Local Dynamics of Quasi-Parabolic Transformations of $\mathbf{C}^n$

Feng Rong

Department of Mathematics
University of Michigan, Ann Arbor

International Workshop on "Local Holomorphic Dynamics"
Pisa, Italy – January 23, 2007

Topics

Object: holomorphic germ f in $\mathbf{C}^n$, fixing the origin O;

Topics:

1. (Holomorphic) Linearization;
2. Existence of attracting sets;

Types of germs:

- Tangent to the identity: the spectrum of df_O consists of only 1's;
- Parabolic: the spectrum of df_O consists of only $e^{2\pi i\theta_j}$ with $\theta_j \in \mathbf{R}\backslash\mathbf{Q}$;
- Quasi-parabolic: the spectrum of df_O consists of both 1's and $e^{2\pi i\theta_j}$ with $\theta_j \in \mathbf{R}\backslash\mathbf{Q}$.

Linearization - Brjuno Condition

Definition (Brjuno condition)

Let $\lambda_j = e^{2\pi i\theta_j}$ *with* $\theta_j \in \mathbf{R}\backslash\mathbf{Q}$, $1 \leq j \leq l$. *We say that* $\{\lambda_j\}$ *satisfies the* *Brjuno condition* *if*

$$\sum_{\nu \geq 0} p_\nu^{-1} \log \omega^{-1}(p_{\nu+1}) < \infty,$$

where

$$\omega(m) = \min_{2 \leq |k| \leq m} \min_{1 \leq j \leq l} |\lambda^k - \lambda_j|, \qquad m \geq 2,$$

and $\{p_\nu\}_{\nu=0}^{\infty}$ *is a sequence of integers with* $1 = p_0 < p_1 < \cdots$.

Linearization - Main Theorem

Theorem

f: quasi-parabolic with $df_O = Diag(I_r, \Lambda_s)$;
M: $f-$invariant, smooth analytic variety through O of dimension r, with $f|_M = Id$;
Assume: 1. $df_p = df_O$ for all $p \in M$;
2. $\{\lambda_j\}$ satisfies the Brjuno condition;
Conclusion: there exists a local holomorphic change of coordinates ψ such that

$$f \circ \psi = \psi \circ \Lambda,$$

where Λ is the linear part of f.

Linearization - Pöschel

Theorem (Pöschel, '86)

f: spectrum of df_O is $\{\lambda_j\}_{j=1}^{n}$;
Assume: $\{\lambda_j\}_{j=r+1}^{n}$ satisfies the "Brjuno condition";
Conclusion: there exists a $f-$invariant, smooth analytic variety M through O of dimension r such that $f|_M$ is linearizable.

"Siegel Disk" $\rightarrow$ "Siegel Cylinder"!

Definition

A parabolic curve for f, a germ of analytic transformation of $(\mathbf{C}^n, O)$, is an injective holomorphic map $\varphi : \Delta \to \mathbf{C}^n$ satisfying the following properties:
(i) Δ is a simply connected domain in $\mathbf{C}$ with $0 \in \partial\Delta$;
(ii) φ is continuous on $\partial\Delta$ and $\varphi(0) = O$;
(iii) $\varphi(\Delta)$ is invariant under f and $f^k(\varphi(\zeta)) \to O$ as $k \to \infty$ for any $\zeta \in \Delta$.
Furthermore, if $[\varphi(\zeta)] \to [v] \in \mathbf{P}^{n-1}$ as $\zeta \to 0$ (where $[\cdot]$ denotes the canonical projection of $\mathbf{C}^n \backslash \{O\}$ onto $\mathbf{P}^{n-1}$), we say that φ is tangent to $[v]$ at the origin.

Parabolic Curves - Review

Theorem (Hakim, '98)

f: tangent to the identity, with order ν;
$[v]$: non-degenerate characteristic direction;
Conclusion: there are $\nu - 1$ parabolic curves tangent to $[v]$ at the origin.

Definitions:

- order ν: $f = f_1 + f_2 + \cdots$, $\nu = \min\{j > 1 : f_j \neq 0\}$;
- characteristic direction $[v]$: $f_\nu(v) = \lambda v$, non-degenerate if $\lambda \neq 0$.

Abate ('01) has shown the existence of parabolic curves for any germs of $\mathbf{C}^2$ tangent to the identity, with the origin as an isolated fixed point. (ref. presentation by Molino on Thursday)

Parabolic Curves - Review, cont.

Theorem (Bracci & Molino, '04)

f: quasi-parabolic germ of $\mathbf{C}^2$;
Assume: f is of order $\nu < \infty$ and dynamically separating;
Conclusion: there exist $\nu - 1$ parabolic curves for f at O tangent to the eigenspace of 1.

Definitions:

- order ν: write $f = (f^1(x,y), f^2(x,y))$.
  $\nu_j = \min\{\text{order of pure x terms in } f^j(x,y)\}$,
  if $\nu_1 \leq \nu_2$ set $\nu = \nu_1$;
- dynamically separating:
  $\mu = \min\{\text{order of x in terms like } x^k y \text{ in } f^2(x,y)\}$,
  f is dynamically separating if $\mu \geq \nu - 1$.

Note: the notion of dynamically separating is needed because those "bad" terms are persistent under blow-ups.

Parabolic Curves - Main Theorem

Theorem

f: quasi-parabolic germ of $\mathbf{C}^n$;
Assume: f is of order $\nu < \infty$ and dynamically separating in the characteristic direction $[v]$;
Conclusion: there exist $\nu - 1$ parabolic curves for f at O tangent to the direction $[v]$.

Note:

- we also have results on "parabolic manifolds";
- when f is given by a global isomorphism of $\mathbf{C}^n$, we also obtain results on "global attracting sets".

Hakim proved similar results for germs tangent to the identity.

Parabolic Curves - Quasi-parabolic, Definitions

Write $f(x, y, z) = (x_1, y_1, z_1)$, $(x, y, z) \in (\mathbf{C}, \mathbf{C}^l, \mathbf{C}^m)$.
ν_1: the least of $i + |j| > 1$ for terms $x^i y^j$ in the expression of x_1, y_1;
ν_2: the least of $i + |j| > 1$ for terms $x^i y^j$ in the expression of z_1;
If $\nu_1 < \infty$ and $\nu_2 \geq \nu_1$, then we say that f is ultra-resonant, and that the order of f is ν.

Note: the order is well-defined.

f: ultra-resonant with order $\nu < \infty$;
A characteristic direction for f is a vector $[v] = [v_1 : \cdots : v_n] \in \mathbf{P}^{n-1}$, with $v_i = 0$ for $l + 1 < i \leq n$, such that $f_\nu(v) = \lambda v$ for some $\lambda \in \mathbf{C}$.
If $\lambda \neq 0$, we say that $[v]$ is non-degenerate.

Parabolic Curves - Quasi-parabolic, Definitions, cont.

Focus: Terms $x^i y^j z^k$ with $|k| = 1$ and $i + |j| + 1 < \nu$ in the expression of z_1;

Let $\Lambda = \mathrm{Diag}(\Lambda_1, \cdots, \Lambda_r)$, $z = (z_{(1)}, \cdots, z_{(r)})$;
Change of coordinates: such that no terms $x^i y^j z^k$ with $|k| = 1$, $i + |j| + 1 < \nu$ and $z^k \neq z_{(p),s}$ in the expression of $z_{1,(p)}$

Definition

We say that f is dynamically separating in the characteristic direction $[v] = [1 : 0 : 0]$ *if there are no terms* $x^i z_{(p),s}$ *with* $i < \nu - 1$ *in the expression of* $z_{1,(p)}$.

Note: the notion of dynamically separating is well-defined.

Parabolic Curves - Method, 1

f: tangent to the identity, with $\nu = 2$;
$[v] = [1 : 0 : \cdots , 0]$: non-degenerate characteristic direction;
Blow-up the origin, in the coordinates $(x = x, y = xu)$, write

$$\begin{cases} x_1 = x - x^2 + O(\|u\|x^2, x^3), \\ u_1 = (I - xA)u + O(\|u\|^2 x, \|u\|x^2) + O(x^2), \end{cases}$$

Note: the class of similarity of A is unique, in particular we can assume that A is in Jordan canonical form;
Want: a parabolic curve, i.e. an analytic curve $u = u(x)$ such that

$$u_1(x) = u(x_1), \text{ with } \lim_{x \to 0} u(x) = 0.$$

Parabolic Curves - Method, 2

We have

$$x_1^{-A} = x^{-A}[(I + xA) + O(\cdots)].$$

Therefore,

$$\begin{aligned} x_1^{-A} u_1 &= x^{-A}[(I + xA) + O(\cdots)][(I - xA)u + O(\cdots)] \\ &= x^{-A} u + x^{-A} O(\cdots). \end{aligned}$$

Thus,

$$u_1(x) = u(x_1) \iff x^{-A} u(x) - x_1^{-A} u(x_1) = x^{-A} H(x, u(x)).$$

Define

$$Tu(x) = x^{A} \sum_{n=0}^{\infty} x_n^{-A} H(x_n, u(x_n)).$$

Want: A fixed point of the operator T.

Parabolic Curves - Method, 3

Questions:

1. Is T well-defined?
2. How to obtain the fixed point?

The answer to Question 1 depends on two facts:

- by local change of coordinates (which might involve the use of $\log(x)$), one can push the order of the pure x terms in the expression of u_1 to arbitrarily big.
- $x_n = O(1/n)$, for n big.

These facts make the summands in the definition of T to be of order n^{-a}, with $a > 1$, thus T well-defined.

Parabolic Curves - Method, 4

The idea to answer Question 2 is as follows.

B: Banach space of functions

$\{u(\cdot) = x^k (\log x)^q h(\cdot);\ h \text{ holomorphic bounded from } D_r \text{ to } \mathbf{C}^{n-1}\},$

$$D_r = \{x \in \mathbf{C};\ |x - r| < r\}, \quad \|u\| = \|h\|_\infty.$$

We know that T is well-defined on B for suitable k, q, r.
To obtain the fixed point, we show that T restricted to a certain closed subset of B is continuous and contracting.

f: quasi-parabolic in $\mathbf{C}^n$;
After a finite number of blow-ups and changes of coordinates:

$$\begin{cases} x_1 = x - x^\nu + O(\cdots), \\ u_1 = (I_l - x^{\nu-1}A)u + O(\cdots), \\ v_1 = (\Lambda - x^{\nu-1}B)v + O(\cdots), \end{cases}$$

Note:

- $B = \mathrm{Diag}(B_1, \cdots, B_r)$;
- the classes of similarity of A and B are unique, in particular, we can assume that both A and $\Lambda^{-1}B$ are in Jordan canonical forms.

Set $N = \text{Diag}(I_l, \Lambda)$, $M = \text{Diag}(A, B)$ and $L = N^{-1}M$. Rewrite

$$\begin{cases} x_1 = x - x^\nu + O(\cdots), \\ w_1 = (N - x^{\nu-1}M)w + O(\cdots). \end{cases}$$

Want: an analytic curve $w = w(x)$ such that

$$w_1(x) = w(x_1), \text{ with } \lim_{x \to 0} w(x) = 0.$$

We have

$$x_1^{-L} = x^{-L}(I + x^{\nu-1}L + O(\cdots)).$$

Therefore,

$$\begin{aligned} x_1^{-L}N^{-1}w_1 &= x^{-L}(I + x^{\nu-1}L + O(\cdots))N^{-1}((N - x^{\nu-1}M)w + O(\cdots)) \\ &= x^{-L}w + x^{-L}O(\cdots). \end{aligned}$$

Lemma

The equality $x_i^{-L}N^{-1} = N^{-1}x_i^{-L}$ holds for all $i \geq 0$.

Therefore,

$$N^{-1}x_1^{-L}w_1 = x^{-L}w + x^{-L}O(\cdots).$$

Thus,

$$w_1(x) = w(x_1) \Longleftrightarrow x^{-L}w(x) - N^{-1}x_1^{-L}w(x_1) = x^{-L}H(x, w(x)).$$

Define

$$Tw(x) = x^L \sum_{n=0}^{\infty} N^{-n}x_n^{-L}H(x_n, w(x_n)).$$

Want: A fixed point of the operator T.

Open Questions

Open questions:

1. tangent to the identity:
   - what is the dynamics in a characteristic direction with zero "index"?
   - what can one say about the dynamics of a general map in $\mathbf{C}^n$, $n > 2$?
2. quasi-parabolic:
   - what can one say about the "non-dynamically-separating" case?
   - are there index theorems in this case?

An Example

Let us take a look at the following map:

$$\begin{cases} z_1 = z - z^3, \\ w_1 = \lambda w - azw + z^{d+1}, \end{cases}$$

where $|\lambda| = 1$, $a \neq 0$ and $d \geq 3$.

Note:

- if $\lambda = 1$, the map is tangent to the identity and $[1 : 0]$ is a degenerate characteristic direction with zero "index";
- if λ is not a root of unity, the map is quasi-parabolic and is "non-dynamically-separating" in the characteristic direction $[1 : 0]$;
- $\{z = 0\}$ is invariant and there are two "parabolic tubes" by one-dimensional theory.

An Example, cont.

Let $D_1 = \{z : |z^2 - \delta| < \delta, \text{Re}z > 0\}$ and
$D_2 = \{z : |z^2 - \delta| < \delta, \text{Re}z < 0\}$, δ small.

By fundamental analysis, we have:

- if $\text{Re}(\lambda \cdot \bar{a}) > 0$, then there is a two-dimensional parabolic domain over D_1 and a parabolic curve over D_2, both tangent to the direction $[1 : 0]$;
- if $\text{Re}(\lambda \cdot \bar{a}) < 0$, then there is a two-dimensional parabolic domain over D_2 and a parabolic curve over D_1, both tangent to the direction $[1 : 0]$.

Note: symmetry break-down!

Parabolic Manifolds - Review

Recall:

f: tangent to the identity, of order ν;
$[v] = [1 : 0 : \cdots, 0]$: non-degenerate characteristic direction;

$$\begin{cases} x_1 = x - x^{\nu} + O(\cdots), \\ u_1 = (I - x^{\nu-1}A)u + O(\cdots), \end{cases}$$

The class of similarity of A is unique,
and A is in Jordan canonical form;

Assume: $A = \mathrm{Diag}(A_1, \cdots, A_r)$, with $A_j = J(\alpha_j)$.
Moreover, $\mathrm{Re}(\alpha_j) > \alpha > 0$, for $1 \le j \le p$.

Parabolic Manifolds - Review, cont.

Theorem (Hakim, '98)

Let E be the sum of the generalized eigenspaces associated to the α_j's, $1 \leq j \leq p$, of dimension d. Then there exist $\nu - 1$ invariant pieces of analytic manifolds of dimension $d + 1$, with O in the boundary and tangent to $\mathbf{C}[v] + E$ at O, such that every point is attracted to the origin in the direction $[v]$.

Note:

- Hakim also assumed that $\text{Re}(\alpha_j) < \alpha$, for $p < j \leq r$, which turns out to be unnecessary. (Rong, '06)
- the proof of the theorem is again to define a suitable operator T and to find a fixed point of T.

Parabolic Manifolds - Quasi-parabolic, 1

f: quasi-parabolic, of order ν,
dynamically separating in the characteristic direction $[1 : 0 : 0]$:

$$\begin{cases} x_1 = x - x^\nu + O(x^\nu \|w\|, x^{\nu+1} \log x), \\ u_1 = (I_l - x^{\nu-1} A)u + O(x^{\nu-1}\|w\|^2, x^\nu \log x \|w\|, x^\mu (\log x)^{q_\mu}), \\ v_1 = (\Lambda - x^{\nu-1} B)v + O(x^{\nu-1}\|w\|^2, x^\nu \log x \|w\|, x^\mu (\log x)^{q_\mu}), \end{cases}$$

Assume:

$A = \mathrm{Diag}(A_1, \cdots, A_r)$, with $A_j = J(\alpha_j)$,
$\Lambda^{-1} B = C = \mathrm{Diag}(C_1, \cdots, C_s)$, with $C_k = J(\beta_k)$,
$\mathrm{Re}(\alpha_j) > \alpha > 0$, for $1 \le j \le p$,
$\mathrm{Re}(\beta_k) > \beta > 0$, for $1 \le k \le p'$.

Set $\gamma = \min\{\alpha, \beta\}$.

Parabolic Manifolds - Quasi-parabolic, 2

Write $u = (u^1, u^2)$, $v = (v^1, v^2)$. Set $\tilde{u} = (u^1, v^1)$, $\tilde{v} = (u^2, v^2)$.
Write $A = (A^1, A^2)$, $B = (B^1, B^2)$, $\Lambda = (\Lambda^1, \Lambda^2)$ and $I_l = (I^1, I^2)$,
Set $N = (I^1, \Lambda^1)$, $M = (A^1, B^1)$, $E = (I^2, \Lambda^2)$ and $D = (A^2, B^2)$.

$$\begin{cases} x_1 = x - x^\nu + P(x, \tilde{u}, \tilde{v}), \\ \tilde{u}_1 = (E - x^{\nu-1} D)\tilde{u} + Q(x, \tilde{u}, \tilde{v}), \\ \tilde{v}_1 = (N - x^{\nu-1} M)\tilde{v} + R(x, \tilde{u}, \tilde{v}). \end{cases}$$

Want: an analytic manifold $\tilde{v} = \tilde{v}(x, \tilde{u})$ such that

$$\tilde{v}_1(x, \tilde{u}) = \tilde{v}(x_1, \tilde{u}_1), \text{ with } \lim_{x \to 0, \tilde{u} \to 0} \tilde{v}(x, \tilde{u}) = 0.$$

Parabolic Manifolds - Quasi-parabolic, 3

Theorem

Let E be the sum of the generalized eigenspaces associated to the α_j's, $1 \leq j \leq p$, and β_k's, $1 \leq k \leq p'$. Then there exist $\nu - 1$ invariant pieces of analytic manifolds of dimension $d + d' + 1$, with O in the boundary and tangent to $\mathbf{C}[v] + E$ at O, such that every point is attracted to the origin in the direction $[v]$.

Note:

- we need technical lemmas to "push" certain terms in P, Q and R;
- the proof of the theorem is again to define a suitable operator T and to find a fixed point of T.

Parabolic Manifolds - Global

f: tangent to the identity at O, global isomorphism of $\mathbf{C}^n$;

Theorem (Hakim, '98)

Let Ω be the set of all points attracted to O through one of the "parabolic manifolds", of dimension $d+1$. Then Ω is isomorphic to $\mathbf{C}^{d+1}$.

Note:

- when $d = n - 1$, we get Fatou-Bieberbach domains;
- the theorem holds in the quasi-parabolic case. (Rong, '06)

Theorem (Nishimura, '83)

f: semi-attractive;
M: $f-$invariant, smooth analytic variety through O of dimension r, with $f|_M = Id$;
Assume: the spectrum of df_p is contained in the unit circle for all $p \in M$;
Conclusion: there exists a local holomorphic change of coordinates ψ such that

$$f \circ \psi = \psi \circ \Lambda(x),$$

where $\Lambda(x) = Diag(I_r, A(x))$.

The main new difficulty is the presence of resonances.
Write $f(z) = \Lambda z + \hat{f}(z)$ and $\psi(w) = w + \hat{\psi}(w)$. Want

$$\hat{\psi} \circ \Lambda - \Lambda \hat{\psi} = \hat{f} \circ \psi.$$

Write

$$\hat{\psi} = \sum_{|k| \geq 2} \psi_k w^k, \qquad \psi_k \in \mathbf{C}^n,$$

and

$$\hat{f} = \sum_{|l| \geq 2} f_l z^l, \qquad f_l \in \mathbf{C}^n,$$

then

$$\sum_{|k| \geq 2} E_k \psi_k w^k = \sum_{|l| \geq 2} f_l \left(\sum_{|m| \geq 1} \psi_m w^m \right)^l, \tag{1}$$

where $E_k = \tilde{\lambda}^k I_n - \Lambda$.

Linearization - Resonances, cont.

Write $E_k = \text{Diag}(E_k^1, E_k^2)$, then $\det E_k^1 = 0$ for

$$k = (k_1, \cdots, k_r, 0, \cdots, 0) \tag{2}$$

and $\det E_k^2 = 0$ for

$$k = (k_1, \cdots, k_r, 0, \cdots, 0, 1, 0, \cdots, 0). \tag{3}$$

Denote by K_1 the set of k's as in (2) and K_2 the set of k's as in (3).
Write $f_l = (f_l^1, f_l^2)$ and $\psi_k = (\psi_k^1, \psi_k^2)$.
Set $\psi_k^1 = 0$ for $k \in K_1$ and $\psi_k^2 = 0$ for $k \in K_1 \cup K_2$.
The right-hand side of (1) produces terms w^k with $k \in K_1$ only when $l \in K_1$ and terms w^k with $k \in K_2$ only when $l \in K_1 \cup K_2$.
But $f_l^1 = 0$ for $l \in K_1$ and $f_l^2 = 0$ for $l \in K_1 \cup K_2$.