}{16A\text{sh}^2\frac{x}{2}} - \frac{(p_\alpha + p_\beta)^2}{16A\text{ch}^2\frac{x}{2}},\\
\mathcal{T}_{\text{int}}^{\text{met}-\text{aff}} &= \frac{p^2}{4(I+A+2B)} + \frac{p_x^2}{I+A}\\
&+ \frac{(p_\alpha - p_\beta)^2}{16(I+A)\text{sh}^2\frac{x}{2}} - \frac{(p_\alpha + p_\beta)^2}{16(I+A)\text{ch}^2\frac{x}{2}} + \frac{Ip_\alpha^2}{I^2-A^2},\\
\mathcal{T}_{\text{int}}^{\text{aff}-\text{met}} &= \frac{p^2}{4(I+A+2B)} + \frac{p_x^2}{I+A}\\
&+ \frac{(p_\alpha - p_\beta)^2}{16(I+A)\text{sh}^2\frac{x}{2}} - \frac{(p_\alpha + p_\beta)^2}{16(I+A)\text{ch}^2\frac{x}{2}} + \frac{Ip_\beta^2}{I^2-A^2}.
\end{aligned}$$

In the special case $n = 2$, it is easily seen that on the level of variables q, x all these geodetic models have identical dynamics. The difference appears only on the level of angular variables α, β. And, just as for the general n, the same is true if we introduce to Hamiltonians some doubly-isotropic potentials $V(q, x)$. In particular, this is true for dilatation-stabilizing potentials $V(q)$, i.e., in a sense, for geodetic invariant models on $\text{SL}(2, \mathbb{R})$ (incompressible bodies).

Quantization of two-dimensional models

Let us now turn to quantization. The Haar measure λ on $\text{GL}(2, \mathbb{R})$ is given by the following expression:

$$d\lambda\left(\alpha; q^1, q^2; \beta\right) = \left|\text{sh}\left(q^1 - q^2\right)\right| d\alpha d\beta dq^1 dq^2,$$

i.e.,

$$d\lambda\left(\alpha; q, x; \beta\right) = \left|\text{sh}x\right| d\alpha d\beta dq dx, \qquad P = \left|\text{sh}x\right|.$$

The Peter-Weyl expansion with respect to the L, R-factors of the two-polar splitting is just the usual double Fourier series:

$$\Psi\left(\alpha; q, x; \beta\right) = \sum_{m,n\in\mathbb{Z}} f^{mn}(q, x) e^{im\alpha} e^{in\beta}.$$

The reduced kinetic Hamiltonian corresponding to $\mathcal{T}_{\text{int}}^{\text{aff}-\text{aff}}$ is as follows:

$$\begin{aligned}
\mathbf{T}^{mn} f^{mn} &= -\frac{\hbar^2}{A}\mathbf{D}_x f^{mn} - \frac{\hbar^2}{4(A+2B)}\frac{\partial^2 f^{mn}}{\partial q^2}\\
&+ \frac{\hbar^2(n-m)^2}{16A\text{sh}^2\frac{x}{2}} f^{mn} - \frac{\hbar^2(n+m)^2}{16A\text{ch}^2\frac{x}{2}} f^{mn},
\end{aligned}$$

where

$$\mathbf{D}_x f^{mn} = \frac{1}{\left|\text{sh}x\right|}\frac{\partial}{\partial x}\left(\left|\text{sh}x\right|\frac{\partial f^{mn}}{\partial x}\right).$$

For the metric-affine and affine-metric models $\mathcal{T}_{\text{int}}^{\text{met}-\text{aff}}$, $\mathcal{T}_{\text{int}}^{\text{aff}-\text{met}}$ we obtain, respectively, the following expressions:

$$\begin{aligned}
\mathbf{T}^{mn} f^{mn} &= -\frac{\hbar^2}{I+A}\mathbf{D}_x f^{mn} - \frac{\hbar^2}{4(I+A+2B)}\frac{\partial^2 f^{mn}}{\partial q^2}\\
&+ \frac{\hbar^2(n-m)^2}{16(I+A)\text{sh}^2\frac{x}{2}} f^{mn} - \frac{\hbar^2(n+m)^2}{16(I+A)\text{ch}^2\frac{x}{2}} f^{mn} + \frac{I\hbar^2 m^2}{I^2-A^2} f^{mn},
\end{aligned}$$

$$\begin{aligned}
\mathbf{T}^{mn} f^{mn} &= -\frac{\hbar^2}{I+A}\mathbf{D}_x f^{mn} - \frac{\hbar^2}{4(I+A+2B)}\frac{\partial^2 f^{mn}}{\partial q^2}\\
&+ \frac{\hbar^2(n-m)^2}{16(I+A)\text{sh}^2\frac{x}{2}} f^{mn} - \frac{\hbar^2(n+m)^2}{16(I+A)\text{ch}^2\frac{x}{2}} f^{mn} + \frac{I\hbar^2 n^2}{I^2-A^2} f^{mn}
\end{aligned}$$

with the same meaning of symbols as previously. It is seen that in all these expressions the complete separation between dilatational and incompressible motion is very effectively described in analytical terms just due to the use of coordinates q, x. Obviously, for geodetic Hamiltonians on $\mathrm{GL}(2,\mathbb{R})$ the energy spectrum is continuous (and classical trajectories are unbounded; in a sense equivalent facts) because dilatational motion is free. As in the general case, this fact is physically avoided by introducing to the Hamiltonian some dilatation-stabilizing potential $V_{\mathrm{dil}}(q)$. On the quantum level the simplest possible model is the potential well.

This is, in a sense, reduction to the geodetic quantum problem on $\mathrm{SL}(2,\mathbb{R})$. Obviously, the problem with $V_{\mathrm{dil}}(q)$ remains explicitly separable. It remains so also for a more general class of doubly isotropic potentials, e.g., for ones explicitly splitting,

$$V(q,x) = V_{\mathrm{dil}}(q) + V_{\mathrm{sh}}(x),$$

but perhaps also for more general ones. Solutions of the corresponding stationary Schrödinger equations may be sought in the following form:

$$f^{mn}(q,x) = \varphi^{mn}(q)\chi^{mn}(x);$$

the problem reduces then to one-dimensional Schrödinger equations for φ^{mn} and χ^{mn}. And now, in the special two-dimensional case, it is explicitly seen that there exists a discrete spectrum (bounded situations) for χ-functions, i.e., for the isochoric $\mathrm{SL}(2,\mathbb{R})$-problem, even in the purely geodetic case without any potential $V_x(x)$. And this is true in spite of the non-compactness of the $\mathrm{SL}(2,\mathbb{R})$-configuration space. Everything depends on the mutual relationship between "rotational" quantum numbers m, n. If $|n+m| > |n-m|$, the attractive ch^{-2}-term prevails at large "distances" $|x| \to \infty$ and the spectrum is discrete. In the opposite case, if $|n+m| < |n-m|$, it is continuous.

For the affine-affine geodetic model on $\mathrm{SL}(2,\mathbb{R})$, the total spectrum (total in the sense of solutions for all possible $m, n \in \mathbb{Z}$) is not bounded from below; this might seem undesirable. For the metric-affine and affine-metric geodetic problems on $\mathrm{SL}(2,\mathbb{R})$, the spectrum may be bounded from below (and so is the corresponding kinetic energy). Everything depends on the mutual relationship between inertial constants I, A, B, which play the role of some controlling parameters.

Two-dimensional d'Alembert models

For comparison, let us quote a few corresponding formulas for the "usual" d'Alembert model in two dimensions. We restrict ourselves to the doubly-isotropic model. The classical kinetic Hamiltonian may be expressed as follows:

$$\mathcal{T}_{\mathrm{int}}^{\mathrm{d.A}} = \frac{1}{2I}\left(P_1^2 + P_2^2\right) + \frac{1}{4I}\frac{\mathsf{m}^2}{\left(Q^1 - Q^2\right)^2} + \frac{1}{4I}\frac{\mathsf{n}^2}{\left(Q^1 + Q^2\right)^2},$$

with the same meaning of symbols as previously. Let us stress that Q^a are diagonal elements of D, and now the variables $q^a = \ln Q^a$ would be completely useless. The quantity P_l is given simply by the following expression:

$$P_l = \left|\left(Q^1\right)^2 - \left(Q^2\right)^2\right| = \left|\left(Q^1 + Q^2\right)\left(Q^1 - Q^2\right)\right|,$$

and the usual Lebesgue measure on $\mathrm{L}(2,\mathbb{R}) \simeq \mathbb{R}^4$ is expressed as follows:

$$dl\left(\alpha; Q^1, Q^2; \beta\right) = P_l\left(Q^1, Q^2\right) d\alpha d\beta dQ^1 dQ^2.$$

As mentioned, geodetic models are non-physical (and, by the way, the above coordinates would be completely artificial for them). There is, however, a class of physically reasonable doubly isotropic potentials $V\left(Q^1, Q^2\right)$ for which the corresponding Hamiltonians $H = \mathcal{T} + V$ describe integrable systems admitting solutions in terms of separation of variables. This fact is obvious when, instead of Q^1, Q^2, the $(\pi/4)$-rotated coordinates Q^+, Q^- on the plane of deformation invariants are used,

$$Q^{\pm} := \frac{1}{\sqrt{2}}\left(Q^1 \pm Q^2\right).$$

The polar and elliptic coordinates on the (Q^+, Q^-)-plane are also convenient,

$$Q^+ = r\cos\varphi, \qquad Q^- = r\sin\varphi$$

and

$$Q^+ = \mathrm{ch}\rho\cos\lambda, \qquad Q^- = \mathrm{sh}\rho\sin\lambda.$$

There exist physically reasonable potentials V for which the corresponding Hamiltonian problems are separable (thus, obviously, integrable) in coordinates (Q^+, Q^-), (r, φ), or (ρ, λ). There are also interesting superintegrable (degenerate) models separable simultaneously in two or even three of the above coordinate systems.

On the quantized level the reduced Schrödinger equation has the following form:

$$\mathbf{H}^{mn} f^{mn} = E^{mn} f^{mn},$$

where

$$\mathbf{H}^{mn} f^{mn} = \mathbf{T}^{mn} f^{mn} + \mathbf{V}\left(Q^1, Q^2\right) f^{mn}$$

$$= -\frac{\hbar^2}{2I}\mathbf{D}_l f^{mn} + \frac{\hbar^2 m^2}{4I(Q^1 - Q^2)^2} f^{mn} + \frac{\hbar^2 n^2}{4I(Q^1 + Q^2)^2} f^{mn} + \mathbf{V}\left(Q^1, Q^2\right) f^{mn}.$$

Obviously,

$$\mathbf{D}_l f = \frac{1}{P_l}\sum_{a=1}^{2}\frac{\partial}{\partial Q^a}\left(P_l \frac{\partial f}{\partial Q^a}\right).$$

Everything said above about separability of the classical problems remains true on the quantized level. Again the coordinate systems (Q^+, Q^-), (r, φ), (ρ, λ) are crucial.

Hamiltonian systems on $\mathbf{U}(n)$

To finish these quantization remarks let us mention briefly about Hamiltonian systems on $\mathrm{U}(n)$, i.e., in a sense, affine systems with "compactified deformation invariants". The resulting kinetic energy operator has the following form:

$$\begin{aligned}\mathbf{T} &= -\frac{\hbar^2}{2A}\mathbf{D}_U + \frac{\hbar^2 B}{2A(A+nB)}\frac{\partial^2}{\partial q^2} \\ &+ \frac{1}{32A}\sum_{a,b}\frac{(\mathbf{M}^a{}_b)^2}{\sin^2\frac{q^a - q^b}{2}} + \frac{1}{32A}\sum_{a,b}\frac{(\mathbf{N}^a{}_b)^2}{\cos^2\frac{q^a - q^b}{2}},\end{aligned}$$

where

$$\begin{aligned}\mathbf{D}_U &= \frac{1}{P_U}\sum_a \frac{\partial}{\partial q^a} P_U \frac{\partial}{\partial q^a} = \sum_a \frac{\partial^2}{\partial (q^a)^2} + \sum_a \frac{\partial \ln P_U}{\partial q^a}\frac{\partial}{\partial q^a}, \\ P_U &= \prod_{a\neq b}\left|\sin(q^a - q^b)\right|.\end{aligned}$$

The Haar measure is given by the following expression:

$$d\lambda_U(L, D, R) = P_U d\mu(L) d\mu(R) dq^1 \cdots dq^n,$$

where μ, as previously, denotes the Haar measure on $\mathrm{SO}(n, \mathbb{R})$.

Obviously, $\mathrm{U}(n)$ is compact, thus, all classical trajectories for geodetic models are bounded and the corresponding quantum spectrum is discrete. Nevertheless, more general models with doubly-isotropic potentials,

$$\mathbf{H} = \mathbf{T} + \mathbf{V}\left(q^1, \ldots, q^n\right),$$

may be also of physical interest.

The problem splits again, just as in the $\mathrm{GL}(n,\mathbb{R})$-case, into the family of reduced problems resulting from the Fourier analysis on $\mathrm{SO}(n,\mathbb{R})$ performed both in the L- and R-variables:

$$\mathbf{H}^{\alpha\beta}f^{\alpha\beta}=E^{\alpha\beta}f^{\alpha\beta},$$

where

$$\begin{aligned}\mathbf{H}^{\alpha\beta}f^{\alpha\beta} &= -\frac{\hbar^2}{2I}\mathbf{D}_U f^{\alpha\beta}-\frac{\hbar^2}{2\beta}\frac{\partial^2 f^{\alpha\beta}}{\partial q^2}+\frac{1}{32A}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta}}{}^{a}{}_{b}-\overrightarrow{S^{\alpha}}{}^{a}{}_{b}\right)^2}{\sin^2\frac{q^a-q^b}{2}}f^{\alpha\beta}\\ &+ \frac{1}{32A}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta}}{}^{a}{}_{b}+\overrightarrow{S^{\alpha}}{}^{a}{}_{b}\right)^2}{\cos^2\frac{q^a-q^b}{2}}f^{\alpha\beta}+\mathbf{V}\left(q^1,\ldots,q^n\right)f^{\alpha\beta},\end{aligned}$$

with the same meaning of symbols as previously.

Just as in the $\mathrm{GL}(n,\mathbb{R})$-models, particularly simple are physical dimensions $n=2,3$. The former one has also certain very peculiar features and admits simple calculations based on integrable models and separability techniques. Namely, $\mathbf{H}^{mn}$ acts as follows:

$$\begin{aligned}\mathbf{H}^{mn}f^{mn} &= -\frac{\hbar^2}{A}\mathbf{D}_x f^{mn}+\frac{\hbar^2(n-m)^2}{16A\sin^2\frac{x}{2}}f^{mn}+\frac{\hbar^2(n+m)^2}{16A\cos^2\frac{x}{2}}f^{mn}\\ &+ \mathbf{V}_x(x)f^{mn}-\frac{\hbar^2}{4(A+2B)}\frac{\partial^2 f^{mn}}{\partial q^2}+\mathbf{V}_q(q)f^{mn},\end{aligned}$$

where, obviously, the Haar measure has the following form:

$$d\lambda_U\left(\alpha;q,x;\beta\right)=|\sin x|d\alpha d\beta dq dx,$$

and

$$\mathbf{D}_x f=\frac{1}{|\sin x|}\frac{\partial}{\partial x}\left(|\sin x|\frac{\partial f}{\partial x}\right).$$

The problem also separates, in particular, for geodetic problems, $V=0$, or for potentials of the above-mentioned form:

$$V(q,x)=V_{\mathrm{dil}}(q)+V_{\mathrm{sh}}(x).$$

4 Systems of Affine Bodies

The configuration space of a single structural element is identified with

$$Q = \mathbb{R}^n \times G = Q_{\rm tr} \times Q_{\rm int},$$

in particular, for elements with affine modes of deformation:

$$Q = \mathbb{R}^n \times {\rm GL}(n, \mathbb{R}).$$

Usually, especially in classical (non-quantized) problems, ${\rm GL}(n, \mathbb{R})$ is replaced by ${\rm GL}^+(n, \mathbb{R})$. The labels "tr" and "int" refer obviously to translational and internal degrees of freedom. The total body (medium) consists of N elements. Its configuration space is obviously given by the Cartesian product

$$Q^N = Q^N_{\rm tr} \times Q^N_{\rm int} \simeq \mathbb{R}^{nN} \times G^N;$$

in our treatment ${\rm GL}(n, \mathbb{R})$ or ${\rm GL}^+(n, \mathbb{R})$ substituted for G. Therefore, the configuration is an array:

$$q = (x_1, \ldots, x_N; \varphi_1, \ldots, \varphi_N)\,,$$

where $x_A \in \mathbb{R}^n$, $\varphi_A \in {\rm GL}(n, \mathbb{R})$, and $A = \overline{1, N}$.

For a single structure element the summation of usual kinetic energies of its constituents gives the usual d'Alembert form

$$T^{\rm d'A} = T^{\rm d'A}_{\rm tr} + T^{\rm d'A}_{\rm int} = \frac{M}{2}{\rm Tr}\left(vv^T\right) + \frac{1}{2}{\rm Tr}\left(\xi J \xi^T\right),$$

where $v \in \mathbb{R}^n$, $\xi \in {\rm L}(n, \mathbb{R})$ denote respectively the translational and internal velocities:

$$v^i = \frac{dx^i}{dt}, \qquad \xi^i{}_k = \frac{d\varphi^i{}_k}{dt},$$

and M, J are constant inertial characteristics. More precisely, M is the total mass of the element and the symmetric positively definite matrix J is its modified inertial tensor, i.e., second-order moment of the mass distribution with respect to co-moving (Lagrange) coordinates,

$$M = \sum_p \mu_p, \qquad J^{kl} = \sum_p \mu_p a^k_p a^l_p.$$

Summation is performed over constituents ("atoms") of the element ("molecule"); μ_p is the mass of the p-th constituent. Sometimes it is convenient to use the symbol of integration with respect to the mass distribution measure μ:

$$J^{kl} = \int a^k a^l d\mu(a).$$

Remark: the above kinetic energy is spatially isotropic, i.e., invariant under the transformations L_A below with A restricted to the orthogonal group ${\rm O}(n, \mathbb{R})$ (spatial rotations). So are its both terms separately. The material rotations R_A preserve $T_{\rm tr}$ trivially, but in general $T_{\rm int}$ is non-invariant under the right-acting ${\rm O}(n, \mathbb{R})$. However, it is invariant under the right actions of ${\rm O}(n, J)$, the subgroup of ${\rm GL}(n, \mathbb{R})$ preserving J. This reduces to the ${\rm O}(n, \mathbb{R})$-invariance, when J is isotropic, i.e., $J = I{\rm Id}_n$; I is a positive constant of internal inertia and ${\rm Id}_n$ is the $n \times n$ identity matrix. Then

$$T^{\rm d'A}_{\rm int} = \frac{I}{2}{\rm Tr}\left(\xi\xi^T\right).$$

The total kinetic energy of the body is given by

$$T^{\rm d'A} = \sum_{A=1}^N T^{\rm d'A}_A = \frac{1}{2}\sum_{A=1}^N M_A {\rm Tr}\left(v_A v_A^T\right) + \frac{1}{2}\sum_{A=1}^N {\rm Tr}\left(\xi_A J_A \xi_A^T\right).$$

Assuming that the body consists of identical structure elements we have that $M_A = M$, $J_A = J$, $A = \overline{1,N}$.

Let us again concentrate on a single element. Its Green and Cauchy deformation tensors are respectively denoted as

$$G[\varphi] = \varphi^T\varphi = G[\varphi]^T, \qquad C[\varphi] = \varphi^{-1T}\varphi^{-1} = C[\varphi]^T,$$

similarly, for their contravariant inverses we write

$$\widetilde{G}[\varphi] = \varphi^{-1}\varphi^{-1T}, \qquad \widetilde{C}[\varphi] = \varphi\varphi^T.$$

Spatial and material transformations are respectively given by left and right regular translations:

$$\varphi \mapsto L_A(\varphi) = A\varphi, \qquad \varphi \mapsto R_A(\varphi) = \varphi A$$

for any fixed $A \in \mathrm{GL}(n,\mathbb{R})$. When $A \in \mathrm{O}(n,\mathbb{R})$, then obviously

$$G[A\varphi] = G[\varphi], \qquad C[\varphi A] = C[\varphi],$$

and for the general $A \in \mathrm{GL}(n,\mathbb{R})$

$$G[\varphi A] = A^T G[\varphi] A, \qquad C[A\varphi] = A^{-1T} C[\varphi] A^{-1}.$$

There is no concise formula for $G[A\varphi]$, $C[\varphi A]$ if A is not orthogonal (does not belong to $\mathrm{O}(n,\mathbb{R})$).

Deformation invariants are scalar functions $f : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}$ invariant under two-side regular orthogonal translations

$$f(A\varphi B) = f(\varphi)$$

for any $A, B \in \mathrm{O}(n,\mathbb{R})$. There are n basic invariants through which all other ones may be expressed. Various choices are possible, e.g., the following frequently used

$$\mathcal{K}_a[\varphi] = \mathrm{Tr}\left(G[\varphi]^a\right) = \mathrm{Tr}\left(C[\varphi]^{-a}\right), \qquad a = \overline{1,n},$$

eigenvalues $\lambda_a[\varphi]$ of $G[\varphi]$,

$$\det\left(G[\varphi] - \lambda[\varphi] I_n\right) = 0,$$

or coefficients $I_p[\varphi]$ of the eigenequation

$$\det\left(G[\varphi] - \lambda I_n\right) = \sum_{k=0}^{n} (-1)^k I_{n-k}[\varphi]\lambda^k;$$

obviously, $I_0 = 1$ is standard. Geometrically speaking, deformation invariants are functions on the manifold of double cosets

$$\mathrm{Inv} := \mathrm{O}(n,\mathbb{R})\backslash\mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n,\mathbb{R}).$$

Deformation invariants are used when constructing potential energy models for a single affine body. When dealing with the system of such bodies we need some basic scalars assigned to pairs of internal configurations. In analogy to Green and Cauchy deformation tensors for any pair $\psi, \varphi \in \mathrm{GL}(n,\mathbb{R})$ we define the quantities

$$G[\psi,\varphi] := \psi^T\varphi, \qquad C[\psi,\varphi] := \varphi^{-1T}\psi^{-1}.$$

Obviously,

$$G[\psi,\psi] = G[\psi], \qquad C[\psi,\psi] = C[\psi],$$

i.e., the above mutual deformation tensors reduce then to the usual ones.

But one can also define another mutual quantities, namely,

$$\Gamma[\psi,\varphi] := \psi^{-1}\varphi, \qquad \Sigma[\psi,\varphi] := \varphi\psi^{-1}.$$

For orthogonal matrices they reduce to the previous ones,

$$\psi, \varphi \in \mathrm{O}(n, \mathbb{R}) \Rightarrow \Gamma[\psi, \varphi] = G[\psi, \varphi], \quad \Sigma[\psi, \varphi] = C[\psi, \varphi].$$

Obviously, $\Gamma[\psi, \varphi]$, $\Sigma[\psi, \varphi]$ are exactly group-theoretical counterparts of the displacement vector in translational degrees of freedom. Indeed, interpreting $\mathbb{R}^n$ as an Abelian group under addition of vectors, we immediately notice that the Γ-prescription in the non-Abelian multiplicative matrix group $\mathrm{GL}(n, \mathbb{R})$ has exactly the same group meaning as $u = y - w$ in $\mathbb{R}^n$.

It is clear that for any $A \in \mathrm{O}(n, \mathbb{R})$

$$G[A\psi, A\varphi] = G[\psi, \varphi], \qquad C[\psi A, \varphi A] = C[\psi, \varphi],$$

i.e., they are respectively invariant under spatial and material isometries. For the general $A \in \mathrm{GL}(n, \mathbb{R})$ we have

$$G[\psi A, \varphi A] = A^T G[\psi, \varphi] A, \qquad C[A\psi, A\varphi] = A^{-1T} C[\psi, \varphi] A^{-1}.$$

And just as previously there is no concise expression for $G[A\psi, A\varphi]$, $C[\psi A, \varphi A]$ if A is non-orthogonal.

Transformation rules for Γ, Σ have another form. Namely, for any $A \in \mathrm{GL}(n, \mathbb{R})$ we have

$$\begin{aligned} \Gamma[A\psi, A\varphi] &= \Gamma[\psi, \varphi], & \Sigma[A\psi, A\varphi] &= A\Sigma[\psi, \varphi]A^{-1}, \\ \Gamma[\psi A, \varphi A] &= A^{-1}\Gamma[\psi, \varphi]A, & \Sigma[\psi A, \varphi A] &= \Sigma[\psi, \varphi]. \end{aligned}$$

Therefore, Γ is invariant under spatial affine transformations and suffers the inverse adjoint rule under material affine transformations. And conversely, Σ transforms according to the adjoint rule under spatial affine mappings and is affinely invariant under material transformations.

The quantities $G[\psi, \varphi]$, $C[\psi, \varphi]$, $\Gamma[\psi, \varphi]$, $\Sigma[\psi, \varphi]$, give rise to scalars which may be used as arguments of the potential energy terms. Typical scalars of this type are given by

$$\mathcal{K}_a[\psi, \varphi] = \mathrm{Tr}\left(G[\psi, \varphi]^a\right) = \mathrm{Tr}\left(C[\psi, \varphi]^{-a}\right), \qquad a = \overline{1, n}.$$

Just as in the case of deformation invariants, these scalars are invariant under spatial and materia rotations (left and right regular translations of φ, ψ by orthogonal matrices):

$$\mathcal{K}_a[A\psi B, A\varphi B] = \mathcal{K}_a[\psi, \varphi], \qquad A, B \in \mathrm{O}(n, \mathbb{R}).$$

One can also use solutions of the eigenequation for $G[\psi, \varphi]$ (or $C[\psi, \varphi]$), or coefficients in the eigenequation as basic invariants. Another kind of invariants is built of Γ, Σ-objects, e.g.,

$$\mathcal{M}_a[\psi, \varphi] = \mathrm{Tr}\left(\Gamma[\psi, \varphi]^a\right) = \mathrm{Tr}\left(\Sigma[\psi, \varphi]^a\right)$$

or, according to the λ_a-, I_a-schemes. These objects are invariant under all affine spatial and material transformations, i.e.,

$$\mathcal{M}_a[A\psi B, A\varphi B] = \mathcal{M}_a[\psi, \varphi], \qquad A, B \in \mathrm{O}(n, \mathbb{R})$$

for any $A, B \in \mathrm{GL}(n, \mathbb{R})$. These scalars measures of the "distance" between internal configurations are affinely invariant. Unlike this, the measures $\mathcal{K}_a$ are only orthogonally invariant, so they are usual Euclidean distances.

Dynamical models. Affine invariance problems. Realistic questions, academic questions, and pure fantasy

For the system of affine bodies Lagrangian has the form:

$$L = T - \mathcal{V},$$

where the kinetic energy is obtained by summation of individual kinetic energies,

$$T = \sum_{A=1}^{N} T_A = T_{\mathrm{tr}} + T_{\mathrm{int}} = \sum_{A=1}^{N} \left(T_{\mathrm{tr}}\right)_A + \sum_{A=1}^{N} \left(T_{\mathrm{int}}\right)_A.$$

The potential energy in typical situations consists of two main terms, the one- and two-body potentials,

$$\mathcal{V} = \mathcal{V}^{(1)} + \mathcal{V}^{(2)}.$$

It is known that in realistic problems it is usually less then 10% of energy that could be assigned to three-body and higher multibody interactions. $\mathcal{V}^{(1)}$ is the sum of terms depending on individual elements,

$$\mathcal{V}^{(1)}\left(\ldots; x_A, \varphi_A; \ldots\right) = \sum_{B=1}^{N} {\mathcal{V}^{(1)}}_B\left(x_B, \varphi_B\right).$$

The over-simplified models where ${\mathcal{V}^{(1)}}_B$ splits into the sum of translational and internal parts,

$${\mathcal{V}^{(1)}}_B\left(x_B, \varphi_B\right) = {\mathcal{V}^{(1)}_{\mathrm{tr}}}_B\left(x_B\right) + {\mathcal{V}^{(1)}_{\mathrm{int}}}_B\left(\varphi_B\right),$$

are not very realistic, nevertheless, they provide some so-to-speak zeroth-order approximation. When the elements are identical, all ${\mathcal{V}^{(1)}}_B$ have the same functional form.

The binary term has the usual form,

$$\mathcal{V}^{(2)}\left(\ldots; x_A, \varphi_A; \ldots\right) = \frac{1}{2}\sum_{K,L=1}^{N} {\mathcal{V}^{(2)}}_{KL}\left(x_K, \varphi_K; x_L, \varphi_L\right).$$

And again the simplest, although rather academic models are those with the separated dependence of $\mathcal{V}^{(2)}$ on translational and internal variables,

$${\mathcal{V}^{(2)}}_{KL}\left(x_K, \varphi_K; x_L, \varphi_L\right) = {\mathcal{V}^{(2)}_{\mathrm{tr}}}_{KL}\left(x_K, x_L\right) + {\mathcal{V}^{(2)}_{\mathrm{int}}}_{KL}\left(\varphi_K, \varphi_L\right).$$

Mutual interactions should be translationally invariant, i.e., ${\mathcal{V}^{(2)}}_{KL}$ depend on x_K, x_L through $\overrightarrow{x_K x_L} = x_L - x_K$. Isotropy of the physical space implies that the radius-vectors $x_L - x_K$ enter ${\mathcal{V}^{(2)}}_{KL}$ only through their lengths $\|x_L - x_K\|$. There is some more discussion concerning the dependence of ${\mathcal{V}^{(2)}}_{KL}$ on internal degrees of freedom. Isotropy of the physical space implies that ${\mathcal{V}^{(2)}_{\mathrm{int}}}_{KL}$ should depend on φ_K, φ_L only through the mutual tensors $G[\varphi_K, \varphi_L]$, $\Gamma[\varphi_K, \varphi_L]$, thus,

$${\mathcal{V}^{(2)}}_{KL}\left(x_k, \varphi_k; x_L, \varphi_L\right) = f_{KL}\left(\|x_L - x_K\|, G\left[\varphi_K, \varphi_L\right], \Gamma\left[\varphi_K, \varphi_L\right]\right)$$

and obviously for the body consisting of identical elements there is no dependence on K, L; $f_{KL} = f$ for some fixed f. And if the dynamics is to be invariant also under simultaneous material rotations, then at the same time, ${\mathcal{V}^{(2)}}_{KL}$ must depend on internal configurations only through $C[\varphi_K, \varphi_L]$, $\Sigma[\varphi_K, \varphi_L]$. But this means that ${\mathcal{V}^{(2)}}_{KL}$ is algebraically built of the mutual invariants, e.g., chosen as $\mathcal{K}_a[\varphi_K, \varphi_L]$, $\mathcal{M}_a[\varphi_K, \varphi_L]$,

$${\mathcal{V}^{(2)}}_{KL}\left(x_K, \varphi_K; x_L, \varphi_L\right) = f_{KL}\left(\|x_L - x_K\|, \mathcal{K}\left[\varphi_K, \varphi_L\right], \mathcal{M}\left[\varphi_K, \varphi_L\right]\right).$$

In the last formula, $\mathcal{K}$, $\mathcal{M}$ are abbreviations for the systems $\mathcal{K}_a$, $\mathcal{M}_a$, $a = \overline{1, n}$.

In our model, geometry of degrees of freedom and kinematics is ruled by the affine group. On the other hand, the dynamics is not invariant either under spatial or material affine transformations. The spatial metric tensor and the inertial moment J break the affine symmetry and restrict it to the Euclidean one in the physical space and to $\mathrm{O}(n, J)$ in the material space. What concerns potential energy of mutual interactions, it is clear that the vector norm $\|x_L - x_K\|$ and transposition-dependent invariants $\mathcal{K}\left[\varphi_K, \varphi_L\right]$ also restrict the spatial symmetry to $\mathrm{O}(n, \mathbb{R})$. But it is well-known that particularly interesting models and successful analytical procedures appear when the group of dynamical symmetries (symmetries of Lagrangian) coincides with the kinematical group, or at least, when it is as large a subgroup as possible. The questions arise as to the formal possibility and physical usefulness of affinely-invariant models. For a single affinely-rigid body such models are in a sense possible. Their

physical usefulness is not yet decided, although there are some arguments supporting it. Namely, it is quite possible that in complex media with a complicated net of internal interactions a single element is more sensitive to its material surrounding than to the "true" metric tensor (produced, according to General Relativity by the gravitational field as its "vacuum" non-excited state). The more so such a mechanism works in defect theory. Let us also mention the concept of effective mass in crystals, where the kinetic energy of electrons is not based on the "true" metric, but on the effective tensor produced by the material surroundings. There are nice mathematical models of the kinetic energy of a single affine body with the kinetic energy based on the Cauchy tensor used as a metric. There are also some physical arguments supporting such a hypothesis.

The material affine invariance may seem perhaps more natural because there exist models of continua based on very rich material symmetry. As mentioned, this is Arnold description of the ideal incompressible fluid. It is based on infinite-dimensional group of volume-preserving diffeomorphism. They act on the right, i.e., as material transformations. In any case, finite-dimensional geodetic models of small grains or suspensions with kinetic energies materially invariant under $\mathrm{SL}(n,\mathbb{R})$ may be considered as on over-simplified, drastically discretized version of the Arnold model.

Obviously, everything said above concerns directly a single affinely-deformable element, nevertheless, just as in the d'Alembert model, applies also immediately to the total system, because the modified kinetic energies are additive. One should only use explicitly the label K referring to structural elements.

And now let us go back to the problem of potential energy. As mentioned, the external one-particle potential $\mathcal{V}^{(1)}$ cannot be affinely invariant (only constant functions may be so). The formula for the doubly isotropic binary potential $\mathcal{V}^{(2)}_{KL}$ seems to suggest something similar for the dynamics of mutual interactions. However, things are not so simple and one can try to find some modifications towards the affine invariance, just as it was done in the case of kinetic energy models. Some at least formally admissible suggestion may be easily formulated.

Let us fix some pair of structural elements labelled by (K,L). Internal configurations $\varphi_K, \varphi_L \in \mathrm{GL}(n,\mathbb{R})$ give rise to the Cauchy tensors $C[\varphi_K]$, $C[\varphi_L]$. In our considerations above we were faced with the idea of using $C[\varphi]$ as a kind of spatial "metric tensor" underlying affinely-invariant kinetic energies of single elements. Let us now introduce the objects

$$C\left[\varphi_K,\varphi_L\right] = \frac{1}{2}\left(C\left[\varphi_K\right] + C\left[\varphi_L\right]\right).$$

It is symmetric in the labels K, L and positively definite. This motivates the temptation to use it as a "metric tensor" underlying some modified "distance" between x_K and x_L, namely,

$$\begin{aligned}\mathcal{D}\left[x_K,\varphi_K;x_L,\varphi_L\right] &= \sqrt{\left(x_K-x_L\right)^T C\left[\varphi_K,\varphi_L\right]\left(x_K-x_L\right)} \\ &= \sqrt{\mathrm{Tr}\left(C\left[\varphi_K,\varphi_L\right]\left(x_K-x_L\right)\left(x_K-x_L\right)^T\right)}.\end{aligned}$$

Obviously, the above prescription is invariant under the spatial action of $\mathrm{GL}(n,\mathbb{R})$:

$$\mathcal{D}\left[Ax_K, A\varphi_K; Ax_L, A\varphi_L\right] = \mathcal{D}\left[x_K,\varphi_K;x_L,\varphi_L\right]$$

for any $A \in \mathrm{GL}(n,\mathbb{R})$. This is a rather curious affinely-invariant "distance". And now we can modify the potential by introducing into it this new distance-like argument in addition to the usual one:

$$\left\|x_K - x_L\right\| = \sqrt{\left(x_K-x_L\right)^T\left(x_K-x_L\right)} = \sqrt{\mathrm{Tr}\left(\left(x_K-x_L\right)\left(x_K-x_L\right)^T\right)}.$$

So, finally, we have

$$\begin{aligned}&\mathcal{V}^{(2)}_{KL}\left(x_K,\varphi_K;x_L,\varphi_L\right) = \\ &= f_{KL}\left(\left\|x_K-x_L\right\|, \mathcal{D}\left[x_K,\varphi_K;x_L,\varphi_L\right], \mathcal{K}\left[\varphi_K,\varphi_L\right], \mathcal{M}\left[\varphi_K,\varphi_L\right]\right),\end{aligned}$$

where in realistic situations all f_{KL} coincide with some fixed f. It is seen that $\mathcal{V}^{(2)}_{KL}$ depends on its configuration arguments through the system of four scalar quantities. Two of them, namely, $\|x_K - x_L\|$ and $\mathcal{K}\left[\varphi_K, \varphi_L\right]$ are invariants of the rotation group $\mathrm{O}(n, \mathbb{R})$. The remaining two, $\mathcal{D}\left[x_K, \varphi_K; x_L, \varphi_L\right]$ and $\mathcal{M}\left[\varphi_K, \varphi_L\right]$, are invariant under the total linear group $\mathrm{GL}(n, \mathbb{R})$. One can expect that the dependence of $\mathcal{V}^{(2)}$ on the latter two scalars is a highly symmetric, affine background of mutual interactions between constituents of the body. Further on, this high affine symmetry is broken and reduced to the orthogonal one $\mathrm{O}(n, \mathbb{R})$ by the arguments $\|x_K - x_L\|$, $\mathcal{K}\left[\varphi_K, \varphi_L\right]$. This may happen in such a way that $\mathcal{V}^{(2)}$ is a sum of some purely affine term dependent only on $\mathcal{D}$, $\mathcal{M}$ and on an appropriate symmetry-restricting metrical term built of $\|\cdot\|$ and $\mathcal{K}$.

It is a very interesting question whether the binary purely affine models

$$\mathcal{V}^{(2)\mathrm{af}}_{KL} = f_{KL}\left(\mathcal{D}, \mathcal{M}\right)$$

may be realistic. The question was not yet touched seriously. Nevertheless, some limitations of applicability of the binary affine paradigm seem to be obvious. Earlier we discussed dynamical models of a single affinely-rigid body, in particular, the purely geodetic models, i.e., ones without potentials. Lagrangian coincides then with the kinetic energies (metric tensors on $\mathrm{GL}(n, \mathbb{R})$) given above. It turns out that for incompressible affine bodies, when the configuration space of internal motion is restricted to $\mathrm{SL}(n, \mathbb{R})$, the purely geodetic affine models predict the existence of an open family of bounded (oscillatory) trajectories within the general solution. However, on the non-restricted $\mathrm{GL}(n, \mathbb{R})$, when the volume changes are admitted, geodetic affine models predict the non-restricted dilatational motion, i.e., unlimited expansion or contraction. This is an evidently non-physical feature of these models. Therefore, at least some dilatations-stabilizing potential $\mathcal{V}_{\mathrm{dil}}(\det\varphi)$ must be assumed. When we deal with systems of affine bodies, then it is clear that for an appropriate choice of f the relative volumes

$$\det\varphi_L / \det\varphi_K = \det\left({\varphi_K}^{-1}\varphi_L\right) = \det\Gamma\left[\varphi_K, \varphi_L\right]$$

are stabilized in the sense of performing bounded motions. However, no binary potential may stabilize the single volumes $\det\varphi_K$ themselves. Their time evolution will be non-bounded although the above ratios are bounded functions of time. To prevent this one should introduce some one-body potential term stabilizing (making bounded) the over-all dilatational behaviour,

$$\mathcal{V}^{(1)}_{\mathrm{dil}}\left(\ldots, \varphi_A, \ldots\right) = \sum_{K=1}^{N} \mathcal{V}^{(1)}_{\mathrm{dil}\,K}\left(\det\varphi_K\right).$$

It is reasonable to assume that all $\mathcal{V}^{(1)}_{\mathrm{dil}\,K}$ are identical (when the body consists of identical elements),

$$\mathcal{V}^{(1)}_{\mathrm{dil}}\left(\ldots, \varphi_A, \ldots\right) = \sum_{K=1}^{N} f\left(\det\varphi_K\right).$$

When $\mathcal{V}^{(2)}_{KL}$ depend on their arguments in a proper way, so that $\det\Gamma[\varphi_K, \varphi_L]$ are bounded functions of time, then in principle it would be sufficient to use $\mathcal{V}^{(1)}_{\mathrm{dil}}$ depending on $\det\varphi_A$ for some fixed label A only. If such $\mathcal{V}^{(1)}(\det\varphi_A)$ stabilizes $\det\varphi_A$, then automatically all volumes $\det\varphi_K$ will be stabilized by $\mathcal{V}^{(2)}$. But of course such a choice of the shape of $\mathcal{V}$ would not be either aesthetic or reasonable.

General quantization ideas

There is a direct logical chain from the atomic and molecular structure to macroscopic properties, constitutive laws and material engineering. The point is particularly delicate on the nano-level, where one is dealing with a very peculiar convolution of quantum and classical concepts. In any case, quantization is necessary then. Also some quasi-classical and correspondence problems are very relevant for these phenomena.

The first step towards quantization is the classical canonical formalism. One should start from Legendre transformations which for potential systems with Lagrangians $L = T - \mathcal{V}(\cdots; x_K, \varphi_K; \cdots)$ are given by

$$p^K{}_i = \frac{\partial L}{\partial v^i{}_K} = \frac{\partial T}{\partial v^i{}_K}, \quad \pi^{Ka}{}_i = \frac{\partial L}{\partial \xi_K{}^i{}_a} = \frac{\partial T}{\partial \xi_K{}^i{}_a}$$

or, alternatively,

$$\widehat{p}^K{}_a = \frac{\partial L}{\partial \widehat{v}^a{}_K} = \frac{\partial T}{\partial \widehat{v}^a{}_K}, \quad \widehat{\Sigma}^{Ka}{}_b = \frac{\partial L}{\partial \widehat{\Omega}_K{}^b{}_a} = \frac{\partial T}{\partial \widehat{\Omega}_K{}^b{}_a}, \quad \Sigma^{Ki}{}_j = \frac{\partial L}{\partial \Omega_K{}^j{}_i} = \frac{\partial T}{\partial \Omega_K{}^j{}_i}.$$

Inverting these formulas and substituting them to the energy expression

$$E = v^i{}_K \frac{\partial L}{\partial v^i{}_K} + \xi_K{}^i{}_a \frac{\partial L}{\partial \xi_K{}^i{}_a} - L$$

one obtains the classical Hamiltonian

$$H = \mathcal{T} + \mathcal{V}.$$

Let us quote the resulting formulas for the geodetic (kinetic) Hamiltonians $\mathcal{T}$. For the "usual" d'Alembert model we obtain

$$\mathcal{T}^{\mathrm{d'A}} = \mathcal{T}^{\mathrm{d'A}}_{\mathrm{tr}} + \mathcal{T}^{\mathrm{d'A}}_{\mathrm{int}} = \frac{1}{2M}\mathrm{Tr}\left(pp^T\right) + \frac{1}{2}\mathrm{Tr}\left(\pi^T J^{-1}\pi\right).$$

This is, as mentioned, the "usual" expression compatible with the d'Alembert principle. Although from some point of view it seems the best-motivated one, in complicated systems with collective modes and strong internal interactions some doubts and just objections may be raised against it. Our idea here was to concentrate on models motivated by symmetry principles, first of all, by affine symmetry. Let us now review Legendre transforms of affine models quoted above.

Of course, the model of translational kinetic energy $T^{\mathrm{is-af}}_{\mathrm{tr}}$ coincides exactly with $T^{\mathrm{d'A}}_{\mathrm{tr}}$, so we have

$$\begin{aligned}\mathcal{T}_{\mathrm{tr}}{}^{\mathrm{is-af}} &= \frac{1}{2M}p^T p = \frac{1}{2M}\mathrm{Tr}\left(pp^T\right)\\ &= \frac{1}{2M}\widehat{p}^T G[\varphi]^{-1}\widehat{p} = \frac{1}{2M}\mathrm{Tr}\left(\widehat{p}\widehat{p}^T G[\varphi]^{-1}\right).\end{aligned}$$

The corresponding expression for $\mathcal{T}_{\mathrm{tr}}{}^{\mathrm{af-is}}$ has the following form:

$$\begin{aligned}\mathcal{T}_{\mathrm{tr}}{}^{\mathrm{af-is}} &= \frac{1}{2M}\widehat{p}^T\widehat{p} = \frac{1}{2M}\mathrm{Tr}\left(\widehat{p}\widehat{p}^T\right)\\ &= \frac{1}{2M}p^T C[\varphi]^{-1}p = \frac{1}{2M}\mathrm{Tr}\left(pp^T C[\varphi]^{-1}\right).\end{aligned}$$

Let us now quote the Legendre transforms of affinely-invariant internal kinetic energies. So we obtain

$$\mathcal{T}^{\mathrm{af-J}}_{\mathrm{int}} = \frac{1}{2}\mathrm{Tr}\left(C[\varphi]^{-1}\pi^T J^{-1}\pi\right) = \frac{1}{2}\mathrm{Tr}\left(\widehat{\Sigma}^T J^{-1}\widehat{\Sigma}\right).$$

In particular, for the isotropic inertial tensor we have the expression:

$$\mathcal{T}^{\mathrm{af-is}}_{\mathrm{int}} = \frac{1}{2I}\mathrm{Tr}\left(\widehat{\Sigma}^T\widehat{\Sigma}\right) = \frac{1}{2I}\mathrm{Tr}\left(C[\varphi]^{-1}\pi^T\pi\right).$$

Let us remind that the configuration space of our N-body system is given by

$$Q^N \simeq Q^N{}_{\mathrm{tr}} \times Q^N{}_{\mathrm{int}} \simeq \mathbb{R}^{nN} \times \mathrm{GL}\left(n,\mathbb{R}\right)^N,$$

i.e., configurations are arrays. The manifold Q^N is obviously an open subset of the linear space

$$\mathbb{R}^{nN} \times \mathrm{L}\left(n,\mathbb{R}\right)^N \simeq \mathbb{R}^{nN} \times \mathbb{R}^{n^2N} \simeq \mathbb{R}^{n(n+1)N}.$$

In any one-element configuration space $\mathbb{R}^n\times \mathrm{GL}(n,\mathbb{R})$ we are given two distinguished measures. One of them is the Haar measure α invariant under left and right group translations. The other one is the usual Lebesgue measure a on $\mathbb{R}^n\times \mathrm{L}(n,\mathbb{R})$. It is invariant under additive translations. In terms of coordinates

$$\begin{aligned} da\left(x,\varphi\right) &= dx^1\cdots dx^n d\varphi^1{}_1\cdots d\varphi^n{}_n, \\ d\alpha\left(x,\varphi\right) &= \left(\det\varphi\right)^{-n-1} da\left(x,\varphi\right) \\ &= \left(\det\varphi\right)^{-n-1} dx^1\cdots dx^n d\varphi^1{}_1\cdots d\varphi^n{}_n. \end{aligned}$$

When we neglect translational motion then the Haar measure λ on $\mathrm{GL}(n,\mathbb{R})$ and the Lebesgue measure l on $\mathrm{L}(n,\mathbb{R})$ are used

$$\begin{aligned} dl\left(\varphi\right) &= d\varphi^1{}_1\cdots d\varphi^n{}_n, \\ d\lambda\left(\varphi\right) &= \left(\det\varphi\right)^{-n-1} dl\left(\varphi\right) = \left(\det\varphi\right)^{-n} d\varphi^1{}_1\cdots d\varphi^n{}_n. \end{aligned}$$

Configuration spaces of the total N-element system are endowed with the N-told tensor products of these measures, $a^{(N)}$, $\alpha^{(N)}$, $l^{(N)}$, $\lambda^{(N)}$.

The quantized theory is formulated in the following Hilbert spaces:

$$\mathrm{L}^2\left(Q^N,\alpha^{(N)}\right),\mathrm{L}^2\left(Q^N,a^{(N)}\right),\mathrm{L}^2\left(\mathrm{GL}(n,\mathbb{R})^N,\lambda^{(N)}\right),\mathrm{L}^2\left(\mathrm{GL}(n,\mathbb{R})^N,l^{(N)}\right).$$

Their elements, i.e., wave functions, are complex probability amplitudes of finding the system at a given classical configuration. Classical quantities depending only on configuration variables are represented in these L^2-spaces as operators of multiplication by real-valued functions, in particular, by coordinates like x^i, $\varphi^i{}_a$, etc. According to the general rules of quantum mechanics all other quantities are also represented by Hermitian or formally Hermitian (symmetric in dense domains) operators in these Hilbert spaces. Usually some ordering problems of non-commuting operators appear then. However, in dynamical applications, when Hamiltonian operators are constructed, one deals usually with some special physical quantities of well-defined geometric interpretation. As a rule, they are generators of symmetry groups underlying the problem. In our model they are just the affine spin in both the spatial and co-moving representation, the usual metrical spin and vorticity, etc.

Linear momentum operators in spatial and co-moving representations are given respectively by

$$\mathbf{p}^K{}_a = \frac{\hbar}{i}\frac{\partial}{\partial x^a{}_K}, \qquad \widehat{\mathbf{p}}^K{}_a = \frac{\hbar}{i}\varphi_K{}^b{}_a\frac{\partial}{\partial x^b{}_K},$$

where, obviously, $K=\overline{1,N}$ is the "particle" label. These operators are formally Hermitian both in $\mathrm{L}^2\left(Q^N,\alpha^{(N)}\right)$ and $\mathrm{L}^2\left(Q^N,a^{(N)}\right)$. The operators

$$\mathbf{\Sigma}_K{}^a{}_b = \frac{\hbar}{i}\varphi_K{}^a{}_c\frac{\partial}{\partial\varphi_K{}^b{}_c}, \qquad \widehat{\mathbf{\Sigma}}_K{}^a{}_b = \frac{\hbar}{i}\varphi_K{}^c{}_b\frac{\partial}{\partial\varphi_K{}^c{}_a}$$

are formally Hermitian (not literally, they are unbounded as all differential operators) in $\mathrm{L}^2\left(Q^N,\alpha^{(N)}\right)$ and in $\mathrm{L}^2\left(\mathrm{GL}(n,\mathbb{R})^N,\lambda^{(N)}\right)$. Therefore, when using these Hilbert spaces we may interpret $\mathbf{\Sigma}_K{}^a{}_b$, $\widehat{\mathbf{\Sigma}}_K{}^a{}_b$ as operators of affine spin respectively in the spatial and co-moving representations.

Just as in classical theory, $\mathbf{p}^K{}_a$ are infinitesimal generators of translations of the K-th constituent. Similarly, $\mathbf{\Sigma}_K{}^a{}_b$ generate spatial affine transformations (rotations and homogeneous deformations) of internal degrees of freedom of the K-th "molecule". $\widehat{\mathbf{\Sigma}}_K{}^a{}_b$ generate material affine transformations of the K-th element. Namely, let us consider the operators

$$\begin{aligned} \mathbf{V}_K\left(y\right) &:= \exp\left(\frac{i}{\hbar}y^a\mathbf{p}^K{}_a\right), \qquad y\in\mathbb{R}^n, \\ \mathbf{L}_K\left(z\right) &:= \exp\left(\frac{i}{\hbar}z^b{}_a\mathbf{\Sigma}_K{}^a{}_b\right), \qquad z\in\mathrm{L}\left(n,\mathbb{R}\right), \end{aligned}$$

where the operator exponent is meant in the usual power-series sense. If this series convergent in the action on some function $\Psi : Q^N \to \mathbb{C}$, then

$$\begin{aligned}
&\left(\mathbf{V}_K\left(y\right)\Psi\right)\left(\ldots,x_A,\ldots;\ldots,\varphi_B,\ldots\right) = \\
&\qquad = \Psi\left(\ldots,x_A+y\delta_{AK},\ldots;\ldots,\varphi_B,\ldots\right), \\
&\left(\mathbf{L}_K\left(z\right)\Psi\right)\left(\ldots,x_A,\ldots;\ldots,\varphi_B,\ldots\right) = \\
&\qquad = \Psi\left(\ldots,x_A\ldots;\ldots,\exp\left(z\delta_{KB}\right)\varphi_B,\ldots\right).
\end{aligned}$$

Similar statements may be formulated about the co-moving objects, e.g., defining

$$\mathbf{R}_K\left(z\right) := \exp\left(\frac{i}{\hbar}z^b{}_a\widehat{\mathbf{\Sigma}}_K{}^a{}_b\right), \qquad z \in \mathrm{L}\left(n,\mathbb{R}\right),$$

we obtain that

$$\begin{aligned}
&\left(\mathbf{R}_K\left(z\right)\Psi\right)\left(\ldots,x_A,\ldots;\ldots,\varphi_{B,}\ldots\right) = \\
&\qquad = \left(\ldots,x_A,\ldots;\ldots,\varphi_B\exp\left(z\delta_{BK}\right),\ldots\right).
\end{aligned}$$

One can act separately on all arguments, nevertheless, the special geometric role is played by transformations acting in the same way on all arguments, e.g.,

$$\begin{aligned}
\mathbf{V}\left(y\right) &= \mathbf{V}_1\left(y\right)\cdots\mathbf{V}_N\left(y\right), \\
\mathbf{L}\left(z\right) &= \mathbf{L}_1\left(z\right)\cdots\mathbf{L}_N\left(z\right), \\
\mathbf{R}\left(z\right) &= \mathbf{R}_1\left(z\right)\cdots\mathbf{R}_N\left(z\right).
\end{aligned}$$

Their generators are respectively identical with the total linear momentum and the total affine spin in the spatial and co-moving representations,

$$\mathbf{p}_a = \sum_{K=1}^{N}\mathbf{p}^K{}_a, \qquad \mathbf{\Sigma}^a{}_b = \sum_{K=1}^{N}\mathbf{\Sigma}_K{}^a{}_b, \qquad \widehat{\mathbf{\Sigma}}^a{}_b = \sum_{K=1}^{N}\widehat{\mathbf{\Sigma}}_K{}^a{}_b.$$

Obviously,

$$\begin{aligned}
\mathbf{V}\left(y\right) &= \exp\left(\frac{i}{\hbar}y^a\mathbf{p}_a\right), \\
\mathbf{L}\left(z\right) &= \exp\left(\frac{i}{\hbar}z^b{}_a\mathbf{\Sigma}^a{}_b\right), \\
\mathbf{R}\left(z\right) &= \exp\left(\frac{i}{\hbar}z^b{}_a\widehat{\mathbf{\Sigma}}^a{}_b\right).
\end{aligned}$$

Obviously, all exponential operators quoted here are unitary in $\mathrm{L}^2\left(Q^N,\alpha^{(N)}\right)$ or $\mathrm{L}^2\left(\mathrm{GL}(n,\mathbb{R})^N,\lambda^{(N)}\right)$. However, $\mathbf{L}_K$, $\mathbf{R}_K$ are not unitary in $\mathrm{L}^2\left(Q^N,a^{(N)}\right)$ and $\mathrm{L}^2\left(\mathrm{GL}(n,\mathbb{R})^N,l^{(N)}\right)$. The reason is that the measures a, l are not invariant under group translations. Also, when working in $\mathrm{L}^2\left(Q^N,a^{(N)}\right)$ and $\mathrm{L}^2\left(\mathrm{GL}(n,\mathbb{R})^N,l^{(N)}\right)$, that is admissible, one must modify the definition of the above unitary operators (introducing some multipliers). The generators $\mathbf{\Sigma}_K{}^a{}_b$, $\widehat{\mathbf{\Sigma}}_K{}^a{}_b$ are not formally Hermitian and to become such they must be modified by some additive corrections:

$${}'\mathbf{\Sigma}_K{}^a{}_b := \mathbf{\Sigma}_K{}^a{}_b + \frac{\hbar n}{2i}\delta^a{}_b, \quad {}'\widehat{\mathbf{\Sigma}}_K{}^a{}_b := \widehat{\mathbf{\Sigma}}_K{}^a{}_b + \frac{\hbar n}{2i}\delta^a{}_b.$$

Let us also quote the formally Hermitian operators

$$\mathbf{J}_K{}^a{}_b = \mathbf{x}_K{}^a\mathbf{p}^K{}_b + \mathbf{\Sigma}_K{}^a{}_b = \mathbf{\Lambda}_K{}^a{}_b + \mathbf{\Sigma}_K{}^a{}_b,$$

which generate affine transformations acting both on translational and internal degrees of freedom of the K-th constituents. $\mathbf{\Lambda}_K$ and $\mathbf{\Sigma}_K$ are respectively the translational (orbital) and internal parts. One can also introduce the total quantities

$$\mathbf{J}^a{}_b = \mathbf{\Lambda}^a{}_b + \mathbf{\Sigma}^a{}_b$$

obtained by the K-summation.

We have also that

$$'\mathbf{\Lambda}_K{}^a{}_b := \mathbf{\Lambda}_K{}^a{}_b + \frac{\hbar}{2i}\delta^a{}_b.$$

Let us observe that for the total quantities we have

$$'\mathbf{\Sigma}^a{}_b = \mathbf{\Sigma}^a{}_b + \frac{\hbar nN}{2i}\delta^a{}_b, \quad '\widehat{\mathbf{\Sigma}}^a{}_b = \widehat{\mathbf{\Sigma}}^a{}_b + \frac{\hbar nN}{2i}\delta^a{}_b.$$

and similarly

$$'\mathbf{\Lambda}^a{}_b = \mathbf{\Lambda}^a{}_b + \frac{\hbar N}{2i}\delta^a{}_b.$$

After quantization the canonical momenta $\pi^{Ka}{}_i$ conjugate to $\varphi_K{}^i{}_a$ become operators:

$$\mathbf{p}^{Ka}{}_j := \frac{\hbar}{i}\frac{\partial}{\partial\varphi_K{}^j{}_a}.$$

They are formally Hermitian in $\mathrm{L}^2\left(Q^N, a^{(N)}\right)$, $\mathrm{L}^2\left(\mathrm{GL}(n,\mathbb{R})^N, l^{(N)}\right)$, but not in $\mathrm{L}^2\left(Q^N, \alpha^{(N)}\right)$, $\mathrm{L}^2\left(\mathrm{GL}(n,\mathbb{R})^N, \lambda^{(N)}\right)$, so now the situation is quite opposite to the previous one.