Parabolic and near-parabolic renormalizations in complex dynamics

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Local holomorphic dynamics
Pisa, January 25, 2007

Parabolic fixed points

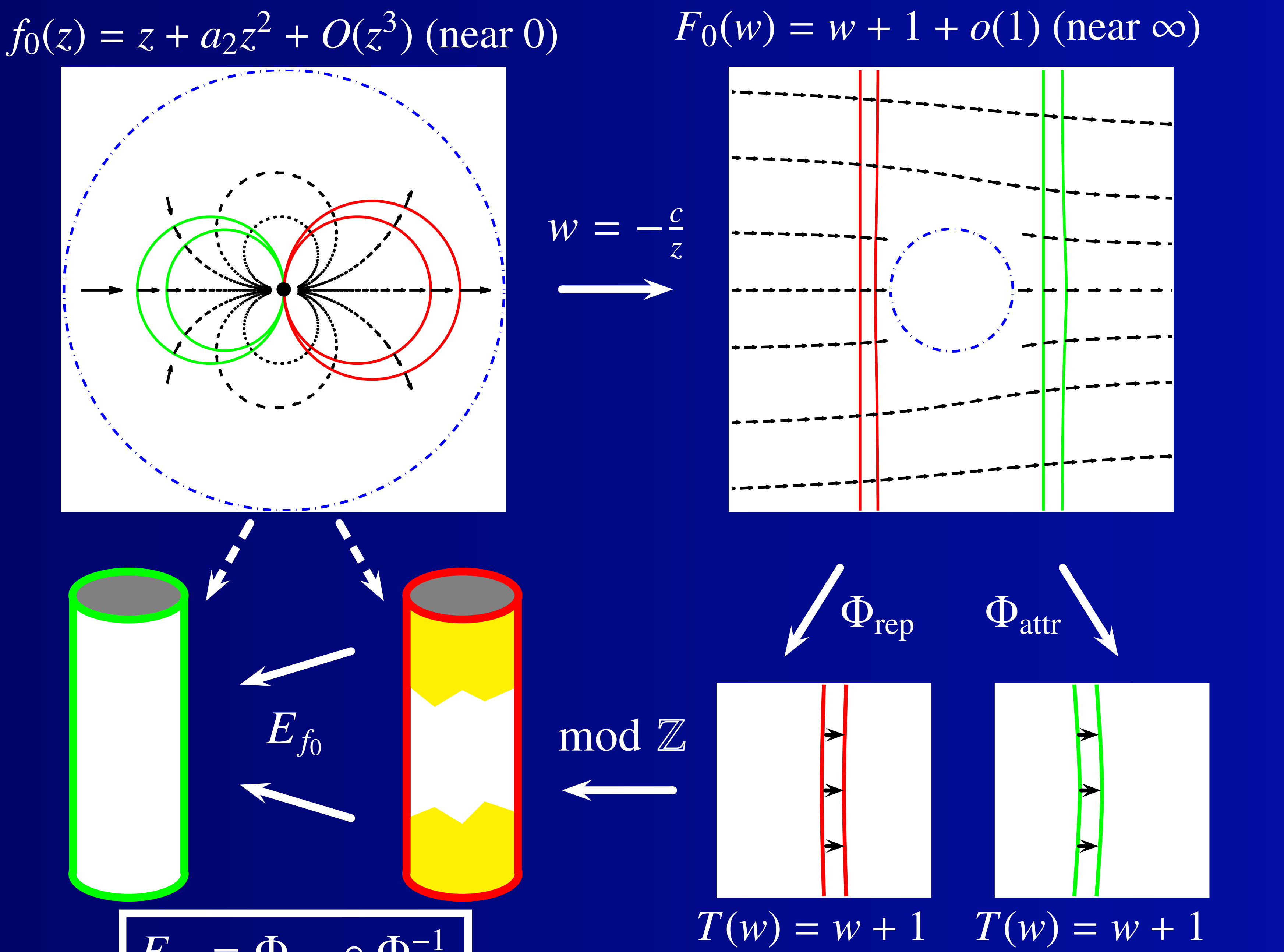
- $f_0(z)$: holomorphic near 0, $f_0(0) = 0$
- multiplier $\lambda = f_0'(0)$
- 0: parabolic $\Leftrightarrow$ λ: root of unity
- We consider the simplest case:

  — 1-parabolic: $\lambda = 1$
  — non-degenerate: $f_0''(0) \neq 0$

  Namely, f_0 has the form

$$f_0(z) = z + a_2 z^2 + O(z^3), \quad a_2 \neq 0$$

Fatou coordinates Φ_{attr}, Φ_{rep} and horn map E_{f_0}



$$E_{f_0} = \Phi_{\mathrm{attr}} \circ \Phi_{\mathrm{rep}}^{-1}$$

Fatou coordinates $\Phi_{\mathrm{attr}}, \Phi_{\mathrm{rep}}$ and horn map E_{f_0}

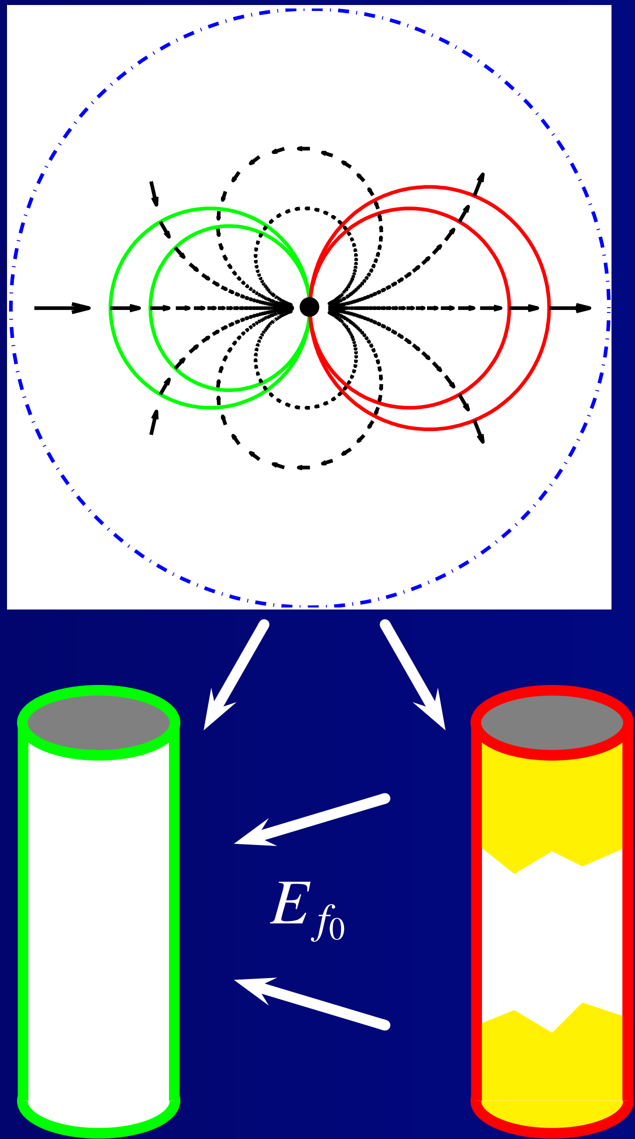
- Fatou coordinates: $\Phi_{\mathrm{attr}}, \Phi_{\mathrm{rep}}$
  $\Phi_*(f_0(z)) = \Phi_*(z) + 1$ ($* = \mathrm{attr},\ \mathrm{rep}$).
- Ambiguity: $\Phi_* + \mathrm{const}_*$.
- Horn map: $E_{f_0} = \Phi_{\mathrm{attr}} \circ \Phi_{\mathrm{rep}}^{-1}$, defined on $|\operatorname{Im} z| \gg 0$.
- Fourier series expansion of E_{f_0}:

$$E_{f_0}(z) = \begin{cases} z + c_+ + \sum_{n>0} a_n^+ e^{2\pi i n z} & \operatorname{Im} z \gg 0, \\ z + c_- + \sum_{n<0} a_n^- e^{2\pi i n z} & \operatorname{Im} z \ll 0. \end{cases}$$

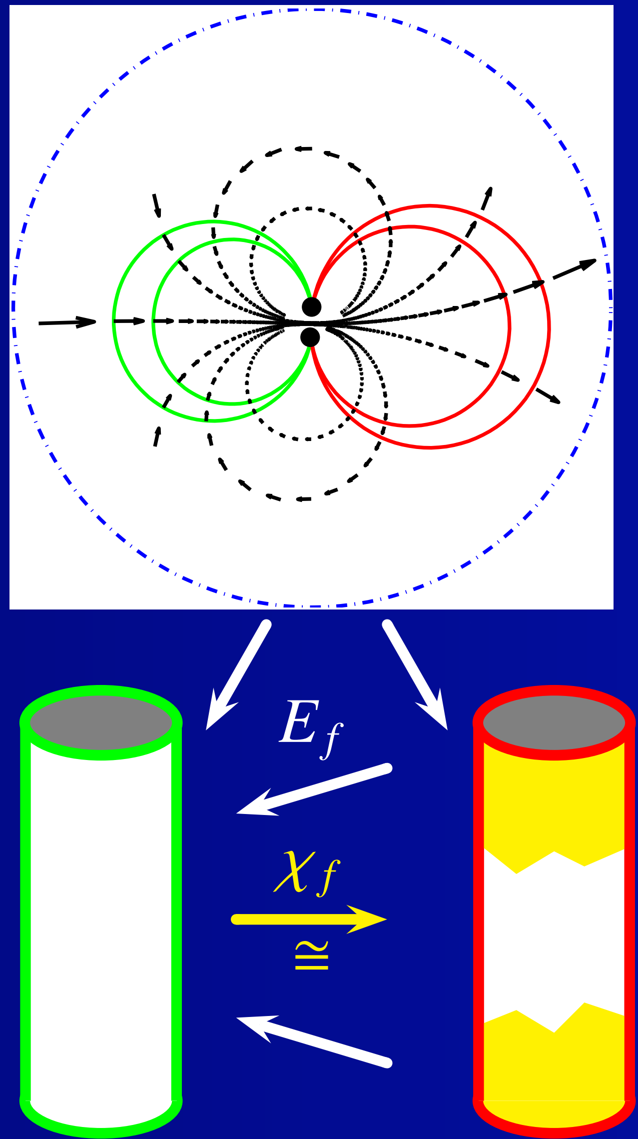
- Ambiguity: $E_{f_0}(z - \mathrm{const}_{\mathrm{rep}}) + \mathrm{const}_{\mathrm{attr}}$.
- E_{f_0} modulo const_*: Ecalle-Voronin invariant (complete invariant for local analytic conjugacy).
- We normalize so that $c_+ = 0$, i.e., $E_{f_0}(z) = z + o(1)$.

Near-parabolic fixed points $f'(0) = e^{2\pi i\alpha}$ (α: small, $|\arg(\pm\alpha)| < \frac{\pi}{4}$)

$f_0(z) = z + a_2 z^2 + O(z^3)$



$f(z) = e^{2\pi i\alpha} z + O(z^2)$



$$\tilde{\mathcal{R}}f = \chi_f \circ E_f$$

Perturbation of parabolic fixed points

- "The gate opens" for a perturbed map $f = e^{2\pi i\alpha} f_0(z)$ and new orbits through the gate induces an isomorphism $\chi_f(z)$ between cylinders $\mathbb{C}/\mathbb{Z}$.
- $\tilde{\mathcal{R}} f = \chi_f \circ E_f$ represents the first return map on the fundamental domain of the Fatou coordinate.
- We normalize the Fatou coordinates so that the following hold:
  - "Parabolic" at the upper end for E_f:
    $$E_f(z) = z + o(1) \quad \text{as } \operatorname{Im} z \to +\infty.$$
  - Continuity on f:
    $$E_f \to E_{f_0} \quad \text{as } f \to f_0.$$
  - $\chi_f(z) = z - \frac{1}{\alpha}$: rigid rotation by $-\frac{1}{\alpha}$.
  - Hence we have
    $$\tilde{\mathcal{R}} f(z) = z - \frac{1}{\alpha} + o(1) \quad \text{as } \operatorname{Im} z \to +\infty.$$

After such a perturbation, orbits through the gate create new complicated dynamics. It is related to many interesting and subtle phenomena.

Example:

- Discontinuous change of the (filled) Julia sets
- Linearization problem of irrationally indifferent fixed points (Siegel, Bruno, Yoccoz. . .)
- Area of Julia sets (Buff-Chéritat)
- Quadratic Julia set having infinite satellite renormalizations

$\tilde{\mathcal{R}}f$ corresponds to a long-time behavior of f. New dynamics can be understood via $\tilde{\mathcal{R}}f$.

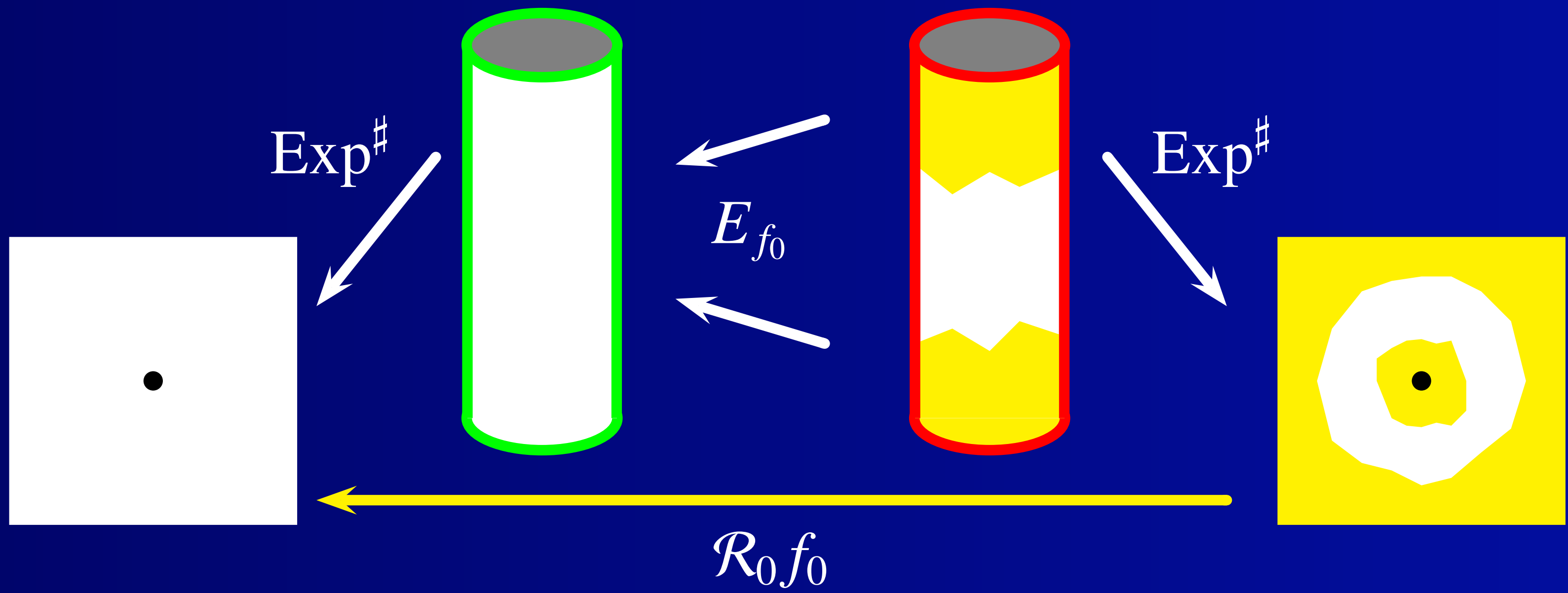
$\rightsquigarrow$ study E_{f_0} first and use continuous dependence of E_f on maps.

Parabolic renormalization

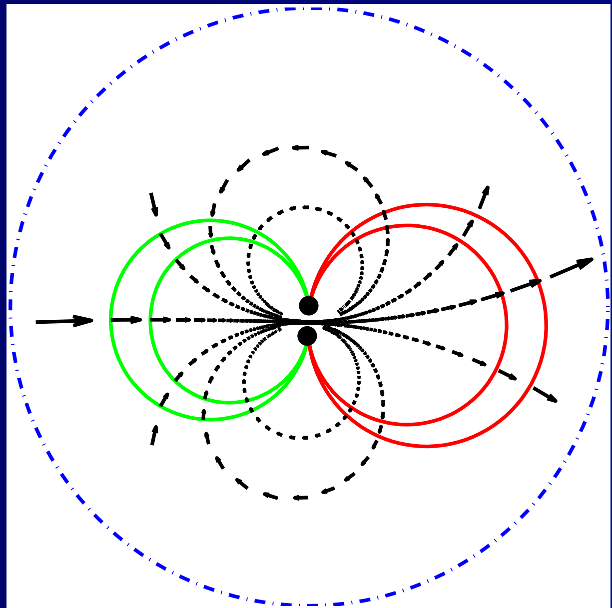
- $\mathrm{Exp}^\sharp(z) = e^{2\pi i z} : \mathbb{C}/\mathbb{Z} \xrightarrow{\cong} \mathbb{C}^*$
- $\mathcal{R}_0 f_0 = \mathrm{Exp}^\sharp \circ E_{f_0} \circ (\mathrm{Exp}^\sharp)^{-1}$: parabolic renormalization of f_0
- $\mathcal{R}_0 f_0$ can be extended to 0 and ∞ holomorphically. They are fixed points and

$$(\mathcal{R}_0 f_0)'(0) = 1$$

Namely, 0 is a 1-parabolic fixed point for $\mathcal{R}_0 f_0$.



Near-parabolic renormalization ($f = e^{2\pi i\alpha} f_0$, f_0: 1-parabolic)



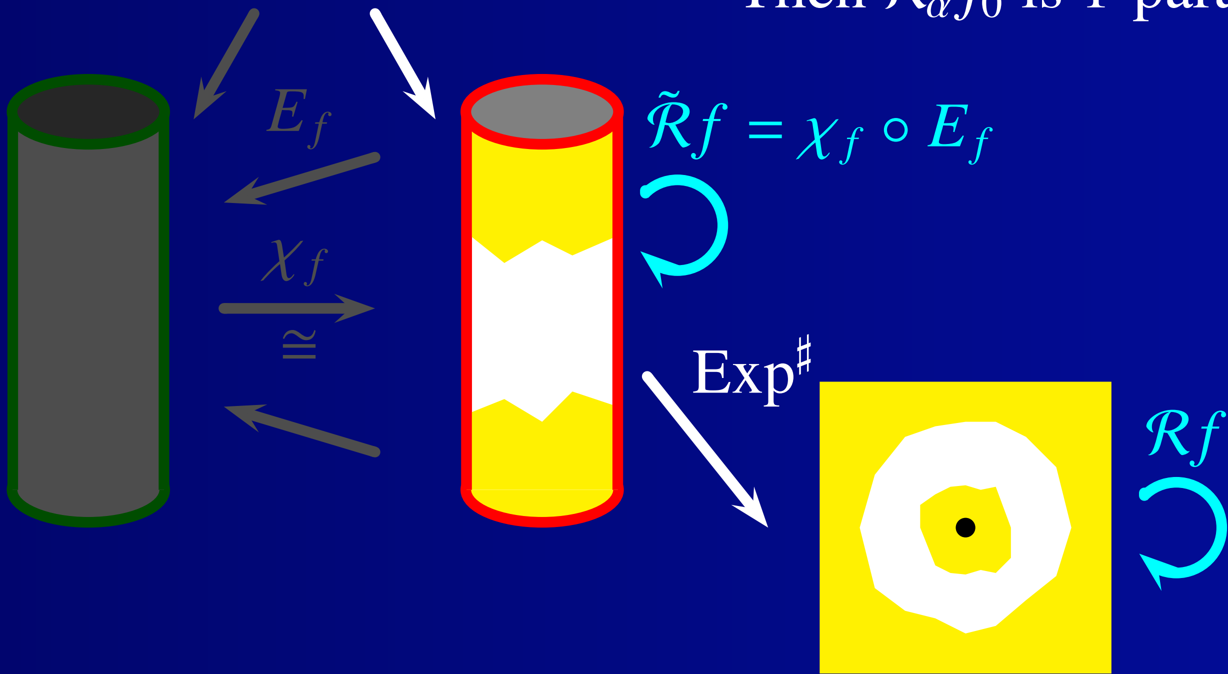
$$\begin{aligned}\mathcal{R}f &= \mathrm{Exp}^\sharp \circ \tilde{\mathcal{R}}f \circ (\mathrm{Exp}^\sharp)^{-1}\\ &= \mathrm{Exp}^\sharp \circ \chi_f \circ E_f \circ (\mathrm{Exp}^\sharp)^{-1}\\ &= e^{2\pi i\beta}\, \mathrm{Exp}^\sharp \circ E_f \circ (\mathrm{Exp}^\sharp)^{-1}\\ &= e^{2\pi i\beta} z + O(z^2),\end{aligned}$$

where $\beta = -\frac{1}{\alpha} \mod \mathbb{Z}$

$\Leftrightarrow \alpha = \frac{1}{m-\beta}$ $(m \in \mathbb{Z})$.

Let $\mathcal{R}_\alpha f_0 = e^{-2\pi i\beta}\mathcal{R}_f$.

Then $\mathcal{R}_\alpha f_0$ is 1-parabolic.



$\mathcal{F}_0$: Invariant space of $\mathcal{R}_0$

$$\mathcal{F}_0 = \left\{ f : U_f \to \mathbb{C} \;\middle|\; \begin{array}{c} 0 \in U_f : \text{ connected open set } \subset \mathbb{C}, \\ f : \text{ holomorphic, } f(0) = 0,\ f'(0) = 1, \\ f : U_f \setminus \{0\} \to \mathbb{C}^* : \text{ branched covering,} \\ \text{with a unique critical value,} \\ \text{local degree at every critical point is 2} \end{array} \right\}$$

- $\mathcal{R}_0 \mathcal{F}_0 \subset \mathcal{F}_0$.
- $z + z^2, \mathcal{R}_0(z + z^2), \cdots \in \mathcal{F}_0$.

This class is used to show that $\dim_{\mathrm{H}}(J(f)) = 2$ for generic $f \in \partial M$ and $\dim_{\mathrm{H}}(\partial M) = 2$ (Shishikura).

To study parabolic bifurcation via $\mathcal{F}_0$, study iteration of $\mathcal{R}_0$ for parabolic maps and then consider perturbations $\mathcal{R}_\alpha$.

Problem. The perturbation size for $\mathcal{R}_0^n$ depends on n. So we can treat only finitely many times of iterations of $\mathcal{R}_\alpha$.

So we want to define a new class of maps where we can iterate $\mathcal{R}_\alpha$ directly.

Main theorems

Theorem 1. *Let $P(z) = z(1+z)^2$. There exist bounded simply connected open sets $0 \in V \Subset V' \subset \mathbb{C}$ such that the class*

$$\mathcal{F}_1 = \left\{ f = P \circ \varphi^{-1} : \varphi(V) \to \mathbb{C} \;\middle|\; \begin{array}{c} \varphi : V \to \mathbb{C} : \text{ univalent}, \\ \varphi(0) = 0,\ \varphi'(0) = 1 \end{array} \right\}$$

satisfies the following.

1. *Every $f \in \mathcal{F}_1$ is non-degenerate;*
2. *$\mathcal{F}_0 \setminus \{\text{quadratic polynomial}\}$ can be naturally embedded into $\mathcal{F}_1$. In particular, $\mathcal{R}_0^n(z+z^2) \in \mathcal{F}_1$ for $n \geq 1$;*
3. *$\mathcal{R}_0$ is defined on $\mathcal{F}_1$ and $\mathcal{R}_0(\mathcal{F}_1) \subset \mathcal{F}_1$;*
4. *let $f \in \mathcal{F}_1$. If we write $\mathcal{R}_0 f = P \circ \psi^{-1}$, then ψ can be extended to a univalent function on V';*
5. *$f \mapsto \mathcal{R}_0 f$ is "holomorphic".*

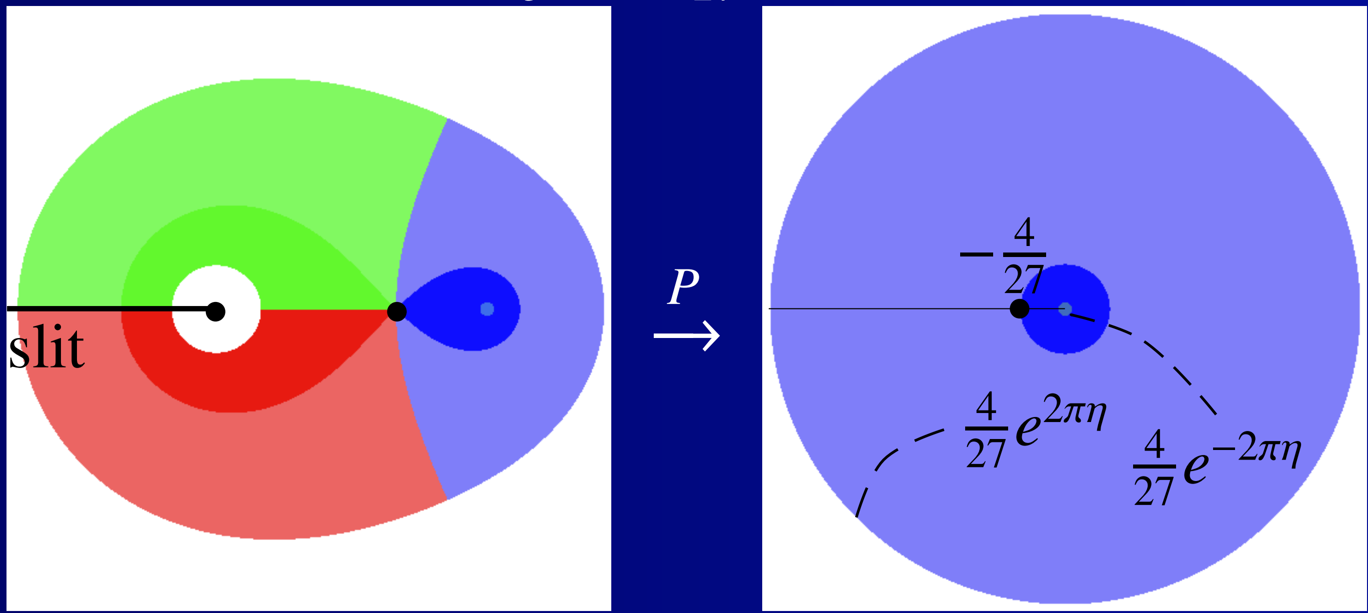
Theorem 2. *The above statements hold for $\mathcal{R}_\alpha$ for α small.*

$P(z) = z(1+z)^2$ and the domains V, V'

$P(0) = 0$, $P'(0) = 1$

Critical points: $-\frac{1}{3}, -1$

Critical values: $P(-\frac{1}{3}) = -\frac{4}{27}$, $P(-1) = 0$



V'

$\boxed{\eta = 2}$ ($\eta = 0.3$ in this figure)

V is a slightly smaller domain than V'.

Proof of Theorem 1

We give an outline of the proof of Theorem 1. (Theorem 2 follows from Theorem 1 and the continuity of E_f on f.)

To prove that a class of maps is invariant, we need a way to recognize that $\mathcal{R}_0 f$ belongs to this class.

We characterize our class by a partial (incomplete) covering property.

Covering property

We say two maps f and g have the same covering property if there exists a univalent map $\varphi : \mathrm{Dom}(f) \to \mathrm{Dom}(g)$ such that $g = f \circ \varphi^{-1}$.

$$\begin{array}{ccc} \mathrm{Dom}(f) & \xrightarrow[\cong]{\varphi} & \mathrm{Dom}(g) \\ f \big\downarrow & & \big\downarrow g \\ \mathbb{C} & = & \mathbb{C} \end{array}$$

$\mathcal{F}_1$ consists of maps with the same covering property as $P|_V$ such that 0 is 1-parabolic.

Coordinate change

We take a coordinate change sending 0 to ∞ because

- maps are close to translation;
- Fatou coordinates are close to the identity.

Hence instead of $P(z)$, we consider

$$Q(z) = z \frac{\left(1 + \frac{1}{z}\right)^6}{\left(1 - \frac{1}{z}\right)^4}.$$

$Q = \psi_0^{-1} \circ P \circ \psi_1$, where $\psi_0(z) = -\frac{4}{z}$, $\psi_1(z) = -\frac{4z}{(1+z)^2}$.

$$\mathcal{F}_1^Q = \left\{ Q \circ \varphi^{-1} \,\middle|\, \begin{array}{l} \varphi : \hat{\mathbb{C}} \setminus E \to \hat{\mathbb{C}} \setminus \{0\} \text{ univalent,} \\ \varphi(\infty) = \infty, \lim_{z\to\infty} \frac{\varphi(z)}{z} = 1 \end{array} \right\}$$

$$E = \left\{ z = x + iy \,\middle|\, \left(\frac{x+0.18}{1.24}\right)^2 + \left(\frac{y}{1.04}\right)^2 \le 1 \right\}$$

$$V = \psi_1(\hat{\mathbb{C}} \setminus E)$$

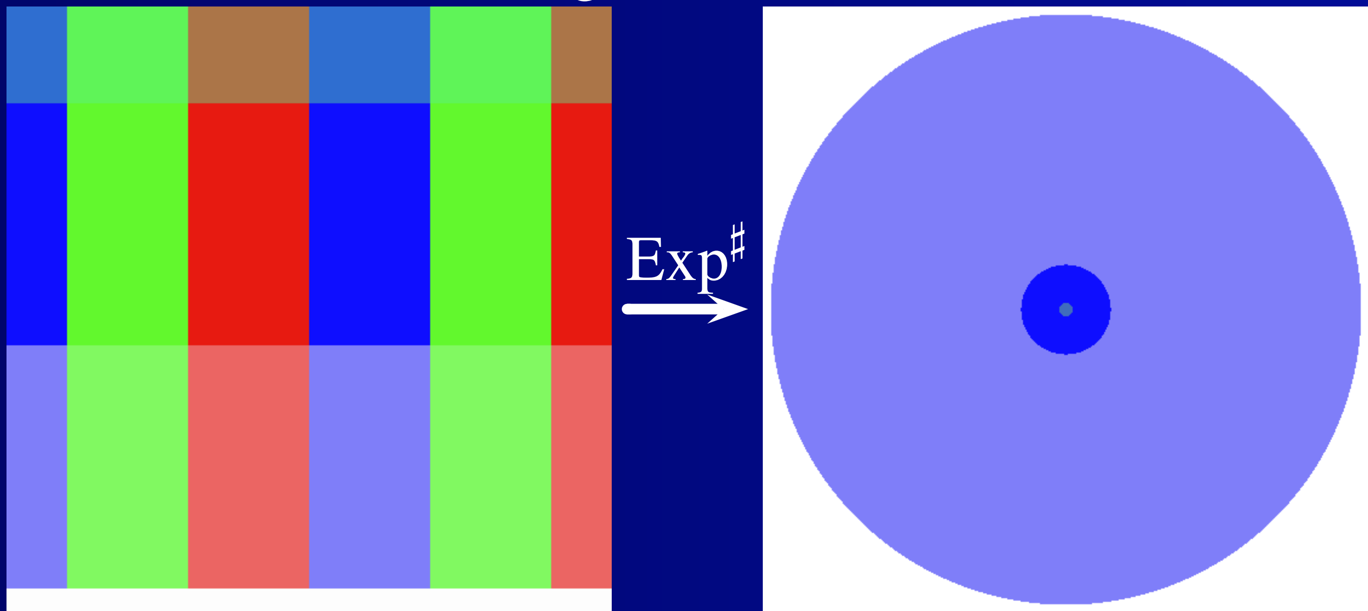
The covering property of $\mathcal{R}_0 f$

How to prove $\mathcal{R}_0 f \in \mathcal{F}_1'$ (replace V by V' in the definition)

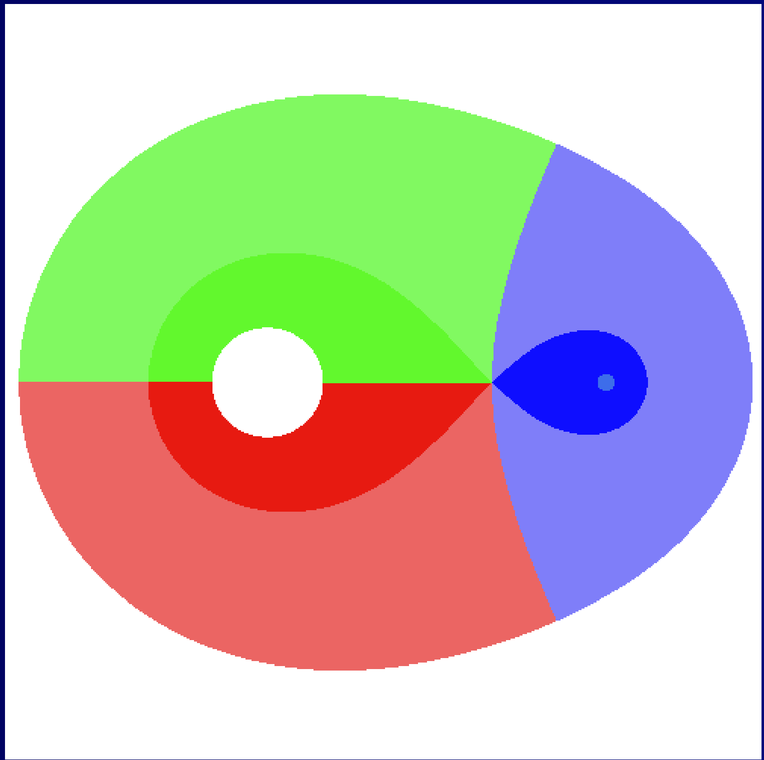
$$\mathcal{R}_0 f = \mathrm{Exp}^\sharp \circ E_f \circ (\mathrm{Exp}^\sharp)^{-1}$$

- Domain of E_f = repelling Fatou coordinate.
- Image of E_f = attracting Fatou coordinate.

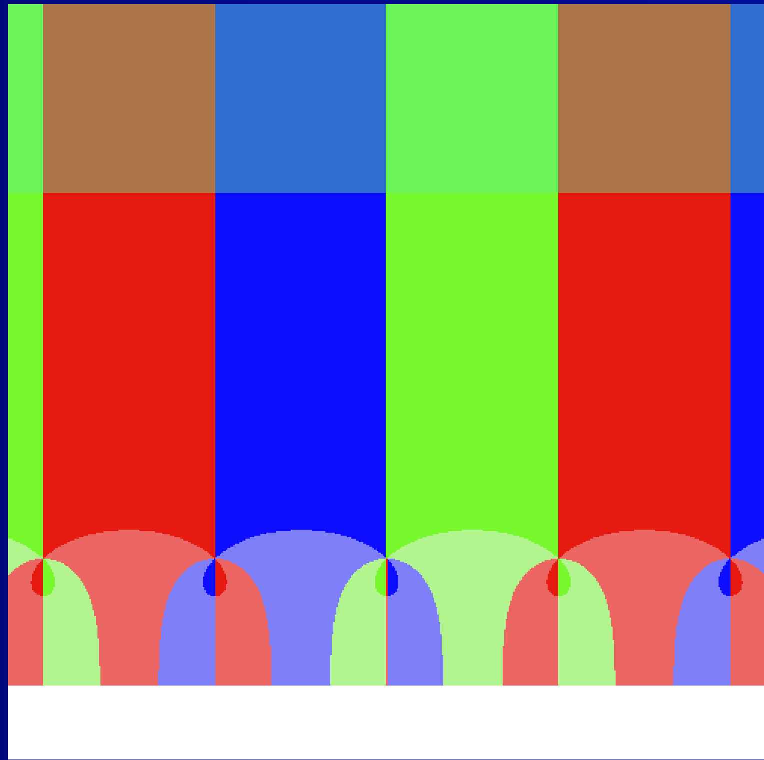
We make a color-tiling in attracting Fatou coordinate (=range) and pull it back by f to the domain of repelling Fatou coordinate. See it is the same as the tiling for P.



The covering property of P and log lift

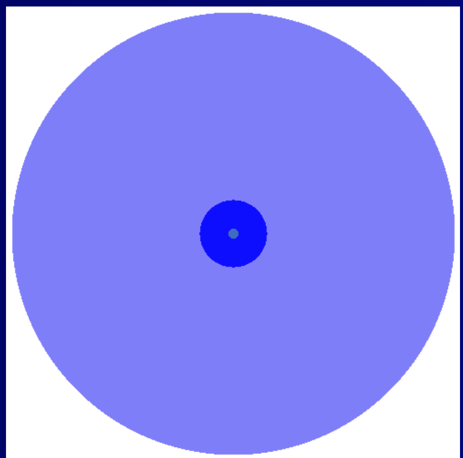


$\xleftarrow{\mathrm{Exp}^\sharp}$

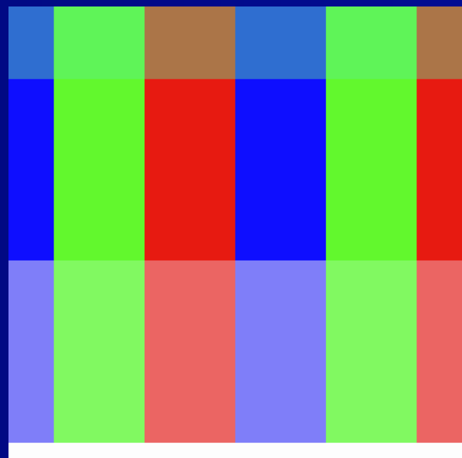


$\downarrow P$

$\downarrow$ The log lift of P



$\xleftarrow{\mathrm{Exp}^\sharp}$



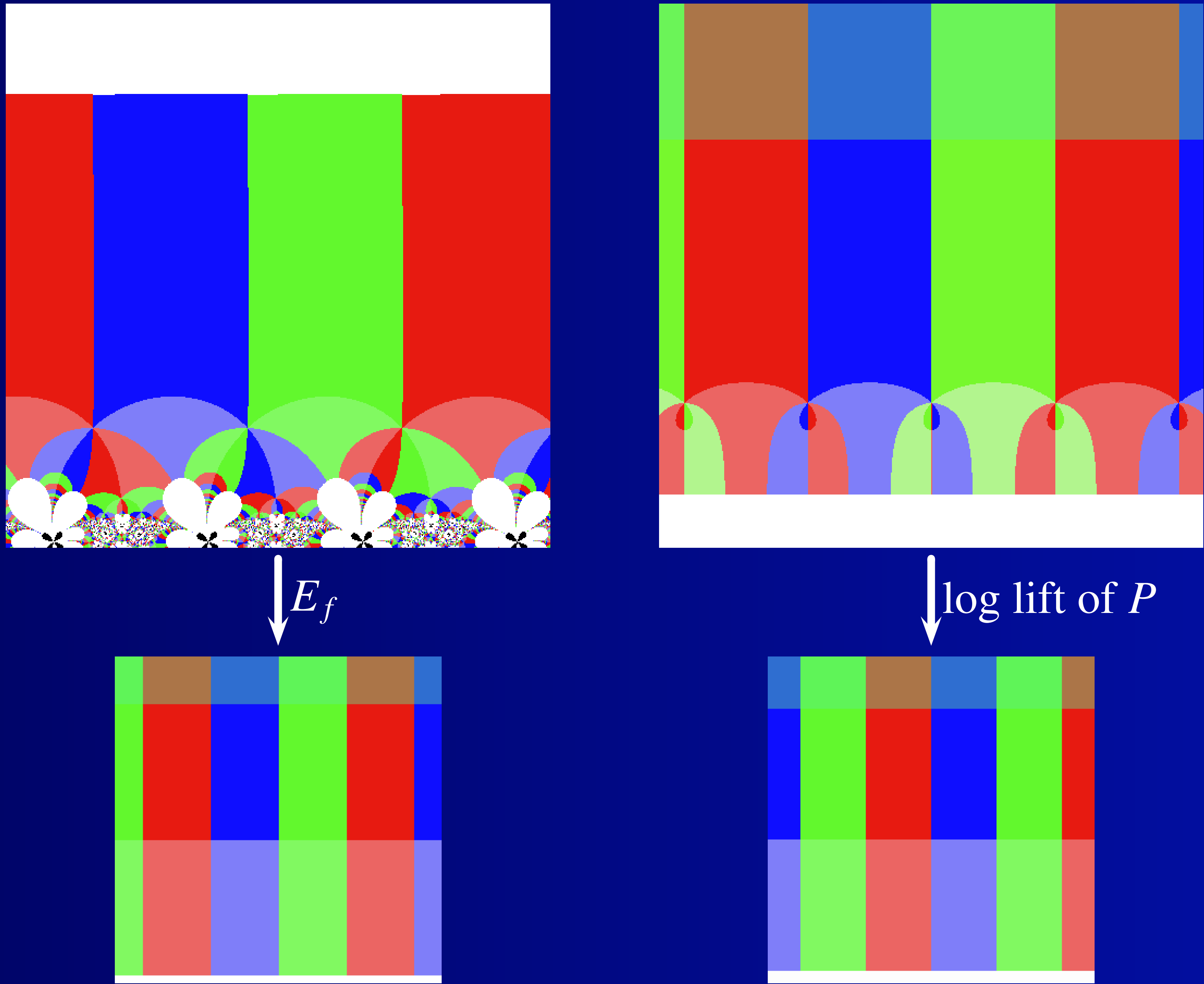
Remark.
$\eta = 0.3$
for P.
$\eta = 2$
for log lift.

$\mathcal{R}_0 f$ and the log lift of P

$f = Q \circ \varphi^{-1}$
$\in \mathcal{F}_1^Q$

repelling side
($\mathrm{Re}\, z \ll 0$)

attracting side
($\mathrm{Re}\, z \gg 0$)

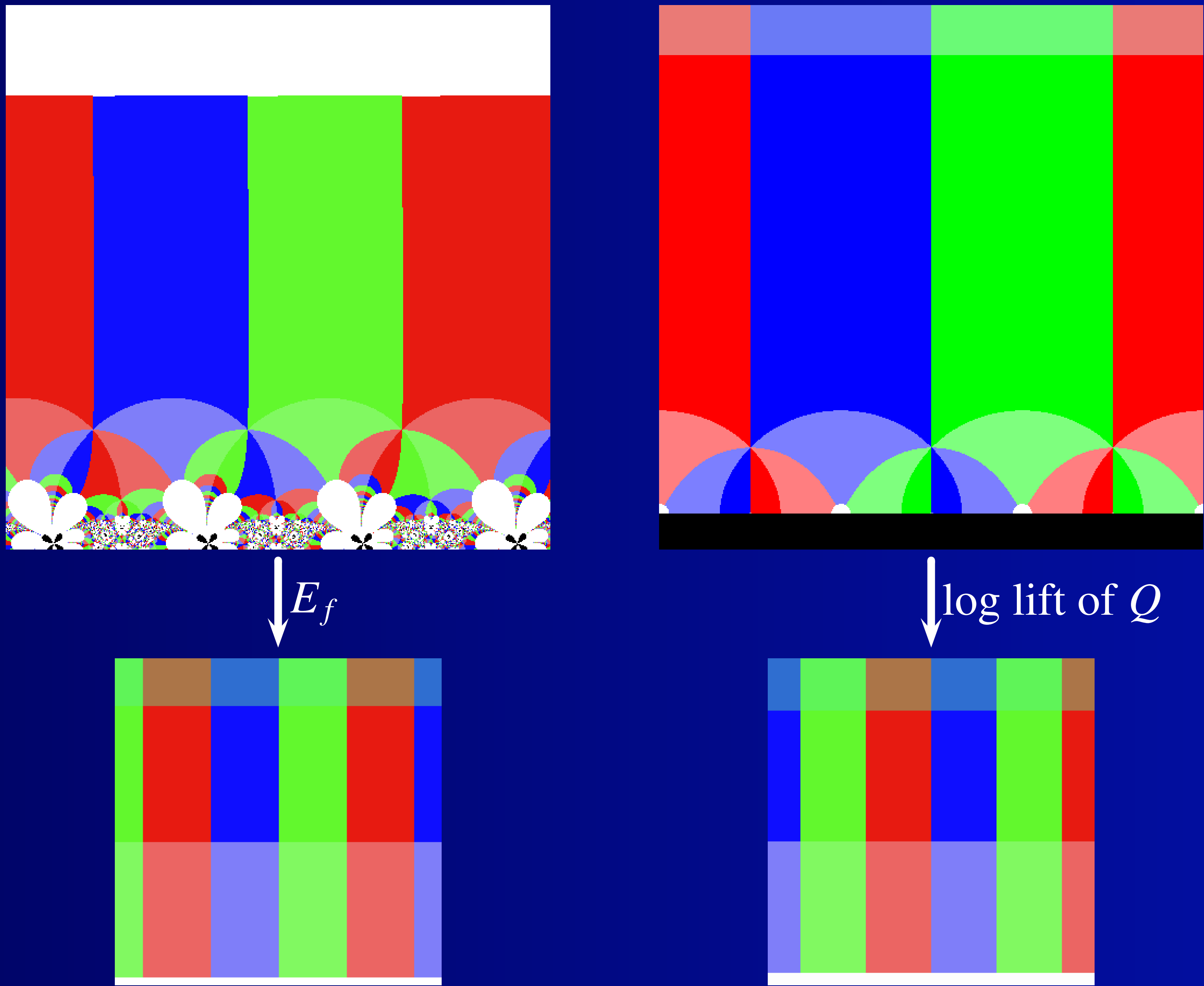


$\mathcal{R}_0 f$ and the log lift of Q

$f = Q \circ \varphi^{-1}$
$\in \mathcal{F}_1^Q$

repelling
side
($\mathrm{Re}\, z \ll 0$)

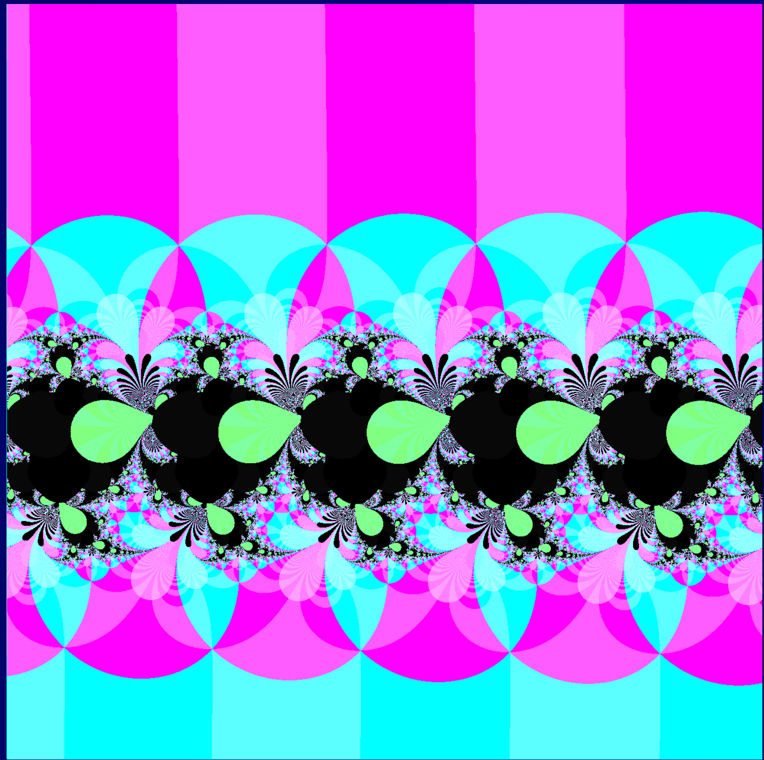
attracting
side
($\mathrm{Re}\, z \gg 0$)

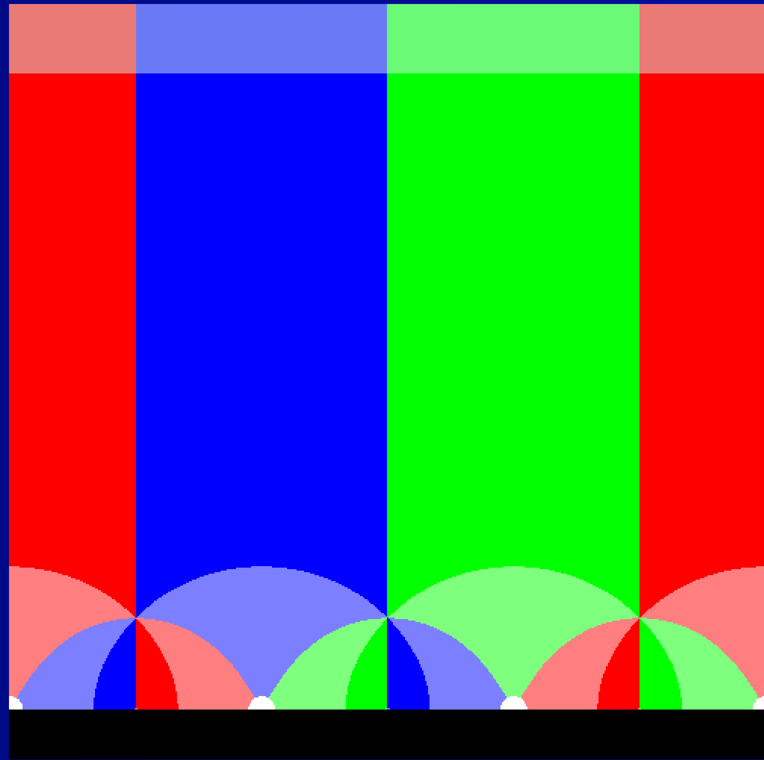


$\mathcal{R}_0^2 f$ and the log lift of Q

$f = Q \circ \varphi^{-1}$
$\in \mathcal{F}_1^Q$

repelling
side
($\operatorname{Re} z \ll 0$)

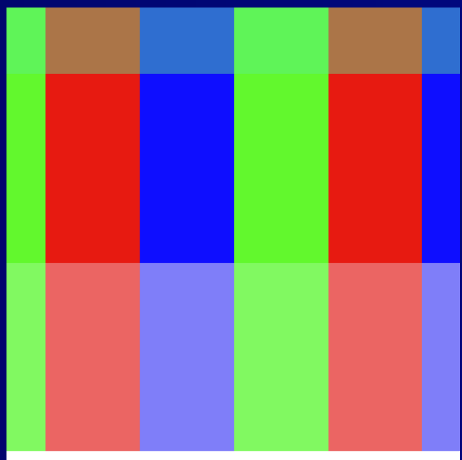


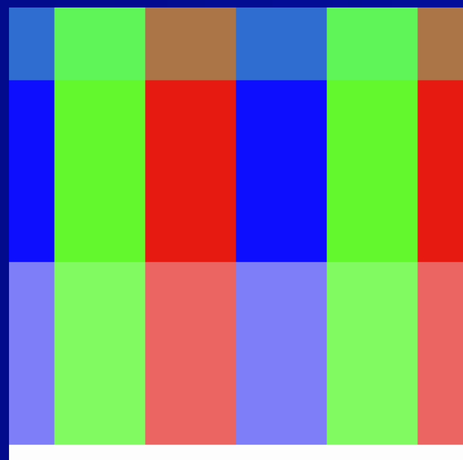


$\downarrow E_f$

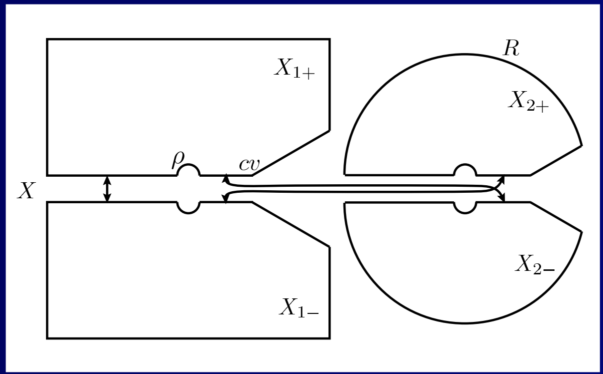
$\downarrow$ log lift of Q

attracting
side
($\operatorname{Re} z \gg 0$)





What we need to do



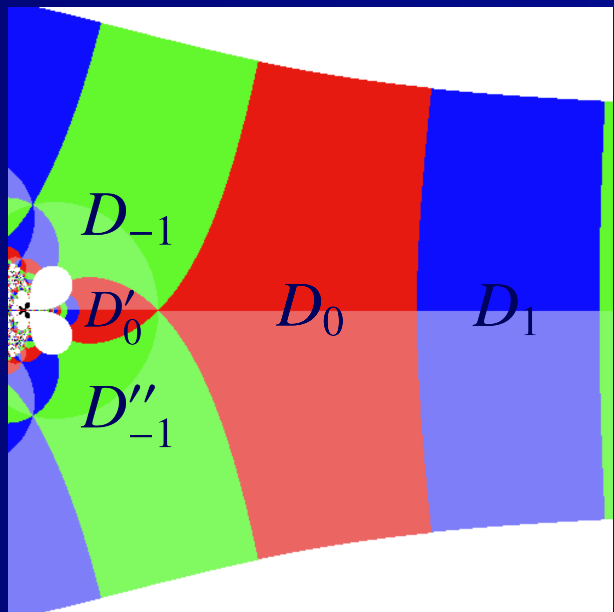
Left

Construct a Riemann surface X where an inverse branch of f can be lifted on X.
Show the repelling Fatou coord. is defined on X.

Show $D_0, D'_0, D_{-1}, D''_{-1}$ are "contained" in X.

Middle

Take inverse images of D_1 $(D_0, D'_0, D_{-1}, D''_{-1})$.
Estimate the position of them.



Right

Estimate the distortion of the attracting Fatou coordinate and the position of D_1.

To do this, determine the region where the attracting Fatou coord. is univalent and apply the Golusin ineq.

Need to check many inequalities (≈30) with help of computers. (Maple, MATLAB+INTLAB)