Non existence of convergent normal form for general germs of unipotent diffeomorphisms

Javier Ribón
jfribon@impa.br

January 2007, Local holomorphic dynamics

Introduction

A normal form is a simple model for a dynamical system

Introduction

A normal form is a simple model for a dynamical system obtained up to formal conjugation

Introduction

A normal form is a simple model for a dynamical system obtained up to formal conjugation

We say that a local diffeomorphism $\varphi \in \mathrm{Diff}\,(\mathbb{C}^n, 0)$ has a convergent normal form if $\varphi \overset{for}{\sim} \exp(X)$ for some $X \in \mathcal{X}(\mathbb{C}^n, 0)$

Normal forms for diffeomorphisms

$$\varphi(y) = \lambda y + \sum_{j=2}^{\infty} \lambda_j y^j \in \mathrm{Diff}\,(\mathbb{C}, 0)$$

Normal forms for diffeomorphisms

$$\varphi(y) = \lambda y + \sum_{j=2}^{\infty} \lambda_j y^j \in \mathrm{Diff}\,(\mathbb{C}, 0)$$

$\lambda \notin e^{2\pi i \mathbb{Q}}$ or φ periodic

Normal forms for diffeomorphisms

$$\varphi(y) = \lambda y + \sum_{j=2}^{\infty} \lambda_j y^j \in \mathrm{Diff}\,(\mathbb{C}, 0)$$

$\lambda \notin e^{2\pi i \mathbb{Q}}$ or φ periodic

$$\varphi \overset{for}{\sim} \lambda y = \exp\left((\ln \lambda) y \frac{\partial}{\partial y}\right)$$

Normal forms for diffeomorphisms

$$\varphi(y) = \lambda y + \sum_{j=2}^{\infty} \lambda_j y^j \in \mathrm{Diff}\,(\mathbb{C}, 0)$$

$\lambda \notin e^{2\pi i \mathbb{Q}}$ or φ periodic

$$\varphi \overset{for}{\sim} \lambda y = \exp\left((\ln \lambda) y \frac{\partial}{\partial y}\right)$$

The linearization problem is a normal form problem

Normal forms for diffeomorphisms

$$\varphi(y) = \lambda y + \sum_{j=2}^{\infty} \lambda_j y^j \in \mathrm{Diff}(\mathbb{C}, 0)$$

$\lambda \notin e^{2\pi i \mathbb{Q}}$ or φ periodic

$$\varphi \overset{for}{\sim} \lambda y = \exp\left((\ln \lambda) y \frac{\partial}{\partial y}\right)$$

The linearization problem is a normal form problem

Determining when the normalizing transformation is analytic

Normal forms for diffeomorphisms

$$\varphi(y) = \lambda y + \sum_{j=2}^{\infty} \lambda_j y^j \in \mathrm{Diff}\,(\mathbb{C}, 0)$$

$\lambda \notin e^{2\pi i \mathbb{Q}}$ or φ periodic

$$\varphi \overset{for}{\sim} \lambda y = \exp\left((\ln \lambda) y \frac{\partial}{\partial y}\right)$$

The linearization problem is a normal form problem
Determining when the normalizing transformation is analytic

$\lambda \in e^{2\pi i \mathbb{Q}}$ and φ non-periodic

Normal forms for diffeomorphisms

$$\varphi(y) = \lambda y + \sum_{j=2}^{\infty} \lambda_j y^j \in \mathrm{Diff}(\mathbb{C}, 0)$$

$\lambda \notin e^{2\pi i \mathbb{Q}}$ or φ periodic

$$\varphi \overset{for}{\sim} \lambda y = \exp\left((\ln \lambda) y \frac{\partial}{\partial y}\right)$$

The linearization problem is a normal form problem

Determining when the normalizing transformation is analytic

$\lambda \in e^{2\pi i \mathbb{Q}}$ and φ non-periodic

$\varphi \neq \exp\left(\hat{\alpha}(y) \frac{\partial}{\partial y}\right)$ for all $\hat{\alpha} \in \mathbb{C}[[y]]$ if $\lambda \neq 1$

Normal forms for diffeomorphisms

$$\varphi(y) = \lambda y + \sum_{j=2}^{\infty} \lambda_j y^j \in \mathrm{Diff}\,(\mathbb{C}, 0)$$

$\lambda \notin e^{2\pi i \mathbb{Q}}$ or φ periodic

$$\varphi \overset{for}{\sim} \lambda y = \exp\left((\ln \lambda) y \frac{\partial}{\partial y}\right)$$

The linearization problem is a normal form problem

Determining when the normalizing transformation is analytic

$\lambda \in e^{2\pi i \mathbb{Q}}$ and φ non-periodic

$$\varphi \neq \exp\left(\hat{\alpha}(y) \frac{\partial}{\partial y}\right) \text{ for all } \hat{\alpha} \in \mathbb{C}[[y]] \text{ if } \lambda \neq 1$$

$$\varphi^q = y + \sum_{j=p+1}^{\infty} \eta_j y^j = \exp\left(\left(\eta_{p+1} y^{p+1} + \sum_{j=p+2}^{\infty} \rho_j y^j\right) \frac{\partial}{\partial y}\right) \in \mathbb{C}[[y]] \frac{\partial}{\partial y}$$

Exponential operator

A formal vector field is a derivation of $\mathbb{C}[[y_1, \ldots, y_n]]$

$$\hat{X}(fg) = g\hat{X}(f) + f\hat{X}(g)$$

$$\begin{array}{rccc} \hat{X}: & \mathbb{C}[[y_1, \ldots, y_n]] & \to & \mathbb{C}[[y_1, \ldots, y_n]] \\ & g & \to & \hat{X}(g) \end{array}$$

$$\hat{X} = \hat{X}(y_1)\frac{\partial}{\partial y_1} + \ldots + \hat{X}(y_n)\frac{\partial}{\partial y_n}$$

Exponential operator

A formal vector field is a derivation of $\mathbb{C}[[y_1, \ldots, y_n]]$

$$\hat{X}(fg) = g\hat{X}(f) + f\hat{X}(g)$$

$$\begin{array}{rccc} \hat{X}: & \mathbb{C}[[y_1, \ldots, y_n]] & \to & \mathbb{C}[[y_1, \ldots, y_n]] \\ & g & \to & \hat{X}(g) \end{array}$$

$$\hat{X} = \hat{X}(y_1)\frac{\partial}{\partial y_1} + \ldots + \hat{X}(y_n)\frac{\partial}{\partial y_n}$$

A formal diffeo is an isomorphism of the $\mathbb{C}$-algebra $\mathbb{C}[[y_1, \ldots, y_n]]$

$$\varphi(fg) = \varphi(f)\varphi(g)$$

$$\begin{array}{rccc} \varphi: & \mathbb{C}[[y_1, \ldots, y_n]] & \to & \mathbb{C}[[y_1, \ldots, y_n]] \\ & g & \to & g \circ \varphi \end{array}$$

$$\varphi = (y_1 \circ \varphi, \ldots, y_n \circ \varphi)$$

Exponential operator

Taylor's formula

$$f(a+t) = f(a) + \sum_{j=1}^{\infty} \frac{\partial^j f}{\partial y^j}(a)\frac{t^j}{j!}$$

Exponential operator

Taylor's formula

$$f(a+t) = f(a) + \sum_{j=1}^{\infty} \frac{\partial^j f}{\partial y^j}(a)\frac{t^j}{j!}$$

$$f \circ \exp(t\frac{\partial}{\partial y}) = \left(\frac{\partial}{\partial y}\right)^0 (f) + \sum_{j=1}^{\infty} \left(\frac{\partial}{\partial y}\right)^j (f)\frac{t^j}{j!}$$

Exponential operator

Taylor's formula

$$f(a+t) = f(a) + \sum_{j=1}^{\infty} \frac{\partial^j f}{\partial y^j}(a)\frac{t^j}{j!}$$

$$f \circ \exp(t\frac{\partial}{\partial y}) = \left(\frac{\partial}{\partial y}\right)^0 (f) + \sum_{j=1}^{\infty} \left(\frac{\partial}{\partial y}\right)^j (f)\frac{t^j}{j!}$$

$$f \circ \exp(tX) = \sum_{j=0}^{\infty} X^j(f)\frac{t^j}{j!}$$

$$\exp(X) = \left(y_1 + \sum_{j=1}^{\infty} \frac{X^j(y_1)}{j!}, \ldots, y_n + \sum_{j=1}^{\infty} \frac{X^j(y_n)}{j!}\right)$$

Exponential operator

Taylor's formula

$$f(a+t) = f(a) + \sum_{j=1}^{\infty} \frac{\partial^j f}{\partial y^j}(a)\frac{t^j}{j!}$$

$$f \circ \exp(t\frac{\partial}{\partial y}) = \left(\frac{\partial}{\partial y}\right)^0 (f) + \sum_{j=1}^{\infty} \left(\frac{\partial}{\partial y}\right)^j (f)\frac{t^j}{j!}$$

$$f \circ \exp(tX) = \sum_{j=0}^{\infty} X^j(f)\frac{t^j}{j!}$$

$$\exp(X) = \left(y_1 + \sum_{j=1}^{\infty} \frac{X^j(y_1)}{j!}, \ldots, y_n + \sum_{j=1}^{\infty} \frac{X^j(y_n)}{j!}\right)$$

Logarithm

$$\Theta := \varphi - Id \Longrightarrow \Theta(g) = g \circ \varphi - g$$

$$(\log \varphi)(g) = \log(Id + \Theta)(g) = \sum_{j=1}^{\infty}(-1)^{j+1}\frac{\Theta^j(g)}{j}$$

Exponential operator

Taylor's formula

$$f \circ \exp(tX) = \sum_{j=0}^{\infty} X^j(f)\frac{t^j}{j!}$$
$$\exp(X) = \left(y_1 + \sum_{j=1}^{\infty} \frac{X^j(y_1)}{j!}, \ldots, y_n + \sum_{j=1}^{\infty} \frac{X^j(y_n)}{j!}\right)$$

Logarithm

$$\Theta := \varphi - Id \Longrightarrow \Theta(g) = g \circ \varphi - g$$
$$(\log \varphi)(g) = \log(Id + \Theta)(g) = \sum_{j=1}^{\infty}(-1)^{j+1}\frac{\Theta^j(g)}{j}$$

$$\begin{array}{rccc} \exp: & \hat{\mathcal{X}}_2(\mathbb{C}^n, 0) & \rightarrow & \text{Diff}_1(\mathbb{C}^n, 0) \\ \log: & \text{Diff}_1(\mathbb{C}^n, 0) & \rightarrow & \hat{\mathcal{X}}_2(\mathbb{C}^n, 0) \end{array}$$

$$\exp \circ \log = Id = \log \circ \exp$$

Quasi-analytic normalizing mappings

$\varphi = y + \eta y^{p+1} + O(y^{p+2})$

$$\varphi = \exp\left(\hat{\alpha}(y)\frac{\partial}{\partial y}\right) \to \frac{dy}{\hat{\alpha}(y)} = d\left(\frac{-1}{p\eta y^{p}} + h.o.t.\right) + \mu\frac{dy}{y}$$

Quasi-analytic normalizing mappings

$\varphi = y + \eta y^{p+1} + O(y^{p+2})$

$$\varphi = \exp\left(\hat{\alpha}(y)\frac{\partial}{\partial y}\right) \to \frac{dy}{\hat{\alpha}(y)} = d\left(\frac{-1}{p\eta y^p} + h.o.t.\right) + \mu\frac{dy}{y}$$

$$\varphi \overset{for}{\sim} \exp(X) \quad \text{with} \quad X = \frac{y^{p+1}}{1+\mu y^p}\frac{\partial}{\partial y}$$

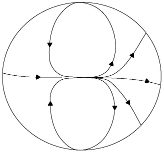
Quasi-analytic normalizing mappings

$\varphi = y + \eta y^{p+1} + O(y^{p+2})$

$$\varphi = \exp\left(\hat{\alpha}(y)\frac{\partial}{\partial y}\right) \to \frac{dy}{\hat{\alpha}(y)} = d\left(\frac{-1}{p\eta y^p} + h.o.t.\right) + \mu\frac{dy}{y}$$

$$\varphi \overset{for}{\sim} \exp(X) \quad \text{with} \quad X = \frac{y^{p+1}}{1+\mu y^p}\frac{\partial}{\partial y}$$

$\varphi = \exp(X) + O(y^{2p+2})$



$$\exists! \hat{\sigma}_\varphi \in \widehat{\mathrm{Diff}}(\mathbb{C}, 0) : \hat{\sigma}_\varphi \circ \varphi = \exp(X) \circ \hat{\sigma}_\varphi \text{ and } \hat{\sigma}_\varphi = y + O(y^{p+2})$$

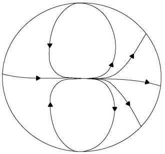
Quasi-analytic normalizing mappings

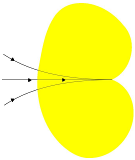
$\varphi = y + \eta y^{p+1} + O(y^{p+2})$

$$\varphi = \exp\left(\hat{\alpha}(y)\frac{\partial}{\partial y}\right) \to \frac{dy}{\hat{\alpha}(y)} = d\left(\frac{-1}{p\eta y^p} + h.o.t.\right) + \mu\frac{dy}{y}$$

$$\varphi \overset{for}{\sim} \exp(X) \quad \text{with} \quad X = \frac{y^{p+1}}{1+\mu y^p}\frac{\partial}{\partial y}$$

$\varphi = \exp(X) + O(y^{2p+2})$





$$\exists! \hat{\sigma}_\varphi \in \widehat{\mathrm{Diff}}(\mathbb{C}, 0) : \hat{\sigma}_\varphi \circ \varphi = \exp(X) \circ \hat{\sigma}_\varphi \text{ and } \hat{\sigma}_\varphi = y + O(y^{p+2})$$

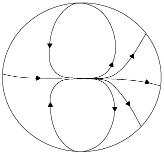
Quasi-analytic normalizing mappings

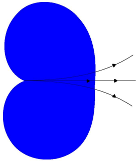
$\varphi = y + \eta y^{p+1} + O(y^{p+2})$

$$\varphi = \exp\left(\hat{\alpha}(y)\frac{\partial}{\partial y}\right) \to \frac{dy}{\hat{\alpha}(y)} = d\left(\frac{-1}{p\eta y^p} + h.o.t.\right) + \mu\frac{dy}{y}$$

$$\varphi \overset{for}{\sim} \exp(X) \quad \text{with} \quad X = \frac{y^{p+1}}{1+\mu y^p}\frac{\partial}{\partial y}$$

$\varphi = \exp(X) + O(y^{2p+2})$





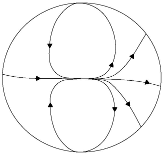
$$\exists! \hat{\sigma}_\varphi \in \widehat{\mathrm{Diff}}(\mathbb{C}, 0) : \hat{\sigma}_\varphi \circ \varphi = \exp(X) \circ \hat{\sigma}_\varphi \text{ and } \hat{\sigma}_\varphi = y + O(y^{p+2})$$

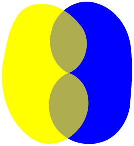
Quasi-analytic normalizing mappings

$\varphi = y + \eta y^{p+1} + O(y^{p+2})$

$$\varphi \overset{for}{\sim} \exp(X) \quad \text{with} \quad X = \frac{y^{p+1}}{1+\mu y^p}\frac{\partial}{\partial y}$$

$\varphi = \exp(X) + O(y^{2p+2})$





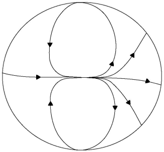
$$\exists ! \hat{\sigma}_{\varphi} \in \widehat{\text{Diff}}(\mathbb{C},0) : \hat{\sigma}_{\varphi} \circ \varphi = \exp(X) \circ \hat{\sigma}_{\varphi} \text{ and } \hat{\sigma}_{\varphi} = y + O(y^{p+2})$$

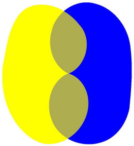
Quasi-analytic normalizing mappings

$\varphi = y + \eta y^{p+1} + O(y^{p+2})$

$$\varphi \overset{for}{\sim} \exp(X) \quad \text{with} \quad X = \frac{y^{p+1}}{1+\mu y^p} \frac{\partial}{\partial y}$$

$\varphi = \exp(X) + O(y^{2p+2})$





$\exists ! \hat{\sigma}_\varphi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}, 0) : \hat{\sigma}_\varphi \circ \varphi = \exp(X) \circ \hat{\sigma}_\varphi$ and $\hat{\sigma}_\varphi = y + O(y^{p+2})$

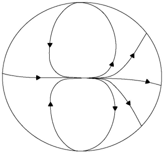
$\exists ! \sigma_{\varphi,V}$ s.t. $\sigma_{\varphi,V} \circ \varphi = \exp(X) \circ \sigma_{\varphi,V}$ and $\hat{\sigma}_\varphi$ is an asympt. devel. of $\sigma_{\varphi,V}$

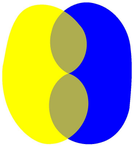
Quasi-analytic normalizing mappings

$\varphi = y + \eta y^{p+1} + O(y^{p+2})$

$$\varphi \overset{for}{\sim} \exp(X) \quad \text{with} \quad X = \frac{y^{p+1}}{1+\mu y^p}\frac{\partial}{\partial y}$$

$\varphi = \exp(X) + O(y^{2p+2})$





$$\exists! \hat{\sigma}_\varphi \in \widehat{\mathrm{Diff}}\,(\mathbb{C},0) : \hat{\sigma}_\varphi \circ \varphi = \exp(X) \circ \hat{\sigma}_\varphi \text{ and } \hat{\sigma}_\varphi = y + O(y^{p+2})$$

$\exists! \sigma_{\varphi,V}$ s.t. $\sigma_{\varphi,V} \circ \varphi = \exp(X) \circ \sigma_{\varphi,V}$ and $\hat{\sigma}_\varphi$ is an asympt. devel. of $\sigma_{\varphi,V}$

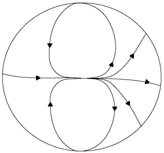
$$\sigma_{\varphi,V} = \lim \exp(X)^{-n} \circ \varphi^n \quad \rightarrow \quad \hat{\sigma}_\varphi \text{ is } p\text{-summable}$$

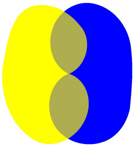
Quasi-analytic normalizing mappings

$\varphi = y + \eta y^{p+1} + O(y^{p+2})$

$$\varphi \overset{for}{\sim} \exp(X) \quad \text{with} \quad X = \frac{y^{p+1}}{1+\mu y^p}\frac{\partial}{\partial y}$$

$\varphi = \exp(X) + O(y^{2p+2})$





$$\exists ! \hat{\sigma}_\varphi \in \widehat{\mathrm{Diff}}\,(\mathbb{C},0) : \hat{\sigma}_\varphi \circ \varphi = \exp(X) \circ \hat{\sigma}_\varphi \text{ and } \hat{\sigma}_\varphi = y + O(y^{p+2})$$

$\exists ! \sigma_{\varphi,V}$ s.t. $\sigma_{\varphi,V} \circ \varphi = \exp(X) \circ \sigma_{\varphi,V}$ and $\hat{\sigma}_\varphi$ is an asympt. devel. of $\sigma_{\varphi,V}$

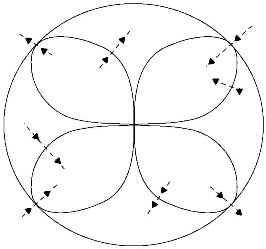
$\sigma_{\varphi,V} = \lim \exp(X)^{-n} \circ \varphi^n \quad \rightarrow \quad$ $\hat{\sigma}_\varphi$ is p-summable

$$\hat{\sigma}_\eta^{-1} \circ \hat{\sigma}_\varphi \in \mathrm{Diff}\,(\mathbb{C},0) \Leftrightarrow \sigma_{\varphi,V(+)} \circ \sigma_{\varphi,V(-)}^{-1} \equiv \sigma_{\eta,V(+)} \circ \sigma_{\eta,V(-)}^{-1}$$

Sub-stability

Unfoldings $\rightarrow \varphi(x, y) = (x, y + iy(y - x)(y + x))$

Consider the flow $Re(iX)$ for $X = iy(y - x)(y + x)\partial/\partial y$

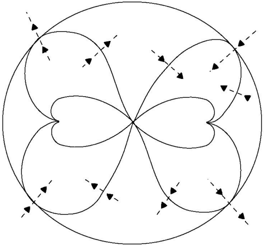


$x = 0$

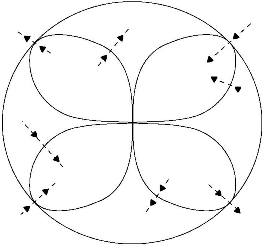
Sub-stability

Unfoldings $\rightarrow \varphi(x, y) = (x, y + iy(y - x)(y + x))$

Consider the flow $Re(iX)$ for $X = iy(y - x)(y + x)\partial/\partial y$



$x_0 \in \mathbb{R}^+$

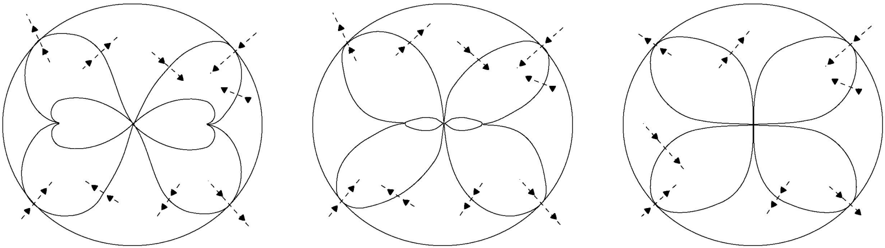


$x = 0$

Sub-stability

Unfoldings $\rightarrow \varphi(x, y) = (x, y + iy(y-x)(y+x))$

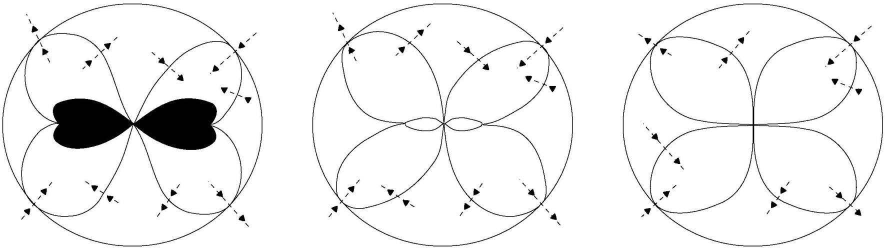
Consider the flow $Re(iX)$ for $X = iy(y-x)(y+x)\partial/\partial y$



$x_0 \in \mathbb{R}^+$ $\quad$ $0 < x_1 << x_0$ $\quad$ $x = 0$

Unfoldings $\rightarrow \varphi(x, y) = (x, y + iy(y-x)(y+x))$

Consider the flow $Re(iX)$ for $X = iy(y-x)(y+x)\partial/\partial y$

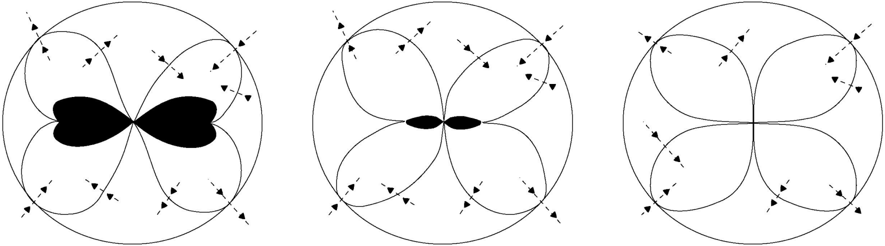


$x_0 \in \mathbb{R}^+$ $\quad$ $0 < x_1 << x_0$ $\quad$ $x = 0$

Sub-stability

Unfoldings $\to \varphi(x, y) = (x, y + iy(y - x)(y + x))$

Consider the flow $Re(iX)$ for $X = iy(y - x)(y + x)\partial/\partial y$

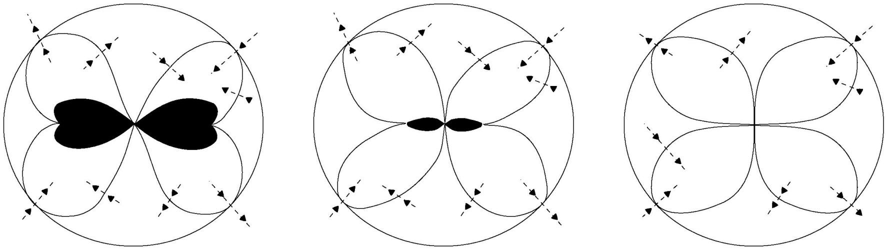


$x_0 \in \mathbb{R}^+$ $\quad$ $0 < x_1 << x_0$ $\quad$ $x = 0$

Sub-stability

Unfoldings $\rightarrow \varphi(x,y) = (x, y + iy(y-x)(y+x))$

Consider the flow $Re(iX)$ for $X = iy(y-x)(y+x)\partial/\partial y$



$x_0 \in \mathbb{R}^+$ $\quad 0 < x_1 << x_0 \quad$ $x = 0$

Studying the dynamics of $Re(\mu X)$ for $\mu \in \mathbb{S}^1 \setminus \{-1, 1\}$ is useful to determine the behavior of the extension of Fatou coordinates in the neighborhood of $Fix\varphi$ and $x = 0$

Normal forms for diffeomorphisms ($\varphi \overset{for}{\sim} \exp(X)$)

$\varphi = \lambda y + O(y^2) \in \mathrm{Diff}\,(\mathbb{C}, 0)$ with $\lambda \notin e^{2\pi i \mathbb{Q}}$

Normal forms for diffeomorphisms ($\varphi \overset{for}{\sim} \exp(X)$)

$\varphi = \lambda y + O(y^2) \in \mathrm{Diff}\,(\mathbb{C}, 0)$ with $\lambda \notin e^{2\pi i \mathbb{Q}}$

$$\varphi \overset{top}{\sim} \lambda z \Leftrightarrow \varphi \overset{an}{\sim} \lambda z$$

Normal forms for diffeomorphisms ($\varphi \overset{for}{\sim} \exp(X)$)

$\varphi = \lambda y + O(y^2) \in \text{Diff}(\mathbb{C}, 0)$ with $\lambda \notin e^{2\pi i \mathbb{Q}}$

$$\varphi \overset{top}{\sim} \lambda z \Leftrightarrow \varphi \overset{an}{\sim} \lambda z$$

Determining whether φ is topologically conjugated to its normal form

Normal forms for diffeomorphisms ($\varphi \overset{for}{\sim} \exp(X)$)

$\varphi = \lambda y + O(y^2) \in \mathrm{Diff}(\mathbb{C}, 0)$ with $\lambda \notin e^{2\pi i \mathbb{Q}}$

$$\varphi \overset{top}{\sim} \lambda z \Leftrightarrow \varphi \overset{an}{\sim} \lambda z$$

Determining whether φ is topologically conjugated to its normal form

Generically $\varphi \overset{top}{\sim} \lambda z$ for $\lambda \in \mathbb{C}^*$ and $\lambda \in \mathbb{S}^1$

Normal forms for diffeomorphisms ($\varphi \overset{for}{\sim} \exp(X)$)

$\varphi = \lambda y + O(y^2) \in \mathrm{Diff}\,(\mathbb{C}, 0)$ with $\lambda \notin e^{2\pi i \mathbb{Q}}$

$$\varphi \overset{top}{\sim} \lambda z \Leftrightarrow \varphi \overset{an}{\sim} \lambda z$$

Determining whether φ is topologically conjugated to its normal form

Generically $\varphi \overset{top}{\sim} \lambda z$ for $\lambda \in \mathbb{C}^*$ and $\lambda \in \mathbb{S}^1$

$\varphi = y + O(y^2) \in \mathrm{Diff}\,(\mathbb{C}, 0)$

Normal forms for diffeomorphisms ($\varphi \overset{for}{\sim} \exp(X)$)

$\varphi = \lambda y + O(y^2) \in \mathrm{Diff}\,(\mathbb{C}, 0)$ with $\lambda \notin e^{2\pi i \mathbb{Q}}$

$$\varphi \overset{top}{\sim} \lambda z \Leftrightarrow \varphi \overset{an}{\sim} \lambda z$$

Determining whether φ is topologically conjugated to its normal form

Generically $\varphi \overset{top}{\sim} \lambda z$ for $\lambda \in \mathbb{C}^*$ and $\lambda \in \mathbb{S}^1$

$\varphi = y + O(y^2) \in \mathrm{Diff}\,(\mathbb{C}, 0)$

$$\varphi \overset{for}{\sim} \exp(X) \implies \varphi \overset{top}{\sim} \exp(X)$$

Normal forms for diffeomorphisms ($\varphi \overset{for}{\sim} \exp(X)$)

$\varphi = \lambda y + O(y^2) \in \mathrm{Diff}\,(\mathbb{C}, 0)$ with $\lambda \notin e^{2\pi i \mathbb{Q}}$

$$\varphi \overset{top}{\sim} \lambda z \Leftrightarrow \varphi \overset{an}{\sim} \lambda z$$

Determining whether φ is topologically conjugated to its normal form

Generically $\varphi \overset{top}{\sim} \lambda z$ for $\lambda \in \mathbb{C}^*$ and $\lambda \in \mathbb{S}^1$

$\varphi = y + O(y^2) \in \mathrm{Diff}\,(\mathbb{C}, 0)$

$$\varphi \overset{for}{\sim} \exp(X) \implies \varphi \overset{top}{\sim} \exp(X)$$

Quasi-analyticity of normalizing mapping → Ecalle-Voronin invariants

Normal forms for diffeomorphisms ($\varphi \overset{for}{\sim} \exp(X)$)

$\varphi = \lambda y + O(y^2) \in \mathrm{Diff}\,(\mathbb{C}, 0)$ with $\lambda \notin e^{2\pi i \mathbb{Q}}$

$$\varphi \overset{top}{\sim} \lambda z \Leftrightarrow \varphi \overset{an}{\sim} \lambda z$$

Determining whether φ is topologically conjugated to its normal form

Generically $\varphi \overset{top}{\not\sim} \lambda z$ for $\lambda \in \mathbb{C}^*$ and $\lambda \in \mathbb{S}^1$

$\varphi = y + O(y^2) \in \mathrm{Diff}\,(\mathbb{C}, 0)$

$$\varphi \overset{for}{\sim} \exp(X) \implies \varphi \overset{top}{\sim} \exp(X)$$

Quasi-analyticity of normalizing mapping $\rightarrow$ Ecalle-Voronin invariants

$\varphi(x, y) = (x, f(x, y)) \in \mathrm{Diff}\,(\mathbb{C}^2, 0)$ with $f(0, y) \in \mathrm{Diff}_1(\mathbb{C}, 0)$

Normal forms for diffeomorphisms ($\varphi \overset{for}{\sim} \exp(X)$)

$\varphi = \lambda y + O(y^2) \in \mathrm{Diff}\,(\mathbb{C}, 0)$ with $\lambda \notin e^{2\pi i \mathbb{Q}}$

$$\varphi \overset{top}{\sim} \lambda z \Leftrightarrow \varphi \overset{an}{\sim} \lambda z$$

Determining whether φ is topologically conjugated to its normal form

Generically $\varphi \overset{top}{\nsim} \lambda z$ for $\lambda \in \mathbb{C}^*$ and $\lambda \in \mathbb{S}^1$

$\varphi = y + O(y^2) \in \mathrm{Diff}\,(\mathbb{C}, 0)$

$$\varphi \overset{for}{\sim} \exp(X) \implies \varphi \overset{top}{\sim} \exp(X)$$

Quasi-analyticity of normalizing mapping $\rightarrow$ Ecalle-Voronin invariants

$\varphi(x, y) = (x, f(x, y)) \in \mathrm{Diff}\,(\mathbb{C}^2, 0)$ with $f(0, y) \in \mathrm{Diff}_1(\mathbb{C}, 0)$

Generically $\varphi \overset{an}{\nsim} \exp(X)$ and $\varphi \overset{top}{\nsim} \exp(X)$

Normal forms for diffeomorphisms ($\varphi \overset{for}{\sim} \exp(X)$)

$\varphi = \lambda y + O(y^2) \in \mathrm{Diff}\,(\mathbb{C}, 0)$ with $\lambda \notin e^{2\pi i \mathbb{Q}}$

$$\varphi \overset{top}{\sim} \lambda z \Leftrightarrow \varphi \overset{an}{\sim} \lambda z$$

Determining whether φ is topologically conjugated to its normal form

Generically $\varphi \overset{top}{\sim} \lambda z$ for $\lambda \in \mathbb{C}^*$ and $\lambda \in \mathbb{S}^1$

$\varphi = y + O(y^2) \in \mathrm{Diff}\,(\mathbb{C}, 0)$

$$\varphi \overset{for}{\sim} \exp(X) \implies \varphi \overset{top}{\sim} \exp(X)$$

Quasi-analyticity of normalizing mapping $\rightarrow$ Ecalle-Voronin invariants

$\varphi(x, y) = (x, f(x, y)) \in \mathrm{Diff}\,(\mathbb{C}^2, 0)$ with $f(0, y) \in \mathrm{Diff}_1(\mathbb{C}, 0)$

Generically $\varphi \overset{an}{\not\sim} \exp(X)$ and $\varphi \overset{top}{\not\sim} \exp(X)$

Determining stable structures of $\varphi \rightarrow$ System of analytic invariants

Setup

Consider $\varphi(x, y) = (x + y(y - x)\Delta(x, y), y + y(y - x)) \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$
Suppose that $\Delta(0, 0) = 0$

Setup

Consider $\varphi(x,y) = (x + y(y-x)\Delta(x,y), y + y(y-x)) \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$
Suppose that $\Delta(0,0) = 0$

$$\exists! \hat{X} \in \hat{\mathcal{X}}_2(\mathbb{C}^2, 0) \text{ with } \varphi = \exp(\hat{X})$$

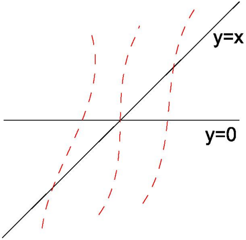
$$Fix\varphi = Sing\hat{X} \to \hat{X} = y(y-x)\left(\frac{\partial}{\partial y} + h.o.t.\right)$$

Setup

Consider $\varphi(x,y) = (x + y(y-x)\Delta(x,y), y + y(y-x)) \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$
Suppose that $\Delta(0,0) = 0$

$$\exists! \hat{X} \in \hat{\mathcal{X}}_2(\mathbb{C}^2,0) \text{ with } \varphi = \exp(\hat{X})$$

$$Fix\varphi = Sing\hat{X} \to \hat{X} = y(y-x)\left(\frac{\partial}{\partial y} + h.o.t.\right)$$

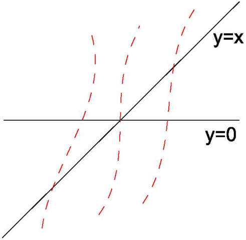


Setup

Consider $\varphi(x,y) = (x + y(y-x)\Delta(x,y), y + y(y-x)) \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$
Suppose that $\Delta(0,0) = 0$

$$\exists! \hat{X} \in \hat{\mathcal{X}}_2(\mathbb{C}^2, 0) \text{ with } \varphi = \exp(\hat{X})$$

$$Fix\varphi = Sing\hat{X} \to \hat{X} = y(y-x)\left(\frac{\partial}{\partial y} + h.o.t.\right)$$



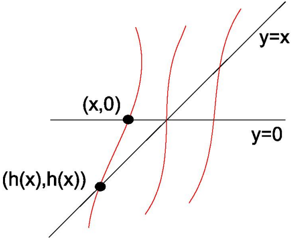
$\exists! f \in \mathbb{C}[[x,y]]$ s. t. $f(x,0) \equiv x$ and $\hat{X}(f) \equiv 0 \implies f(x,x) = x + O(x^2)$

Transport mapping

Consider $\varphi(x, y) = (x + y(y - x)\Delta(x, y), y + y(y - x)) = \exp(\hat{X})$

Transport mapping

Consider $\varphi(x,y) = (x + y(y-x)\Delta(x,y), y + y(y-x)) = \exp(\hat{X})$



Suppose $\hat{X} \in \mathcal{X}(\mathbb{C}^2, 0)$

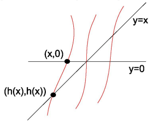
The points $(x,0)$ and $(h(x),h(x))$ are in the same trajectory of $\hat{X}$

$$\Updownarrow$$

$$f(x,0) \equiv f(x,x) \circ h(x)$$

Transport mapping

Consider $\varphi(x, y) = (x + y(y - x)\Delta(x, y), y + y(y - x)) = \exp(\hat{X})$



Suppose $\hat{X} \in \mathcal{X}(\mathbb{C}^2, 0)$

The points $(x, 0)$ and $(h(x), h(x))$ are in the same trajectory of $\hat{X}$

$$\Updownarrow$$

$$f(x, 0) \equiv f(x, x) \circ h(x)$$

Transport mapping

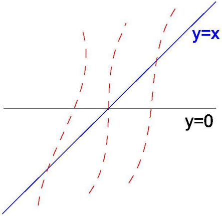
$$\begin{array}{llll} Tr: & y = 0 & \rightarrow & y = x \\ & (x, 0) & \rightarrow & ((f(x,x))^{-1} \circ f(x,0), (f(x,x))^{-1} \circ f(x,0)) \end{array}$$

Transport mapping up to formal conjugacy

Consider $\varphi = (x + y(y-x)\Delta(x,y), y + y(y-x)) = \exp(\log\varphi)$
Suppose $\sigma \circ \varphi = \eta \circ \sigma$ for some $\eta \in \mathrm{Diff}(\mathbb{C}^2, 0)$ and $\sigma \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$

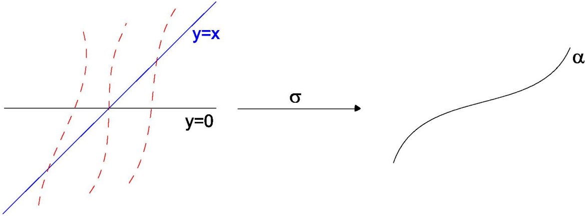
Transport mapping up to formal conjugacy

Consider $\varphi = (x + y(y - x)\Delta(x, y), y + y(y - x)) = \exp(\log \varphi)$
Suppose $\sigma \circ \varphi = \eta \circ \sigma$ for some $\eta \in \mathrm{Diff}\,(\mathbb{C}^2, 0)$ and $\sigma \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^2, 0)$



Transport mapping up to formal conjugacy

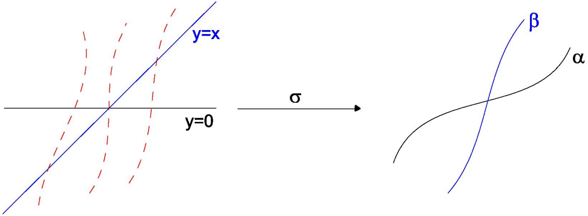
Consider $\varphi = (x + y(y-x)\Delta(x,y), y + y(y-x)) = \exp(\log \varphi)$
Suppose $\sigma \circ \varphi = \eta \circ \sigma$ for some $\eta \in \text{Diff}(\mathbb{C}^2, 0)$ and $\sigma \in \widehat{\text{Diff}}(\mathbb{C}^2, 0)$



$Fix\varphi \overset{for}{\sim} Fix\eta$ $\qquad$ $\log\varphi \overset{for}{\sim} \log\eta$ $\qquad$ $\sigma_{|y=x} \circ Tr_{\varphi} = Tr_{\eta} \circ \sigma_{|y=0}$

Transport mapping up to formal conjugacy

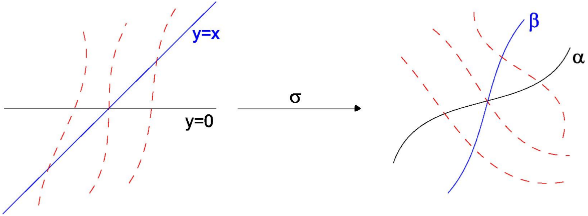
Consider $\varphi = (x + y(y - x)\Delta(x, y), y + y(y - x)) = \exp(\log \varphi)$
Suppose $\sigma \circ \varphi = \eta \circ \sigma$ for some $\eta \in \mathrm{Diff}(\mathbb{C}^2, 0)$ and $\sigma \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$



$Fix\varphi \overset{for}{\sim} Fix\eta$ $\qquad$ $\log \varphi \overset{for}{\sim} \log \eta$ $\qquad$ $\sigma_{|y=x} \circ Tr_{\varphi} = Tr_{\eta} \circ \sigma_{|y=0}$

Transport mapping up to formal conjugacy

Consider $\varphi = (x + y(y-x)\Delta(x,y), y + y(y-x)) = \exp(\log\varphi)$
Suppose $\sigma \circ \varphi = \eta \circ \sigma$ for some $\eta \in \text{Diff}(\mathbb{C}^2, 0)$ and $\sigma \in \widehat{\text{Diff}}(\mathbb{C}^2, 0)$



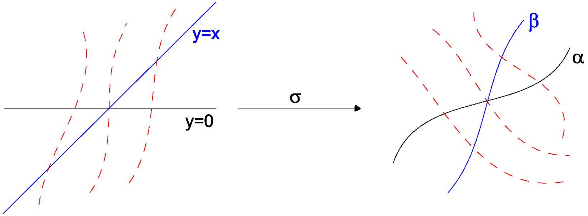
$Fix\varphi \overset{for}{\sim} Fix\eta$

$\log\varphi \overset{for}{\sim} \log\eta \implies \sigma_{|y=x} \circ Tr_\varphi = Tr_\eta \circ \sigma_{|y=0}$

Transport mapping up to formal conjugacy

Consider $\varphi = (x + y(y-x)\Delta(x,y), y + y(y-x)) = \exp(\log \varphi)$

Suppose $\sigma \circ \varphi = \eta \circ \sigma$ for some $\eta \in \text{Diff}(\mathbb{C}^2, 0)$ and $\sigma \in \widehat{\text{Diff}}(\mathbb{C}^2, 0)$



$Fix\varphi \overset{for}{\sim} Fix\eta$

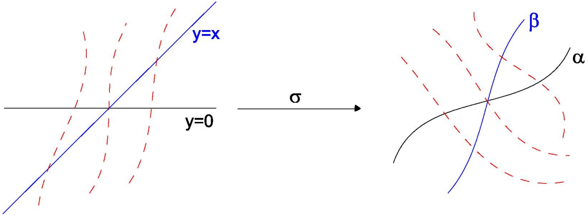
$\log \varphi \overset{for}{\sim} \log \eta \implies \sigma_{|y=x} \circ Tr_{\varphi} = Tr_{\eta} \circ \sigma_{|y=0}$

$$(\mathcal{J}_{\sigma} \circ \varphi)\mathcal{J}_{\varphi} = (\mathcal{J}_{\eta} \circ \sigma)\mathcal{J}_{\sigma}$$

Transport mapping up to formal conjugacy

Consider $\varphi = (x + y(y-x)\Delta(x,y), y + y(y-x)) = \exp(\log \varphi)$

Suppose $\sigma \circ \varphi = \eta \circ \sigma$ for some $\eta \in \mathrm{Diff}(\mathbb{C}^2, 0)$ and $\sigma \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$



$Fix\varphi \overset{for}{\sim} Fix\eta$

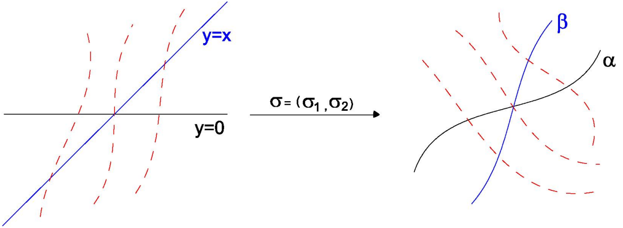
$\log \varphi \overset{for}{\sim} \log \eta \implies \sigma_{|y=x} \circ Tr_\varphi = Tr_\eta \circ \sigma_{|y=0}$

$$(\mathcal{J}_\sigma \circ \varphi)\mathcal{J}_\varphi = (\mathcal{J}_\eta \circ \sigma)\mathcal{J}_\sigma$$

$$((\mathcal{J}_\sigma \circ \varphi)\mathcal{J}_\varphi)_{|Fix\varphi} = ((\mathcal{J}_\eta \circ \sigma)\mathcal{J}_\sigma)_{|Fix\varphi} \implies \begin{cases} (\mathcal{J}_\varphi)_{|y=0} = (\mathcal{J}_\eta)_{|\alpha} \circ \sigma_{|y=0} \\ (\mathcal{J}_\varphi)_{|y=x} = (\mathcal{J}_\eta)_{|\beta} \circ \sigma_{|y=x} \end{cases}$$

Transport mapping up to formal conjugacy

Consider $\varphi = (x + y(y-x)\Delta(x,y), y + y(y-x)) = \exp(\log \varphi)$

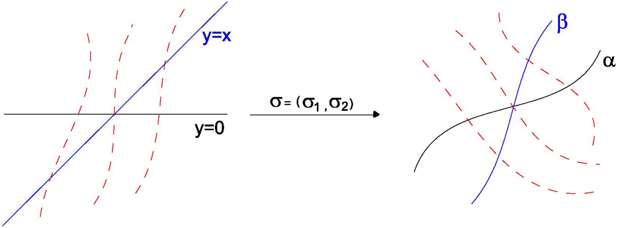


$$\varphi \overset{for}{\sim} \eta \implies \sigma_{|y=x} \circ Tr_{\varphi} = Tr_{\eta} \circ \sigma_{|y=0}$$

$$(\mathcal{J}_{\varphi})_{|y=0} = (\mathcal{J}_{\eta})_{|\alpha} \circ \sigma_{|y=0} \text{ and } (\mathcal{J}_{\varphi})_{|y=x} = (\mathcal{J}_{\eta})_{|\beta} \circ \sigma_{|y=x}$$

Transport mapping up to formal conjugacy

Consider $\varphi = (x + y(y-x)\Delta(x,y), y + y(y-x)) = \exp(\log \varphi)$



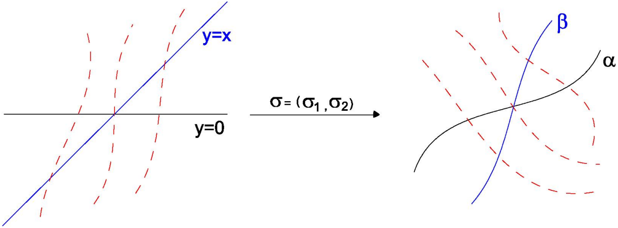
$$\varphi \overset{for}{\sim} \eta \implies \sigma_{|y=x} \circ Tr_{\varphi} = Tr_{\eta} \circ \sigma_{|y=0}$$

$(\mathcal{J}_{\varphi})_{|y=0} = (\mathcal{J}_{\eta})_{|\alpha} \circ \sigma_{|y=0}$ and $(\mathcal{J}_{\varphi})_{|y=x} = (\mathcal{J}_{\eta})_{|\beta} \circ \sigma_{|y=x}$

$\mathcal{J}_{\varphi}(x,0) = 1 - x$ and $\mathcal{J}_{\varphi}(x,x) = 1 + x - x\Delta(x,x)$

Transport mapping up to formal conjugacy

Consider $\varphi = (x + y(y-x)\Delta(x,y), y + y(y-x)) = \exp(\log \varphi)$



$$\varphi \overset{for}{\sim} \eta \implies \sigma_{|y=x} \circ Tr_\varphi = Tr_\eta \circ \sigma_{|y=0}$$

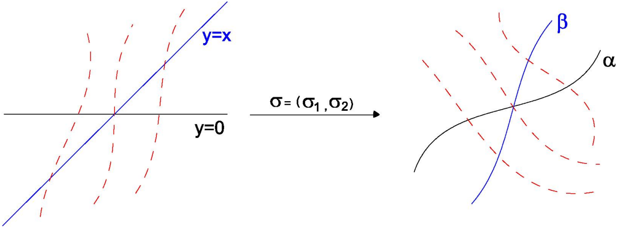
$(\mathcal{J}_\varphi)_{|y=0} = (\mathcal{J}_\eta)_{|\alpha} \circ \sigma_{|y=0}$ and $(\mathcal{J}_\varphi)_{|y=x} = (\mathcal{J}_\eta)_{|\beta} \circ \sigma_{|y=x}$

$\mathcal{J}_\varphi(x,0) = 1 - x$ and $\mathcal{J}_\varphi(x,x) = 1 + x - x\Delta(x,x)$

$\sigma_{|y=0}$ is analytic, i.e. $\sigma_1(x,0) \in \mathbb{C}\{x\} \ni \sigma_2(x,0)$

Transport mapping up to formal conjugacy

Consider $\varphi = (x + y(y-x)\Delta(x,y), y + y(y-x)) = \exp(\log \varphi)$



$$\varphi \overset{for}{\sim} \eta \implies \sigma_{|y=x} \circ Tr_{\varphi} = Tr_{\eta} \circ \sigma_{|y=0}$$

$$(\mathcal{J}_{\varphi})_{|y=0} = (\mathcal{J}_{\eta})_{|\alpha} \circ \sigma_{|y=0} \text{ and } (\mathcal{J}_{\varphi})_{|y=x} = (\mathcal{J}_{\eta})_{|\beta} \circ \sigma_{|y=x}$$

$$\mathcal{J}_{\varphi}(x,0) = 1 - x \text{ and } \mathcal{J}_{\varphi}(x,x) = 1 + x - x\Delta(x,x)$$

$\sigma_{|y=0}$ is analytic, i.e. $\sigma_1(x,0) \in \mathbb{C}\{x\} \ni \sigma_2(x,0)$

$\sigma_{|y=x}$ is analytic, i.e. $\sigma_1(x,x) \in \mathbb{C}\{x\} \ni \sigma_2(x,x)$

Transport mapping up to formal conjugacy

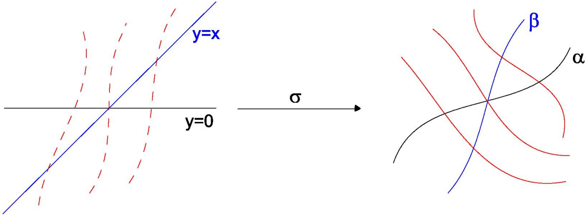
Theorem

Let $\varphi = (x + y(y-x)\Delta(x,y), y + y(y-x))$. *Consider* $\eta \in \mathrm{Diff}\,(\mathbb{C}^2, 0)$. *Then* $\varphi \overset{for}{\sim} \eta \implies Tr_\varphi \overset{ana}{\sim} Tr_\eta$.

Transport mapping up to formal conjugacy

Theorem

Let $\varphi = (x + y(y-x)\Delta(x,y), y + y(y-x))$. *Consider* $\eta \in \mathrm{Diff}\,(\mathbb{C}^2, 0)$.
Then $\varphi \overset{for}{\sim} \eta \implies Tr_\varphi \overset{ana}{\sim} Tr_\eta$.



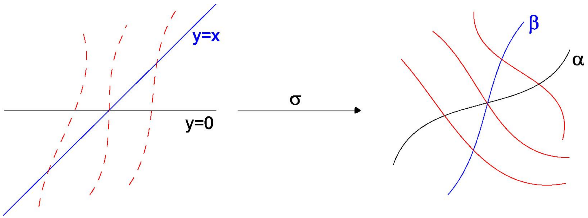
Corollary

Let $\varphi = (x + y(y-x)\Delta(x,y), y + y(y-x))$. *Then* φ *has convergent normal form* $\Rightarrow$ *the transport mapping* Tr_φ *is analytic.*

Transport mapping up to formal conjugacy

Theorem

Let $\varphi = (x + y(y - x)\Delta(x, y), y + y(y - x))$. *Consider* $\eta \in \mathrm{Diff}\,(\mathbb{C}^2, 0)$. *Then* $\varphi \overset{for}{\sim} \eta \implies Tr_\varphi \overset{ana}{\sim} Tr_\eta$.



Corollary

Let $\varphi = (x + y(y - x)\Delta(x, y), y + y(y - x))$. *Then* φ *has convergent normal form* $\Leftrightarrow$ *the transport mapping* Tr_φ *is analytic.*

Methods to divergence. The infinitesimal generator

Consider $X = a(y)\partial/\partial y = (y^{p+1} + h.o.t)\partial/\partial y$

Methods to divergence. The infinitesimal generator

Consider $X = a(y)\partial/\partial y = (y^{p+1} + h.o.t)\partial/\partial y$

$\varphi = \exp(tX)(t, y) \circ (1 + K(y), y)$ with $K \in (y^{p+1})$

Methods to divergence. The infinitesimal generator

Consider $X = a(y)\partial/\partial y = (y^{p+1} + h.o.t)\partial/\partial y$

$\varphi = \exp(tX)(t, y) \circ (1 + K(y), y)$ with $K \in (y^{p+1})$

$\varphi(y) - \exp(X) \in (y^{2(p+1)}) \implies \varphi \overset{for}{\sim} \exp(X)$

Methods to divergence. The infinitesimal generator

Consider $X = a(y)\partial/\partial y = (y^{p+1} + h.o.t)\partial/\partial y$

$\varphi = \exp(tX)(t, y) \circ (1 + K(y), y)$ with $K \in (y^{p+1})$

$$\varphi(y) - \exp(X) \in (y^{2(p+1)}) \implies \varphi \overset{for}{\sim} \exp(X)$$

Denote $\omega = dy/a(y)$. Then $\omega(X) = 1$ implies $\int_{y_0}^{\varphi(y_0)} \omega = 1 + K(y_0)$

Methods to divergence. The infinitesimal generator

Consider $X = a(y)\partial/\partial y = (y^{p+1} + h.o.t)\partial/\partial y$

$\varphi = \exp(tX)(t,y) \circ (1 + K(y), y)$ with $K \in (y^{p+1})$

$\varphi(y) - \exp(X) \in (y^{2(p+1)}) \implies \varphi \overset{for}{\sim} \exp(X)$

Denote $\omega = dy/a(y)$. Then $\omega(X) = 1$ implies $\int_{y_0}^{\varphi(y_0)} \omega = 1 + K(y_0)$

$\log \varphi$ is analytic $\Leftrightarrow \exists \epsilon \in \mathbb{C}\{y\}$ such that $\int_{y_0}^{\varphi(y_0)} (\omega + d\epsilon) = 1$

Methods to divergence. The infinitesimal generator

Consider $X = a(y)\partial/\partial y = (y^{p+1} + h.o.t)\partial/\partial y$

$\varphi = \exp(tX)(t, y) \circ (1 + K(y), y)$ with $K \in (y^{p+1})$

$\varphi(y) - \exp(X) \in (y^{2(p+1)}) \implies \varphi \overset{for}{\sim} \exp(X)$

Denote $\omega = dy/a(y)$. Then $\omega(X) = 1$ implies $\int_{y_0}^{\varphi(y_0)} \omega = 1 + K(y_0)$

$\log \varphi$ is analytic $\Leftrightarrow \exists \epsilon \in \mathbb{C}\{y\}$ such that $\int_{y_0}^{\varphi(y_0)} (\omega + d\epsilon) = 1$

$\int_{y_0}^{\varphi(y_0)} (\omega + d\epsilon) = 1 \Leftrightarrow \epsilon - \epsilon \circ \varphi = K$

Methods to divergence. The infinitesimal generator

Consider $X = a(y)\partial/\partial y = (y^{p+1} + h.o.t)\partial/\partial y$

$\varphi = \exp(tX)(t,y) \circ (1 + K(y), y)$ with $K \in (y^{p+1})$

$\varphi(y) - \exp(X) \in (y^{2(p+1)}) \implies \varphi \overset{for}{\sim} \exp(X)$

Denote $\omega = dy/a(y)$. Then $\omega(X) = 1$ implies $\int_{y_0}^{\varphi(y_0)} \omega = 1 + K(y_0)$

$\log \varphi$ is analytic $\Leftrightarrow \exists \epsilon \in \mathbb{C}\{y\}$ such that $\int_{y_0}^{\varphi(y_0)} (\omega + d\epsilon) = 1$

$\int_{y_0}^{\varphi(y_0)} (\omega + d\epsilon) = 1 \Leftrightarrow \epsilon - \epsilon \circ \varphi = K$

$\varphi_\lambda = \exp(tX)(t,y) \circ (1 + \lambda K(y), y)$ with $K \in (y^{p+1})$ and $\lambda \in \mathbb{C}$

$\log \varphi_\lambda$ is analytic $\Leftrightarrow \exists \epsilon_\lambda \in \mathbb{C}\{y\}$ such that $\epsilon_\lambda - \epsilon_\lambda \circ \varphi_\lambda = \lambda K$

The homological equation $\epsilon - \epsilon \circ \exp(X) = K$

$\varphi_\lambda = \exp(tX)(t, y) \circ (1 + \lambda K(y), y)$ with $K \in (y^{p+1})$ and $\lambda \in \mathbb{C}$
$\log \varphi_\lambda$ is analytic $\Leftrightarrow \exists \epsilon_\lambda \in \mathbb{C}\{y\}$ such that $\epsilon_\lambda - \epsilon_\lambda \circ \varphi_\lambda = \lambda K$

The homological equation $\epsilon - \epsilon \circ \exp(X) = K$

$\varphi_\lambda = \exp(tX)(t, y) \circ (1 + \lambda K(y), y)$ with $K \in (y^{p+1})$ and $\lambda \in \mathbb{C}$
$\log \varphi_\lambda$ is analytic $\Leftrightarrow \exists \epsilon_\lambda \in \mathbb{C}\{y\}$ such that $\epsilon_\lambda - \epsilon_\lambda \circ \varphi_\lambda = \lambda K$

- $\epsilon_\lambda = \sum_{j=1}^{\infty} \lambda \epsilon_j(\lambda) y^j$ with $\epsilon_j \in \mathbb{C}[\lambda]$ and $\deg \epsilon_j \leq j - 1$ for all $j \in \mathbb{N}$

The homological equation $\epsilon - \epsilon \circ \exp(X) = K$

$\varphi_\lambda = \exp(tX)(t, y) \circ (1 + \lambda K(y), y)$ with $K \in (y^{p+1})$ and $\lambda \in \mathbb{C}$
$\log \varphi_\lambda$ is analytic $\Leftrightarrow \exists \epsilon_\lambda \in \mathbb{C}\{y\}$ such that $\epsilon_\lambda - \epsilon_\lambda \circ \varphi_\lambda = \lambda K$

- $\epsilon_\lambda = \sum_{j=1}^{\infty} \lambda \epsilon_j(\lambda) y^j$ with $\epsilon_j \in \mathbb{C}[\lambda]$ and $\deg \epsilon_j \leq j - 1$ for all $j \in \mathbb{N}$
- $\frac{\partial(\epsilon_\lambda - \epsilon_\lambda \circ \varphi_\lambda)}{\partial \lambda}_{|\lambda=0} = K \Leftrightarrow \frac{\partial \epsilon_\lambda}{\partial \lambda}_{|\lambda=0} - \frac{\partial \epsilon_\lambda}{\partial \lambda}_{|\lambda=0} \circ \exp(X) = K$

The homological equation $\epsilon - \epsilon \circ \exp(X) = K$

$\varphi_\lambda = \exp(tX)(t, y) \circ (1 + \lambda K(y), y)$ with $K \in (y^{p+1})$ and $\lambda \in \mathbb{C}$
$\log \varphi_\lambda$ is analytic $\Leftrightarrow \exists \epsilon_\lambda \in \mathbb{C}\{y\}$ such that $\epsilon_\lambda - \epsilon_\lambda \circ \varphi_\lambda = \lambda K$

- $\epsilon_\lambda = \sum_{j=1}^{\infty} \lambda \epsilon_j(\lambda) y^j$ with $\epsilon_j \in \mathbb{C}[\lambda]$ and $\deg \epsilon_j \leq j - 1$ for all $j \in \mathbb{N}$
- $\frac{\partial(\epsilon_\lambda - \epsilon_\lambda \circ \varphi_\lambda)}{\partial \lambda}_{|\lambda=0} = K \Leftrightarrow \frac{\partial \epsilon_\lambda}{\partial \lambda}_{|\lambda=0} - \frac{\partial \epsilon_\lambda}{\partial \lambda}_{|\lambda=0} \circ \exp(X) = K$

Theorem (Pérez Marco)

Let $P = \sum_{j \geq 0} P_j(\lambda) y^j \in \mathbb{C}[\lambda][[y]]$ where $\deg P_j \leq Aj + B$ for some $A, B \in \mathbb{R}$ and all $j \in \mathbb{N}$. Then either $P(\lambda, y)$ is convergent in a neighborhood of $y = 0$ or $P(\lambda_0, x) \notin \mathbb{C}\{x\}$ $\forall \lambda_0 \in \mathbb{C}$ outside a polar set.

The homological equation $\epsilon - \epsilon \circ \exp(X) = K$

$\varphi_\lambda = \exp(tX)(t,y) \circ (1 + \lambda K(y), y)$ with $K \in (y^{p+1})$ and $\lambda \in \mathbb{C}$
$\log \varphi_\lambda$ is analytic $\Leftrightarrow \exists \epsilon_\lambda \in \mathbb{C}\{y\}$ such that $\epsilon_\lambda - \epsilon_\lambda \circ \varphi_\lambda = \lambda K$

- $\epsilon_\lambda = \sum_{j=1}^{\infty} \lambda \epsilon_j(\lambda) y^j$ with $\epsilon_j \in \mathbb{C}[\lambda]$ and $\deg \epsilon_j \leq j - 1$ for all $j \in \mathbb{N}$
- $\frac{\partial(\epsilon_\lambda - \epsilon_\lambda \circ \varphi_\lambda)}{\partial \lambda}_{|\lambda=0} = K \Leftrightarrow \frac{\partial \epsilon_\lambda}{\partial \lambda}_{|\lambda=0} - \frac{\partial \epsilon_\lambda}{\partial \lambda}_{|\lambda=0} \circ \exp(X) = K$

Theorem (Pérez Marco)

Let $P = \sum_{j \geq 0} P_j(\lambda) y^j \in \mathbb{C}[\lambda][[y]]$ where $\deg P_j \leq Aj + B$ for some $A, B \in \mathbb{R}$ and all $j \in \mathbb{N}$. Then either $P(\lambda, y)$ is convergent in a neighborhood of $y = 0$ or $P(\lambda_0, x) \notin \mathbb{C}\{x\}$ $\forall \lambda_0 \in \mathbb{C}$ outside a polar set.

Corollary

Suppose that $\epsilon - \epsilon \circ \exp(X) = K$ has divergent solutions. Then $\log \varphi_\lambda$ is divergent for all $\lambda \in \mathbb{C}$ outside a polar set.

Calculations

$$X = \frac{\eta y^{p+1}}{1+\lambda y^p} \frac{\partial}{\partial y}$$

Consider $\epsilon(K) - \epsilon(K) \circ \exp(X) = K$ with $\epsilon(K) = \sum_{j=1}^{\infty} \epsilon_j(K) y^j \in \mathbb{C}[[y]]$

Calculations

$$X = \frac{\eta y^{p+1}}{1+\lambda y^p}\frac{\partial}{\partial y}$$

Consider $\epsilon(K) - \epsilon(K) \circ \exp(X) = K$ with $\epsilon(K) = \sum_{j=1}^{\infty} \epsilon_j(K) y^j \in \mathbb{C}[[y]]$

$K = y^{p+k}$ with $k \geq 1$

$\epsilon_k(y^{p+k}) = \frac{-1}{k\eta}$

Calculations

$$X = \frac{\eta y^{p+1}}{1+\lambda y^p} \frac{\partial}{\partial y}$$

Consider $\epsilon(K) - \epsilon(K) \circ \exp(X) = K$ with $\epsilon(K) = \sum_{j=1}^{\infty} \epsilon_j(K) y^j \in \mathbb{C}[[y]]$

$K = y^{p+k}$ with $k \geq 1$

$\epsilon_k(y^{p+k}) = \frac{-1}{k\eta}$

$\epsilon_{p+k}(y^{p+k}) = \frac{1}{2} + O\left(\frac{1}{k}\right)$

Calculations

$$X = \frac{\eta y^{p+1}}{1+\lambda y^p} \frac{\partial}{\partial y}$$

Consider $\epsilon(K) - \epsilon(K) \circ \exp(X) = K$ with $\epsilon(K) = \sum_{j=1}^{\infty} \epsilon_j(K) y^j \in \mathbb{C}[[y]]$

$K = y^{p+k}$ with $k \geq 1$

$\epsilon_k(y^{p+k}) = \frac{-1}{k\eta}$

$\epsilon_{p+k}(y^{p+k}) = \frac{1}{2} + O\left(\frac{1}{k}\right)$

$\epsilon_{2p+k}(y^{p+k}) = \frac{-k\eta}{12} + O(1)$

Uniform boundness principle

$$X = \frac{\eta y^{p+1}}{1+\lambda y^p} \frac{\partial}{\partial y}$$

Consider $\epsilon(K) - \epsilon(K) \circ \exp(X) = K$ with $\epsilon(K) = \sum_{j=1}^{\infty} \epsilon_j(K) y^j \in \mathbb{C}[[y]]$

Uniform boundness principle

$$X = \frac{\eta y^{p+1}}{1+\lambda y^p} \frac{\partial}{\partial y}$$

Consider $\epsilon(K) - \epsilon(K) \circ \exp(X) = K$ with $\epsilon(K) = \sum_{j=1}^{\infty} \epsilon_j(K) y^j \in \mathbb{C}[[y]]$

$B :=$ set of functions holomorphic in $B(0,1)$ and continuous in $\overline{B(0,1)}$

B is a Banach space with the norm $||f|| = \sup_{y \in \overline{B}(0,1)} |f(y)|$

Uniform boundness principle

$$X = \frac{\eta y^{p+1}}{1+\lambda y^p}\frac{\partial}{\partial y}$$

Consider $\epsilon(K) - \epsilon(K) \circ \exp(X) = K$ with $\epsilon(K) = \sum_{j=1}^{\infty} \epsilon_j(K) y^j \in \mathbb{C}[[y]]$

$B :=$ set of functions holomorphic in $B(0,1)$ and continuous in $\overline{B(0,1)}$

B is a Banach space with the norm $||f|| = \sup_{y \in \overline{B}(0,1)} |f(y)|$

The functional $\epsilon_j : y^{p+1}B \to \mathbb{C}$ is continuous for all $j \in \mathbb{N}$

Uniform boundness principle

$$X = \frac{\eta y^{p+1}}{1+\lambda y^p}\frac{\partial}{\partial y}$$

Consider $\epsilon(K) - \epsilon(K) \circ \exp(X) = K$ with $\epsilon(K) = \sum_{j=1}^{\infty} \epsilon_j(K) y^j \in \mathbb{C}[[y]]$

$B :=$ set of functions holomorphic in $B(0,1)$ and continuous in $\overline{B(0,1)}$

B is a Banach space with the norm $||f|| = \sup_{y \in \overline{B}(0,1)} |f(y)|$

The functional $\epsilon_j : y^{p+1}B \to \mathbb{C}$ is continuous for all $j \in \mathbb{N}$

If $\varphi \overset{for}{\sim} \exp(X)$ implies $\log \varphi \in \mathcal{X}(\mathbb{C}, 0)$ then $\epsilon(y^{p+1}B) \subset \mathbb{C}\{y\}$

Uniform boundness principle

$$X = \frac{\eta y^{p+1}}{1+\lambda y^p}\frac{\partial}{\partial y}$$

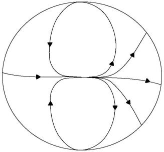
Consider $\epsilon(K) - \epsilon(K) \circ \exp(X) = K$ with $\epsilon(K) = \sum_{j=1}^{\infty} \epsilon_j(K) y^j \in \mathbb{C}[[y]]$

$B :=$ set of functions holomorphic in $B(0,1)$ and continuous in $\overline{B(0,1)}$

B is a Banach space with the norm $||f|| = \sup_{y \in \overline{B}(0,1)} |f(y)|$

The functional $\epsilon_j : y^{p+1}B \to \mathbb{C}$ is continuous for all $j \in \mathbb{N}$

If $\varphi \overset{for}{\sim} \exp(X)$ implies $\log \varphi \in \mathcal{X}(\mathbb{C}, 0)$ then $\epsilon(y^{p+1}B) \subset \mathbb{C}\{y\}$



$$\epsilon(y^{p+1}B) \subset \mathbb{C}\{y\} \implies \epsilon(y^{p+1}B) \subset B$$

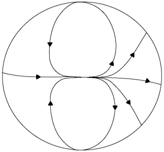
Uniform boundness principle

$$X = \frac{\eta y^{p+1}}{1+\lambda y^p} \frac{\partial}{\partial y}$$

Consider $\epsilon(K) - \epsilon(K) \circ \exp(X) = K$ with $\epsilon(K) = \sum_{j=1}^{\infty} \epsilon_j(K) y^j \in \mathbb{C}[[y]]$

The functional $\epsilon_j : y^{p+1}B \to \mathbb{C}$ is continuous for all $j \in \mathbb{N}$

If $\varphi \overset{for}{\sim} \exp(X)$ implies $\log \varphi \in \mathcal{X}(\mathbb{C}, 0)$ then $\epsilon(y^{p+1}B) \subset \mathbb{C}\{y\}$



$$\epsilon(y^{p+1}B) \subset \mathbb{C}\{y\} \implies \epsilon(y^{p+1}B) \subset B$$

$$\epsilon_j(K) = \frac{1}{2\pi i} \int_{\partial B(0,1)} \frac{\epsilon(K)}{y^{j+1}} dy \Rightarrow |\epsilon_j(K)| \leq ||\epsilon(K)||$$

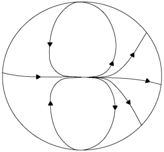
Uniform boundness principle

$$X = \frac{\eta y^{p+1}}{1+\lambda y^p}\frac{\partial}{\partial y}$$

Consider $\epsilon(K) - \epsilon(K) \circ \exp(X) = K$ with $\epsilon(K) = \sum_{j=1}^{\infty} \epsilon_j(K) y^j \in \mathbb{C}[[y]]$

The functional $\epsilon_j : y^{p+1}B \to \mathbb{C}$ is continuous for all $j \in \mathbb{N}$

If $\varphi \overset{for}{\sim} \exp(X)$ implies $\log \varphi \in \mathcal{X}(\mathbb{C}, 0)$ then $\epsilon(y^{p+1}B) \subset \mathbb{C}\{y\}$



$$\epsilon(y^{p+1}B) \subset \mathbb{C}\{y\} \implies \epsilon(y^{p+1}B) \subset B$$

$$\epsilon_j(K) = \frac{1}{2\pi i}\int_{\partial B(0,1)} \frac{\epsilon(K)}{y^{j+1}} dy \Rightarrow |\epsilon_j(K)| \leq ||\epsilon(K)||$$

$\sup_{j\in\mathbb{N}} |\epsilon_j(K)| < \infty$ for all $K \in y^{p+1}B \implies \sup_{j\in\mathbb{N}} ||\epsilon_j|| < \infty$

Generic divergence of the infinitesimal generator

$$X = \frac{\eta y^{p+1}}{1+\lambda y^p} \frac{\partial}{\partial y}$$

Consider $\epsilon(K) - \epsilon(K) \circ \exp(X) = K$ with $\epsilon(K) = \sum_{j=1}^{\infty} \epsilon_j(K) y^j \in \mathbb{C}[[y]]$

$\epsilon_k(y^{p+k}) = \frac{-1}{k\eta}$

$\epsilon_{p+k}(y^{p+k}) = \frac{1}{2} + O\left(\frac{1}{k}\right)$

$\epsilon_{2p+k}(y^{p+k}) = \frac{-k\eta}{12} + O(1)$

Generic divergence of the infinitesimal generator

$$X = \frac{\eta y^{p+1}}{1+\lambda y^p}\frac{\partial}{\partial y}$$

Consider $\epsilon(K) - \epsilon(K) \circ \exp(X) = K$ with $\epsilon(K) = \sum_{j=1}^{\infty} \epsilon_j(K) y^j \in \mathbb{C}[[y]]$

$\epsilon_k(y^{p+k}) = \frac{-1}{k\eta}$

$\epsilon_{p+k}(y^{p+k}) = \frac{1}{2} + O\left(\frac{1}{k}\right)$

$\epsilon_{2p+k}(y^{p+k}) = \frac{-k\eta}{12} + O(1) \implies \lim_{j\to\infty} ||\epsilon_j|| = \infty$

Generic divergence of the infinitesimal generator

$$X = \frac{\eta y^{p+1}}{1+\lambda y^p}\frac{\partial}{\partial y}$$

Consider $\epsilon(K) - \epsilon(K) \circ \exp(X) = K$ with $\epsilon(K) = \sum_{j=1}^{\infty} \epsilon_j(K) y^j \in \mathbb{C}[[y]]$

$\epsilon_k(y^{p+k}) = \frac{-1}{k\eta}$

$\epsilon_{p+k}(y^{p+k}) = \frac{1}{2} + O\left(\frac{1}{k}\right)$

$\epsilon_{2p+k}(y^{p+k}) = \frac{-k\eta}{12} + O(1) \implies \lim_{j\to\infty} ||\epsilon_j|| = \infty$

$\nexists \varphi$ with $\varphi \overset{for}{\sim} \exp(X)$ and $\log \varphi$ divergent $\Rightarrow \sup_{j\in\mathbb{N}} ||\epsilon_j|| < \infty$

Generic divergence of the infinitesimal generator

$$X = \frac{\eta y^{p+1}}{1+\lambda y^p} \frac{\partial}{\partial y}$$

Consider $\epsilon(K) - \epsilon(K) \circ \exp(X) = K$ with $\epsilon(K) = \sum_{j=1}^{\infty} \epsilon_j(K) y^j \in \mathbb{C}[[y]]$

$\epsilon_k(y^{p+k}) = \frac{-1}{k\eta}$

$\epsilon_{p+k}(y^{p+k}) = \frac{1}{2} + O\left(\frac{1}{k}\right)$

$\epsilon_{2p+k}(y^{p+k}) = \frac{-k\eta}{12} + O(1) \implies \lim_{j \to \infty} ||\epsilon_j|| = \infty$

$$\nexists \varphi \text{ with } \varphi \overset{for}{\sim} \exp(X) \text{ and } \log \varphi \text{ divergent} \Rightarrow \sup_{j \in \mathbb{N}} ||\epsilon_j|| < \infty$$

Theorem

Let $X \in \mathcal{X}_2(\mathbb{C}, 0)$. There exists $\varphi \in \mathrm{Diff}_1(\mathbb{C}, 0)$ in the formal class of $\exp(X)$ such that its infinitesimal generator is divergent.

Resume

$\exists\varphi$ whose infinitesimal generator is divergent in the formal class of X

Resume

$\exists \varphi$ whose infinitesimal generator is divergent in the formal class of X

$\Uparrow$ (Potential theory)

Divergence of the homological equation

Resume

$\exists\varphi$ whose infinitesimal generator is divergent in the formal class of X

$\Uparrow$ (Potential theory)

Divergence of the homological equation

$\Uparrow$ (Uniform boundness principle)

Topological dynamics of $Re(X)$

Resume

$\exists\varphi$ whose infinitesimal generator is divergent in the formal class of X

$\Uparrow$ (Potential theory)

Divergence of the homological equation

$\Uparrow$ (Uniform boundness principle)

Topological dynamics of $Re(X)$

$\alpha \in \mathbb{R} \setminus \mathbb{Q}$

$$\exists\varphi = e^{2\pi i\alpha}y + O(y^2) \in \mathrm{Diff}\,(\mathbb{C}, 0) \text{ s. t. } \varphi \overset{an}{\not\sim} e^{2\pi i\alpha}y$$

Resume

$\exists\varphi$ whose infinitesimal generator is divergent in the formal class of X

$\Uparrow$ (Potential theory)

Divergence of the homological equation

$\Uparrow$ (Uniform boundness principle)

Topological dynamics of $Re(X)$

$\alpha \in \mathbb{R} \setminus \mathbb{Q}$

$$\exists \varphi = e^{2\pi i \alpha} y + O(y^2) \in \mathrm{Diff}\,(\mathbb{C}, 0) \text{ s. t. } \varphi \overset{an}{\not\sim} e^{2\pi i \alpha} y$$

$$\Uparrow$$

$$\exists \text{ a divergent solution of } \epsilon - \epsilon(e^{2\pi i \alpha} y) = K(y)$$

Resume

$\exists \varphi$ whose infinitesimal generator is divergent in the formal class of X

$\Uparrow$ (Potential theory)

Divergence of the homological equation

$\Uparrow$ (Uniform boundness principle)

Topological dynamics of $Re(X)$

$\alpha \in \mathbb{R} \setminus \mathbb{Q}$

$$\exists \varphi = e^{2\pi i \alpha} y + O(y^2) \in \mathrm{Diff}\,(\mathbb{C}, 0) \text{ s. t. } \varphi \overset{an}{\not\sim} e^{2\pi i \alpha} y \Leftrightarrow \sum_{j \geq 0} \frac{\log q_{j+1}}{q_j} = \infty$$

$$\Uparrow$$

$$\exists \text{ a divergent solution of } \epsilon - \epsilon(e^{2\pi i \alpha} y) = K(y) \Leftrightarrow \sup_{j \geq 0} \frac{\log q_{j+1}}{q_j} = \infty$$

Resume

$\exists\varphi$ whose infinitesimal generator is divergent in the formal class of X

$\Uparrow$ (Potential theory)

Divergence of the homological equation

$\Uparrow$ (Uniform boundness principle)

Topological dynamics of $Re(X)$

$\alpha \in \mathbb{R} \setminus \mathbb{Q}$

$$\exists\varphi = e^{2\pi i\alpha}y + O(y^2) \in \mathrm{Diff}(\mathbb{C},0) \text{ s. t. } \varphi \overset{an}{\not\sim} e^{2\pi i\alpha}y \Leftrightarrow \sum_{j\geq 0} \frac{\log q_{j+1}}{q_j} = \infty$$

$$\Uparrow$$

$\exists$ a divergent solution of $\epsilon - \epsilon(e^{2\pi i\alpha}y) = K(y) \Leftrightarrow \sup_{j\geq 0} \frac{\log q_{j+1}}{q_j} = \infty$

$\epsilon_{2p+k}(y^{p+k}) = \frac{-k\eta}{12} + O(1) \to \varphi = y + \eta y^{p+1} + O(y^{p+2})$ is p-summable

Resume

$\exists\varphi$ whose infinitesimal generator is divergent in the formal class of X

$\Uparrow$ (Potential theory)

Divergence of the homological equation

$\Uparrow$ (Uniform boundness principle)

Topological dynamics of $Re(X)$

$\alpha \in \mathbb{R} \setminus \mathbb{Q}$

$$\exists\varphi = e^{2\pi i\alpha}y + O(y^2) \in \mathrm{Diff}\,(\mathbb{C},0) \text{ s. t. } \varphi \overset{an}{\not\sim} e^{2\pi i\alpha}y \Leftrightarrow \sum_{j\geq 0} \frac{\log q_{j+1}}{q_j} = \infty$$

$$\Uparrow$$

$$\exists \text{ a divergent solution of } \epsilon - \epsilon(e^{2\pi i\alpha}y) = K(y) \Leftrightarrow \sup_{j\geq 0} \frac{\log q_{j+1}}{q_j} = \infty$$

$\epsilon_{2p+k}(y^{p+k}) = \frac{-k\eta}{12} + O(1) \rightarrow \varphi = y + \eta y^{p+1} + O(y^{p+2})$ is p-summable

Finite determination world $\rightarrow$ Finite world

Towards the divergence of the transport mapping

$$\varphi_\lambda = (x + \lambda y(y - x)\Delta(x, y), y + y(y - x)) = \exp(\hat{X}_\lambda)$$

$\exists! f_\lambda \in \mathbb{C}[[x, y]]$ such that $\hat{X}_\lambda(f_\lambda) \equiv 0$ and $f_\lambda(x, 0) \equiv x$

Towards the divergence of the transport mapping

$$\varphi_\lambda = (x + \lambda y(y-x)\Delta(x,y), y + y(y-x)) = \exp(\hat{X}_\lambda)$$

$\exists! f_\lambda \in \mathbb{C}[[x,y]]$ such that $\hat{X}_\lambda(f_\lambda) \equiv 0$ and $f_\lambda(x,0) \equiv x$

$$Tr_{\varphi_\lambda}(x,0) = (f_\lambda(x,x)^{-1} \circ f_\lambda(x,0), f_\lambda(x,x)^{-1} \circ f_\lambda(x,0))$$

Towards the divergence of the transport mapping

$$\varphi_\lambda = (x + \lambda y(y-x)\Delta(x,y), y + y(y-x)) = \exp(\hat{X}_\lambda)$$

$$\exists! f_\lambda \in \mathbb{C}[[x,y]] \text{ such that } \hat{X}_\lambda(f_\lambda) \equiv 0 \text{ and } f_\lambda(x,0) \equiv x$$

$$Tr_{\varphi_\lambda}(x,0) = (f_\lambda(x,x)^{-1} \circ f_\lambda(x,0), f_\lambda(x,x)^{-1} \circ f_\lambda(x,0))$$

φ_λ has convergent normal form $\Leftrightarrow$ Tr_{φ_λ} is convergent $\Leftrightarrow$ $f_\lambda(x,x) \in \mathbb{C}\{x\}$

Towards the divergence of the transport mapping

$$\varphi_\lambda = (x + \lambda y(y-x)\Delta(x,y), y + y(y-x)) = \exp(\hat{X}_\lambda)$$

$\exists ! f_\lambda \in \mathbb{C}[[x,y]]$ such that $\hat{X}_\lambda(f_\lambda) \equiv 0$ and $f_\lambda(x,0) \equiv x$

$$Tr_{\varphi_\lambda}(x,0) = (f_\lambda(x,x)^{-1} \circ f_\lambda(x,0), f_\lambda(x,x)^{-1} \circ f_\lambda(x,0))$$

φ_λ has convergent normal form $\Leftrightarrow$ Tr_{φ_λ} is convergent $\Leftrightarrow$ $f_\lambda(x,x) \in \mathbb{C}\{x\}$

$f_\lambda(x,x) = x + \lambda \sum_{j=2}^{\infty} f_j(\lambda) x^j \in \mathbb{C}[\lambda][[x]]$ with $\deg f_j \leq j-2$ for all $j \in \mathbb{N}$

Towards the divergence of the transport mapping

$$\varphi_\lambda = (x + \lambda y(y-x)\Delta(x,y), y + y(y-x)) = \exp(\hat{X}_\lambda)$$

$\exists! f_\lambda \in \mathbb{C}[[x,y]]$ such that $\hat{X}_\lambda(f_\lambda) \equiv 0$ and $f_\lambda(x,0) \equiv x$

$$Tr_{\varphi_\lambda}(x,0) = (f_\lambda(x,x)^{-1} \circ f_\lambda(x,0), f_\lambda(x,x)^{-1} \circ f_\lambda(x,0))$$

φ_λ has convergent normal form $\Leftrightarrow$ Tr_{φ_λ} is convergent $\Leftrightarrow$ $f_\lambda(x,x) \in \mathbb{C}\{x\}$

$f_\lambda(x,x) = x + \lambda \sum_{j=2}^{\infty} f_j(\lambda) x^j \in \mathbb{C}[\lambda][[x]]$ with $\deg f_j \leq j-2$ for all $j \in \mathbb{N}$

The homological equation $\epsilon - \epsilon \circ \varphi_0 = y(y-x)\Delta(x,y)$

$$\hat{X}_\lambda(f_\lambda) \equiv 0 \Leftrightarrow f_\lambda \circ \varphi_\lambda \equiv f_\lambda$$

Towards the divergence of the transport mapping

$$\varphi_\lambda = (x + \lambda y(y-x)\Delta(x,y), y + y(y-x)) = \exp(\hat{X}_\lambda)$$

$\exists! f_\lambda \in \mathbb{C}[[x,y]]$ such that $\hat{X}_\lambda(f_\lambda) \equiv 0$ and $f_\lambda(x,0) \equiv x$

φ_λ has convergent normal form $\Leftrightarrow$ Tr_{φ_λ} is convergent $\Leftrightarrow$ $f_\lambda(x,x) \in \mathbb{C}\{x\}$

$f_\lambda(x,x) = x + \lambda \sum_{j=2}^{\infty} f_j(\lambda) x^j \in \mathbb{C}[\lambda][[x]]$ with $\deg f_j \leq j-2$ for all $j \in \mathbb{N}$

The homological equation $\epsilon - \epsilon \circ \varphi_0 = y(y-x)\Delta(x,y)$

$$\hat{X}_\lambda(f_\lambda) \equiv 0 \Leftrightarrow f_\lambda \circ \varphi_\lambda \equiv f_\lambda$$

$$\frac{\partial(f_\lambda \circ \varphi_\lambda)}{\partial \lambda}_{|\lambda=0} = \frac{\partial f_\lambda}{\partial \lambda}_{|\lambda=0} \Leftrightarrow \frac{\partial f_\lambda}{\partial \lambda}_{|\lambda=0} - \frac{\partial f_\lambda}{\partial \lambda}_{|\lambda=0} \circ \varphi_0 = y(y-x)\Delta(x,y)$$

Towards the divergence of the transport mapping

$$\varphi_\lambda = (x + \lambda y(y-x)\Delta(x,y), y + y(y-x)) = \exp(\hat{X}_\lambda)$$

$\exists! f_\lambda \in \mathbb{C}[[x,y]]$ such that $\hat{X}_\lambda(f_\lambda) \equiv 0$ and $f_\lambda(x,0) \equiv x$

φ_λ has convergent normal form $\Leftrightarrow$ Tr_{φ_λ} is convergent $\Leftrightarrow$ $f_\lambda(x,x) \in \mathbb{C}\{x\}$

$f_\lambda(x,x) = x + \lambda \sum_{j=2}^{\infty} f_j(\lambda) x^j \in \mathbb{C}[\lambda][[x]]$ with $\deg f_j \leq j-2$ for all $j \in \mathbb{N}$

The homological equation $\epsilon - \epsilon \circ \varphi_0 = y(y-x)\Delta(x,y)$

$$\hat{X}_\lambda(f_\lambda) \equiv 0 \Leftrightarrow f_\lambda \circ \varphi_\lambda \equiv f_\lambda$$

$$\frac{\partial (f_\lambda \circ \varphi_\lambda)}{\partial \lambda}_{|\lambda=0} = \frac{\partial f_\lambda}{\partial \lambda}_{|\lambda=0} \Leftrightarrow \frac{\partial f_\lambda}{\partial \lambda}_{|\lambda=0} - \frac{\partial f_\lambda}{\partial \lambda}_{|\lambda=0} \circ \varphi_0 = y(y-x)\Delta(x,y)$$

Linear transport mapping $L(\Delta)$

$$\exists! \text{ solution } \epsilon \text{ such that } \epsilon(x,0) \equiv 0 \implies \epsilon = \frac{\partial f_\lambda}{\partial \lambda}_{|\lambda=0}$$

Towards the divergence of the transport mapping

$$\varphi_\lambda = (x + \lambda y(y-x)\Delta(x,y), y + y(y-x)) = \exp(\hat{X}_\lambda)$$

$\exists! f_\lambda \in \mathbb{C}[[x,y]]$ such that $\hat{X}_\lambda(f_\lambda) \equiv 0$ and $f_\lambda(x,0) \equiv x$

φ_λ has convergent normal form $\Leftrightarrow$ Tr_{φ_λ} is convergent $\Leftrightarrow$ $f_\lambda(x,x) \in \mathbb{C}\{x\}$

$f_\lambda(x,x) = x + \lambda \sum_{j=2}^{\infty} f_j(\lambda) x^j \in \mathbb{C}[\lambda][[x]]$ with $\deg f_j \leq j-2$ for all $j \in \mathbb{N}$

The homological equation $\epsilon - \epsilon \circ \varphi_0 = y(y-x)\Delta(x,y)$

$$\hat{X}_\lambda(f_\lambda) \equiv 0 \Leftrightarrow f_\lambda \circ \varphi_\lambda \equiv f_\lambda$$

$$\frac{\partial(f_\lambda \circ \varphi_\lambda)}{\partial \lambda}_{|\lambda=0} = \frac{\partial f_\lambda}{\partial \lambda}_{|\lambda=0} \Leftrightarrow \frac{\partial f_\lambda}{\partial \lambda}_{|\lambda=0} - \frac{\partial f_\lambda}{\partial \lambda}_{|\lambda=0} \circ \varphi_0 = y(y-x)\Delta(x,y)$$

Linear transport mapping $L(\Delta)$

$\exists!$ solution ϵ such that $\epsilon(x,0) \equiv 0 \implies \epsilon = \frac{\partial f_\lambda}{\partial \lambda}_{|\lambda=0}$

$L(\Delta)(x) := \epsilon(x,x)$. We will prove that $L(\Delta)$ is divergent for Δ generic

Calculations

$$\epsilon - \epsilon \circ (x, y + y(y-x)) = y(y-x)\Delta(x,y)$$
$$\epsilon(\Delta) = \sum_{j=1}^{\infty} \epsilon_j(\Delta)$$
$$L(\Delta) = \sum_{j=1}^{\infty} L_j(\Delta)x^j = \epsilon(x,x) - \epsilon(x,0)$$

Calculations

$$\epsilon - \epsilon \circ (x, y + y(y-x)) = y(y-x)\Delta(x,y)$$
$$\epsilon(\Delta) = \sum_{j=1}^{\infty} \epsilon_j(\Delta)$$
$$L(\Delta) = \sum_{j=1}^{\infty} L_j(\Delta)x^j = \epsilon(x,x) - \epsilon(x,0)$$

$\epsilon_{k+1}(y^k) = \frac{-y^{k+1}}{k+1}$

Calculations

$$\epsilon - \epsilon \circ (x, y + y(y-x)) = y(y-x)\Delta(x,y)$$
$$\epsilon(\Delta) = \sum_{j=1}^{\infty} \epsilon_j(\Delta)$$
$$L(\Delta) = \sum_{j=1}^{\infty} L_j(\Delta)x^j = \epsilon(x,x) - \epsilon(x,0)$$

$$\epsilon_{k+1}(y^k) = \frac{-y^{k+1}}{k+1}$$
$$\epsilon_{k+2}(y^k) = y^{k+1}\left(\frac{k}{2(k+2)}y - \frac{k}{2(k+1)}x\right)$$

Calculations

$$\epsilon - \epsilon \circ (x, y + y(y-x)) = y(y-x)\Delta(x,y)$$
$$\epsilon(\Delta) = \sum_{j=1}^{\infty} \epsilon_j(\Delta)$$
$$L(\Delta) = \sum_{j=1}^{\infty} L_j(\Delta)x^j = \epsilon(x,x) - \epsilon(x,0)$$

$$\epsilon_{k+1}(y^k) = \frac{-y^{k+1}}{k+1}$$
$$\epsilon_{k+2}(y^k) = y^{k+1}\left(\frac{k}{2(k+2)}y - \frac{k}{2(k+1)}x\right)$$
$$\epsilon_{k+3}(y^k) = y^{k+1}\left(\frac{-k(k+5)}{12(k+3)}y^2 + \frac{k(2k+7)}{12(k+2)}xy - \frac{k(k+2)}{12(k+1)}x^2\right)$$

Calculations

$$\epsilon - \epsilon \circ (x, y + y(y-x)) = y(y-x)\Delta(x,y)$$
$$\epsilon(\Delta) = \sum_{j=1}^{\infty} \epsilon_j(\Delta)$$
$$L(\Delta) = \sum_{j=1}^{\infty} L_j(\Delta) x^j = \epsilon(x,x) - \epsilon(x,0)$$

$$\epsilon_{k+1}(y^k) = \frac{-y^{k+1}}{k+1}$$
$$\epsilon_{k+2}(y^k) = y^{k+1}\left(\frac{k}{2(k+2)}y - \frac{k}{2(k+1)}x\right)$$
$$\epsilon_{k+3}(y^k) = y^{k+1}\left(\frac{-k(k+5)}{12(k+3)}y^2 + \frac{k(2k+7)}{12(k+2)}xy - \frac{k(k+2)}{12(k+1)}x^2\right)$$

$$L_{k+1}(y^k) = \frac{-1}{k+1}$$

Calculations

$$\epsilon - \epsilon \circ (x, y + y(y-x)) = y(y-x)\Delta(x,y)$$
$$\epsilon(\Delta) = \sum_{j=1}^{\infty} \epsilon_j(\Delta)$$
$$L(\Delta) = \sum_{j=1}^{\infty} L_j(\Delta) x^j = \epsilon(x,x) - \epsilon(x,0)$$

$$\epsilon_{k+1}(y^k) = \frac{-y^{k+1}}{k+1}$$
$$\epsilon_{k+2}(y^k) = y^{k+1}\left(\frac{k}{2(k+2)}y - \frac{k}{2(k+1)}x\right)$$
$$\epsilon_{k+3}(y^k) = y^{k+1}\left(\frac{-k(k+5)}{12(k+3)}y^2 + \frac{k(2k+7)}{12(k+2)}xy - \frac{k(k+2)}{12(k+1)}x^2\right)$$

$$L_{k+1}(y^k) = \frac{-1}{k+1}$$
$$L_{k+2}(y^k) = \frac{-k}{(k+1)(k+2)}$$

Calculations

$$\epsilon - \epsilon \circ (x, y + y(y-x)) = y(y-x)\Delta(x,y)$$
$$\epsilon(\Delta) = \sum_{j=1}^{\infty} \epsilon_j(\Delta)$$
$$L(\Delta) = \sum_{j=1}^{\infty} L_j(\Delta) x^j = \epsilon(x,x) - \epsilon(x,0)$$

$$\epsilon_{k+1}(y^k) = \frac{-y^{k+1}}{k+1}$$
$$\epsilon_{k+2}(y^k) = y^{k+1}\left(\frac{k}{2(k+2)}y - \frac{k}{2(k+1)}x\right)$$
$$\epsilon_{k+3}(y^k) = y^{k+1}\left(\frac{-k(k+5)}{12(k+3)}y^2 + \frac{k(2k+7)}{12(k+2)}xy - \frac{k(k+2)}{12(k+1)}x^2\right)$$

$$L_{k+1}(y^k) = \frac{-1}{k+1}$$
$$L_{k+2}(y^k) = \frac{-k}{(k+1)(k+2)}$$
$$L_{k+3}(y^k) = \frac{k(k-20)}{12(k+1)(k+2)(k+3)}$$

Action of the one parameter group

$$\varphi_0 = (x, y + y(y - x)) = \exp(\hat{X}) = \exp\left(y(y - x)(1 + h.o.t.)\frac{\partial}{\partial y}\right)$$

From a difference equation to a differential equation

Action of the one parameter group

$$\varphi_0 = (x, y + y(y-x)) = \exp(\hat{X}) = \exp\left(y(y-x)(1 + h.o.t.)\frac{\partial}{\partial y} \right)$$

$$\epsilon - \epsilon \circ \varphi_0 = y(y-x)\Delta(x,y) \to L = \epsilon(x,x) - \epsilon(x,0)$$

From a difference equation to a differential equation

Action of the one parameter group

$$\varphi_0 = (x, y + y(y-x)) = \exp(\hat{X}) = \exp\left(y(y-x)(1 + h.o.t.)\frac{\partial}{\partial y} \right)$$

$$\epsilon - \epsilon \circ \varphi_0 = y(y-x)\Delta(x,y) \to L = \epsilon(x,x) - \epsilon(x,0)$$

$$(\epsilon \circ \varphi_0^t) - (\epsilon \circ \varphi_0^t) \circ \varphi_0 = (y(y-x)\Delta(x,y)) \circ \varphi_0^t$$

From a difference equation to a differential equation

Action of the one parameter group

$$\varphi_0 = (x, y + y(y-x)) = \exp(\hat{X}) = \exp\left(y(y-x)(1 + h.o.t.)\frac{\partial}{\partial y}\right)$$

$$\epsilon - \epsilon \circ \varphi_0 = y(y-x)\Delta(x,y) \to L = \epsilon(x,x) - \epsilon(x,0)$$

$$(\epsilon \circ \varphi_0^t) - (\epsilon \circ \varphi_0^t) \circ \varphi_0 = (y(y-x)\Delta(x,y)) \circ \varphi_0^t$$

$$\downarrow$$

$$L = \epsilon \circ \varphi_0^t(x,x) - \epsilon \circ \varphi_0^t(x,0) = \epsilon(x,x) - \epsilon(x,0)$$

From a difference equation to a differential equation

Action of the one parameter group

$$\varphi_0 = (x, y + y(y-x)) = \exp(\hat{X}) = \exp\left(y(y-x)(1+h.o.t.)\frac{\partial}{\partial y}\right)$$

$$\epsilon - \epsilon \circ \varphi_0 = y(y-x)\Delta(x,y) \to L = \epsilon(x,x) - \epsilon(x,0)$$

$$(\epsilon \circ \varphi_0^t) - (\epsilon \circ \varphi_0^t) \circ \varphi_0 = (y(y-x)\Delta(x,y)) \circ \varphi_0^t$$

$$\downarrow$$

$$L = \epsilon \circ \varphi_0^t(x,x) - \epsilon \circ \varphi_0^t(x,0) = \epsilon(x,x) - \epsilon(x,0)$$

$$\epsilon - \epsilon \circ \varphi_0 = \hat{X}(y(y-x)\Delta) = \lim_{t \to 0} \frac{(y(y-x)\Delta(x,y)) \circ \varphi_0^t - y(y-x)\Delta(x,y)}{t}$$

From a difference equation to a differential equation

Action of the one parameter group

$$\varphi_0 = (x, y + y(y-x)) = \exp(\hat{X}) = \exp\left(y(y-x)(1 + h.o.t.)\frac{\partial}{\partial y}\right)$$

$$\epsilon - \epsilon \circ \varphi_0 = y(y-x)\Delta(x,y) \to L = \epsilon(x,x) - \epsilon(x,0)$$

$$(\epsilon \circ \varphi_0^t) - (\epsilon \circ \varphi_0^t) \circ \varphi_0 = (y(y-x)\Delta(x,y)) \circ \varphi_0^t$$

$$\downarrow$$

$$L = \epsilon \circ \varphi_0^t(x,x) - \epsilon \circ \varphi_0^t(x,0) = \epsilon(x,x) - \epsilon(x,0)$$

$$\epsilon - \epsilon \circ \varphi_0 = \hat{X}(y(y-x)\Delta) = \lim_{t \to 0} \frac{(y(y-x)\Delta(x,y)) \circ \varphi_0^t - y(y-x)\Delta(x,y)}{t}$$

has vanishing linear transport mapping

From a difference equation to a differential equation

Action of the one parameter group

$$\varphi_0 = (x, y + y(y-x)) = \exp(\hat{X}) = \exp\left(y(y-x)(1 + h.o.t.)\frac{\partial}{\partial y}\right)$$

$$\epsilon - \epsilon \circ \varphi_0 = y(y-x)\Delta(x,y) \rightarrow L = \epsilon(x,x) - \epsilon(x,0)$$

$$(\epsilon \circ \varphi_0^t) - (\epsilon \circ \varphi_0^t) \circ \varphi_0 = (y(y-x)\Delta(x,y)) \circ \varphi_0^t$$

$$\downarrow$$

$$L = \epsilon \circ \varphi_0^t(x,x) - \epsilon \circ \varphi_0^t(x,0) = \epsilon(x,x) - \epsilon(x,0)$$

$$\epsilon - \epsilon \circ \varphi_0 = \hat{X}(y(y-x)\Delta) = \lim_{t \to 0} \frac{(y(y-x)\Delta(x,y)) \circ \varphi_0^t - y(y-x)\Delta(x,y)}{t}$$

has vanishing linear transport mapping

Consider a solution $\epsilon_0 \in \mathbb{C}[[x,y]]$ of $\hat{X}(\epsilon_0) = -y(y-x)\Delta(x,y)$

From a difference equation to a differential equation

Action of the one parameter group

$$\varphi_0 = (x, y + y(y-x)) = \exp(\hat{X}) = \exp\left(y(y-x)(1 + h.o.t.)\frac{\partial}{\partial y}\right)$$

$$\epsilon - \epsilon \circ \varphi_0 = y(y-x)\Delta(x,y) \to L = \epsilon(x,x) - \epsilon(x,0)$$

$$(\epsilon \circ \varphi_0^t) - (\epsilon \circ \varphi_0^t) \circ \varphi_0 = (y(y-x)\Delta(x,y)) \circ \varphi_0^t$$

$$\downarrow$$

$$L = \epsilon \circ \varphi_0^t(x,x) - \epsilon \circ \varphi_0^t(x,0) = \epsilon(x,x) - \epsilon(x,0)$$

$$\epsilon - \epsilon \circ \varphi_0 = \hat{X}(y(y-x)\Delta) = \lim_{t \to 0} \frac{(y(y-x)\Delta(x,y)) \circ \varphi_0^t - y(y-x)\Delta(x,y)}{t}$$

has vanishing linear transport mapping

Consider a solution $\epsilon_0 \in \mathbb{C}[[x,y]]$ of $\hat{X}(\epsilon_0) = -y(y-x)\Delta(x,y)$

$$\epsilon_0 - \epsilon_0 \circ \varphi_0 = y(y-x)\Delta - \sum_{j=2}^{\infty} \frac{\hat{X}^j(\epsilon_0)}{j!} \Rightarrow L(\Delta) = \epsilon_0(x,x) - \epsilon_0(x,0)$$

From a difference equation to a differential equation

Action of the one parameter group

$$\varphi_0 = (x, y + y(y-x)) = \exp(\hat{X}) = \exp\left(y(y-x)(1 + h.o.t.)\frac{\partial}{\partial y} \right)$$
$$\epsilon - \epsilon \circ \varphi_0 = y(y-x)\Delta(x,y) \rightarrow L = \epsilon(x,x) - \epsilon(x,0)$$
$$(\epsilon \circ \varphi_0^t) - (\epsilon \circ \varphi_0^t) \circ \varphi_0 = (y(y-x)\Delta(x,y)) \circ \varphi_0^t$$
$$\downarrow$$
$$L = \epsilon \circ \varphi_0^t(x,x) - \epsilon \circ \varphi_0^t(x,0) = \epsilon(x,x) - \epsilon(x,0)$$
$$\epsilon - \epsilon \circ \varphi_0 = \hat{X}(y(y-x)\Delta) = \lim_{t \to 0} \frac{(y(y-x)\Delta(x,y)) \circ \varphi_0^t - y(y-x)\Delta(x,y)}{t}$$

has vanishing linear transport mapping

Consider a solution $\epsilon_0 \in \mathbb{C}[[x,y]]$ of $\hat{X}(\epsilon_0) = -y(y-x)\Delta(x,y)$

$$\epsilon_0 - \epsilon_0 \circ \varphi_0 = y(y-x)\Delta - \sum_{j=2}^{\infty} \frac{\hat{X}^j(\epsilon_0)}{j!} \Rightarrow L(\Delta) = \epsilon_0(x,x) - \epsilon_0(x,0)$$

We replace $\epsilon - \epsilon \circ \varphi_0 = y(y-x)\Delta(x,y)$ with $\hat{X}(\epsilon_0) = -y(y-x)\Delta(x,y)$

From a difference equation to a differential equation

Theorem

Let $\Delta \in \mathbb{C}\{x, y\}$ with $\Delta(0,0) = 0$. Suppose that

$$\epsilon_0(x,x) - \epsilon_0(x,0) \notin \mathbb{C}\{x\}$$

for a solution $\epsilon_0 \in \mathbb{C}[[x, y]]$ of $(\log \varphi_0)(\epsilon_0) = -y(y-x)\Delta(x,y)$. Then $\varphi_\lambda = (x + \lambda y(y-x)\Delta(x,y), y + y(y-x))$ has no convergent normal form for all $\lambda \in \mathbb{C}$ outside a polar set.

From a difference equation to a differential equation

Theorem

Let $\Delta \in \mathbb{C}\{x, y\}$ with $\Delta(0,0) = 0$. Suppose that

$$\epsilon_0(x, x) - \epsilon_0(x, 0) \notin \mathbb{C}\{x\}$$

for a solution $\epsilon_0 \in \mathbb{C}[[x, y]]$ of $(\log \varphi_0)(\epsilon_0) = -y(y - x)\Delta(x, y)$. Then $\varphi_\lambda = (x + \lambda y(y - x)\Delta(x, y), y + y(y - x))$ has no convergent normal form for all $\lambda \in \mathbb{C}$ outside a polar set.

The theorem above is still true if we consider

$$\varphi_\lambda = \varphi_{\lambda,u} = (x + \lambda y(y - x)\Delta, y + y(y - x)u) \text{ with } u \in \mathbb{C}\{x, y\} \setminus (x, y)$$

From a difference equation to a differential equation

Theorem

Let $\Delta \in \mathbb{C}\{x, y\}$ with $\Delta(0,0) = 0$. Suppose that
$$\epsilon_0(x,x) - \epsilon_0(x,0) \notin \mathbb{C}\{x\}$$
for a solution $\epsilon_0 \in \mathbb{C}[[x,y]]$ of $(\log \varphi_0)(\epsilon_0) = -y(y-x)\Delta(x,y)$. Then $\varphi_\lambda = (x + \lambda y(y-x)\Delta(x,y), y + y(y-x))$ has no convergent normal form for all $\lambda \in \mathbb{C}$ outside a polar set.

The theorem above is still true if we consider
$$\varphi_\lambda = \varphi_{\lambda,u} = (x + \lambda y(y-x)\Delta, y + y(y-x)u) \text{ with } u \in \mathbb{C}\{x,y\} \setminus (x,y)$$

$\log \varphi_{0,u}$ is analytic $\implies$ Linear transport mapping is convergent

From a difference equation to a differential equation

Theorem

Let $\Delta \in \mathbb{C}\{x,y\}$ with $\Delta(0,0)=0$. Suppose that

$$\epsilon_0(x,x) - \epsilon_0(x,0) \notin \mathbb{C}\{x\}$$

for a solution $\epsilon_0 \in \mathbb{C}[[x,y]]$ of $(\log \varphi_0)(\epsilon_0) = -y(y-x)\Delta(x,y)$. Then $\varphi_\lambda = (x + \lambda y(y-x)\Delta(x,y), y + y(y-x))$ has no convergent normal form for all $\lambda \in \mathbb{C}$ outside a polar set.

The theorem above is still true if we consider

$$\varphi_\lambda = \varphi_{\lambda,u} = (x + \lambda y(y-x)\Delta, y + y(y-x)u) \text{ with } u \in \mathbb{C}\{x,y\} \setminus (x,y)$$

$\log \varphi_{0,u}$ is analytic $\implies$ Linear transport mapping is convergent

Divergence of linear transport mapping is not a finite determination property

The differential equation $(\log \varphi_0)(\epsilon_0) = y(y-x)\Delta$

$$\varphi_0 = (x, y + y(y-x)) = \exp\left(y(y-x)\hat{u}(x,y)\frac{\partial}{\partial y}\right), \ \hat{u} \in \mathbb{C}[[x,y]] \setminus (x,y)$$

Linear transport mapping $\to L(\Delta) = \epsilon_0(x,x) - \epsilon_0(x,0)$ with $\frac{\partial \epsilon_0}{\partial y} = \frac{\Delta(x,y)}{\hat{u}(x,y)}$

$$L(\Delta) = \sum_{j=1}^{\infty} L_j(\Delta)x^j$$

The differential equation $(\log \varphi_0)(\epsilon_0) = y(y-x)\Delta$

$$\varphi_0 = (x, y + y(y-x)) = \exp\left(y(y-x)\hat{u}(x,y)\frac{\partial}{\partial y} \right), \ \ \hat{u} \in \mathbb{C}[[x,y]] \setminus (x,y)$$

Linear transport mapping $\rightarrow L(\Delta) = \epsilon_0(x,x) - \epsilon_0(x,0)$ with $\frac{\partial \epsilon_0}{\partial y} = \frac{\Delta(x,y)}{\hat{u}(x,y)}$

$$L(\Delta) = \sum_{j=1}^{\infty} L_j(\Delta) x^j$$

Let B be the Banach space $\{\Delta = \sum \Delta_{jk} x^j y^k \in \mathbb{C}\{x,y\} : \sum |\Delta_{jk}| < \infty\}$

The differential equation $(\log \varphi_0)(\epsilon_0) = y(y-x)\Delta$

$$\varphi_0 = (x, y + y(y-x)) = \exp\left(y(y-x)\hat{u}(x,y)\frac{\partial}{\partial y}\right), \ \hat{u} \in \mathbb{C}[[x,y]] \setminus (x,y)$$

Linear transport mapping $\rightarrow L(\Delta) = \epsilon_0(x,x) - \epsilon_0(x,0)$ with $\frac{\partial \epsilon_0}{\partial y} = \frac{\Delta(x,y)}{\hat{u}(x,y)}$

$$L(\Delta) = \sum_{j=1}^{\infty} L_j(\Delta) x^j$$

Let B be the Banach space $\{\Delta = \sum \Delta_{jk} x^j y^k \in \mathbb{C}\{x,y\} : \sum |\Delta_{jk}| < \infty\}$

$L_j : B \rightarrow \mathbb{C}$ is a continuous operator $\Leftarrow$ Cauchy's formula

The differential equation $(\log \varphi_0)(\epsilon_0) = y(y-x)\Delta$

$$\varphi_0 = (x, y + y(y-x)) = \exp\left(y(y-x)\hat{u}(x,y)\frac{\partial}{\partial y}\right), \ \hat{u} \in \mathbb{C}[[x,y]] \setminus (x,y)$$

Linear transport mapping $\to L(\Delta) = \epsilon_0(x,x) - \epsilon_0(x,0)$ with $\frac{\partial \epsilon_0}{\partial y} = \frac{\Delta(x,y)}{\hat{u}(x,y)}$

$$L(\Delta) = \sum_{j=1}^{\infty} L_j(\Delta) x^j$$

Let B be the Banach space $\{\Delta = \sum \Delta_{jk} x^j y^k \in \mathbb{C}\{x,y\} : \sum |\Delta_{jk}| < \infty\}$

$L_j : B \to \mathbb{C}$ is a continuous operator $\Leftarrow$ Cauchy's formula

Uniform boundness principle

Either $L(\Delta) \notin \mathbb{C}\{x\}$ for all Δ in a dense subset of H or $\limsup_{j \to \infty} \sqrt[j]{||L_j||} < \infty$

The differential equation $(\log \varphi_0)(\epsilon_0) = y(y-x)\Delta$

$$\varphi_0 = (x, y + y(y-x)) = \exp\left(y(y-x)\hat{u}(x,y)\frac{\partial}{\partial y}\right), \ \hat{u} \in \mathbb{C}[[x,y]] \setminus (x,y)$$

$L(\Delta) = \epsilon_0(x,x) - \epsilon_0(x,0)$ with $\frac{\partial \epsilon_0}{\partial y} = \frac{\Delta(x,y)}{\hat{u}(x,y)}$. Denote $\frac{1}{\hat{u}(x,y)} = \sum v_{jk} x^j y^k$

The differential equation $(\log \varphi_0)(\epsilon_0) = y(y-x)\Delta$

$$\varphi_0 = (x, y + y(y-x)) = \exp\left(y(y-x)\hat{u}(x,y)\frac{\partial}{\partial y}\right), \ \hat{u} \in \mathbb{C}[[x,y]] \setminus (x,y)$$

$L(\Delta) = \epsilon_0(x,x) - \epsilon_0(x,0)$ with $\frac{\partial \epsilon_0}{\partial y} = \frac{\Delta(x,y)}{\hat{u}(x,y)}$. Denote $\frac{1}{\hat{u}(x,y)} = \sum v_{jk} x^j y^k$

Suppose $L(\Delta) \in \mathbb{C}\{x\}$ for all $\Delta \in \mathbb{C}\{x,y\} \implies \limsup_{j\to\infty} \sqrt[j]{||L_j||} < \infty$

$$\begin{pmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{k+1} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{k+2} \\ \vdots & \vdots & & \vdots \\ \frac{1}{k+1} & \frac{1}{k+2} & \cdots & \frac{1}{2k+1} \end{pmatrix} \begin{pmatrix} v_{k,0} \\ v_{k-1,1} \\ \vdots \\ v_{0,k} \end{pmatrix} = \begin{pmatrix} L_{k+1}(1) \\ L_{k+2}(y) \\ \vdots \\ L_{2k+1}(y^k) \end{pmatrix}$$

The differential equation $(\log \varphi_0)(\epsilon_0) = y(y-x)\Delta$

$$\varphi_0 = (x, y + y(y-x)) = \exp\left(y(y-x)\hat{u}(x,y)\frac{\partial}{\partial y}\right), \ \hat{u} \in \mathbb{C}[[x,y]] \setminus (x,y)$$

$L(\Delta) = \epsilon_0(x,x) - \epsilon_0(x,0)$ with $\frac{\partial \epsilon_0}{\partial y} = \frac{\Delta(x,y)}{\hat{u}(x,y)}$. Denote $\frac{1}{\hat{u}(x,y)} = \sum v_{jk} x^j y^k$

Suppose $L(\Delta) \in \mathbb{C}\{x\}$ for all $\Delta \in \mathbb{C}\{x,y\} \implies \limsup_{j\to\infty} \sqrt[j]{||L_j||} < \infty$

$$\begin{pmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{k+1} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{k+2} \\ \vdots & \vdots & & \vdots \\ \frac{1}{k+1} & \frac{1}{k+2} & \cdots & \frac{1}{2k+1} \end{pmatrix} \begin{pmatrix} v_{k,0} \\ v_{k-1,1} \\ \vdots \\ v_{0,k} \end{pmatrix} = \begin{pmatrix} L_{k+1}(1) \\ L_{k+2}(y) \\ \vdots \\ L_{2k+1}(y^k) \end{pmatrix}$$

$$||v_{k,0}, v_{k-1,1}, \ldots, v_{0,k}||_2 \leq ||Hilb_k^{-1}||_2 C^{2(k+1)}$$

The differential equation $(\log \varphi_0)(\epsilon_0) = y(y-x)\Delta$

$\varphi_0 = (x, y + y(y-x)) = \exp\left(y(y-x)\hat{u}(x,y)\frac{\partial}{\partial y}\right)$, $\hat{u} \in \mathbb{C}[[x,y]] \setminus (x,y)$

$L(\Delta) = \epsilon_0(x,x) - \epsilon_0(x,0)$ with $\frac{\partial \epsilon_0}{\partial y} = \frac{\Delta(x,y)}{\hat{u}(x,y)}$. Denote $\frac{1}{\hat{u}(x,y)} = \sum v_{jk} x^j y^k$

Suppose $L(\Delta) \in \mathbb{C}\{x\}$ for all $\Delta \in \mathbb{C}\{x,y\} \implies \limsup_{j\to\infty} \sqrt[j]{||L_j||} < \infty$

$$\begin{pmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{k+1} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{k+2} \\ \vdots & \vdots & & \vdots \\ \frac{1}{k+1} & \frac{1}{k+2} & \cdots & \frac{1}{2k+1} \end{pmatrix} \begin{pmatrix} v_{k,0} \\ v_{k-1,1} \\ \vdots \\ v_{0,k} \end{pmatrix} = \begin{pmatrix} L_{k+1}(1) \\ L_{k+2}(y) \\ \vdots \\ L_{2k+1}(y^k) \end{pmatrix}$$

$$||v_{k,0}, v_{k-1,1}, \ldots, v_{0,k}||_2 \leq ||Hilb_k^{-1}||_2 C^{2(k+1)}$$

$||\mathrm{Hilb}_k^{-1}||_2 = \frac{\rho^{4k}}{K\sqrt{k}}(1+o(1))$ (Kalyabin) where $K \in \mathbb{R}^+$ and $\rho = 1+\sqrt{2}$

The differential equation $(\log \varphi_0)(\epsilon_0) = y(y-x)\Delta$

$\varphi_0 = (x, y + y(y-x)) = \exp\left(y(y-x)\hat{u}(x,y)\frac{\partial}{\partial y}\right)$, $\hat{u} \in \mathbb{C}[[x,y]] \setminus (x,y)$

$L(\Delta) = \epsilon_0(x,x) - \epsilon_0(x,0)$ with $\frac{\partial \epsilon_0}{\partial y} = \frac{\Delta(x,y)}{\hat{u}(x,y)}$. Denote $\frac{1}{\hat{u}(x,y)} = \sum v_{jk} x^j y^k$

Suppose $L(\Delta) \in \mathbb{C}\{x\}$ for all $\Delta \in \mathbb{C}\{x,y\}$ $\implies$ $\limsup_{j\to\infty} \sqrt[j]{||L_j||} < \infty$

$$\begin{pmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{k+1} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{k+2} \\ \vdots & \vdots & & \vdots \\ \frac{1}{k+1} & \frac{1}{k+2} & \cdots & \frac{1}{2k+1} \end{pmatrix} \begin{pmatrix} v_{k,0} \\ v_{k-1,1} \\ \vdots \\ v_{0,k} \end{pmatrix} = \begin{pmatrix} L_{k+1}(1) \\ L_{k+2}(y) \\ \vdots \\ L_{2k+1}(y^k) \end{pmatrix}$$

$$||v_{k,0}, v_{k-1,1}, \ldots, v_{0,k}||_2 \leq ||Hilb_k^{-1}||_2 C^{2(k+1)}$$

$||\mathrm{Hilb}_k^{-1}||_2 = \frac{\rho^{4k}}{K\sqrt{k}}(1 + o(1))$ (Kalyabin) where $K \in \mathbb{R}^+$ and $\rho = 1 + \sqrt{2}$

$\log \varphi_0$ is convergent

Main theorem

Theorem

There exists $\Delta \in \mathbb{C}\{x, y\}$ *such that* $\varphi = (x + y(y - x)\Delta, y + y(y - x))$ *does not have a convergent normal form*

Main theorem

Theorem

There exists $\Delta \in \mathbb{C}\{x, y\}$ *such that* $\varphi = (x + y(y - x)\Delta, y + y(y - x))$ *does not have a convergent normal form*

$\log(x, y + y(y - x))$ is divergent

Main theorem

Theorem

There exists $\Delta \in \mathbb{C}\{x, y\}$ such that $\varphi = (x + y(y - x)\Delta, y + y(y - x))$ does not have a convergent normal form

$\log(x, y + y(y - x))$ is divergent

$\Downarrow$ (Group action + Unif. bound. principle)

$L(\Delta) \notin \mathbb{C}\{x\}$ for generic $\Delta \in \mathbb{C}\{x, y\}$

Main theorem

Theorem

There exists $\Delta \in \mathbb{C}\{x, y\}$ *such that* $\varphi = (x + y(y-x)\Delta, y + y(y-x))$ *does not have a convergent normal form*

$\log(x, y + y(y-x))$ is divergent

$\Downarrow$ (Group action + Unif. bound. principle)

$L(\Delta) \notin \mathbb{C}\{x\}$ for generic $\Delta \in \mathbb{C}\{x, y\}$

$\Downarrow$ (Potential theory + Homological equation)

$\varphi_\lambda = (x + \lambda y(y-x)\Delta, y + y(y-x))$

does not have convergent normal form for generic $\lambda \in \mathbb{C}$

Resume

Convergent normal forms

- Determining whether normalizing mappings are analytic

Resume

Convergent normal forms

- Determining whether normalizing mappings are analytic
- Exploiting the quasi-analiticity of normalizing mappings

Resume

Convergent normal forms

- Determining whether normalizing mappings are analytic
- Exploiting the quasi-analiticity of normalizing mappings
- Determining stable structures

Resume

Convergent normal forms

- Determining whether normalizing mappings are analytic
- Exploiting the quasi-analiticity of normalizing mappings
- Determining stable structures

Example of diffeomorphism with no convergent normal form

Based on the rigidity of dicritic structures

Resume

Convergent normal forms

- Determining whether normalizing mappings are analytic
- Exploiting the quasi-analiticity of normalizing mappings
- Determining stable structures

Example of diffeomorphism with no convergent normal form

Based on the rigidity of dicritic structures

Methods to prove divergence

No convergent normal form

Resume

Convergent normal forms

- Determining whether normalizing mappings are analytic
- Exploiting the quasi-analiticity of normalizing mappings
- Determining stable structures

Example of diffeomorphism with no convergent normal form

Based on the rigidity of dicritic structures

Methods to prove divergence

$$\begin{array}{c}\text{No convergent normal form}\\ \downarrow \\ \text{Divergence of infinitesimal generator}\end{array}$$

Resume

Convergent normal forms

- Determining whether normalizing mappings are analytic
- Exploiting the quasi-analiticity of normalizing mappings
- Determining stable structures

Example of diffeomorphism with no convergent normal form

Based on the rigidity of dicritic structures

Methods to prove divergence

$$\begin{array}{ccc} \text{No convergent normal form} & \longrightarrow & \text{Infinite determination} \\ \downarrow & & \\ \text{Divergence of infinitesimal generator} & & \end{array}$$

Resume

Convergent normal forms

- Determining whether normalizing mappings are analytic
- Exploiting the quasi-analiticity of normalizing mappings
- Determining stable structures

Example of diffeomorphism with no convergent normal form

Based on the rigidity of dicritic structures

Methods to prove divergence

$$\begin{array}{ccc} \text{No convergent normal form} & \rightarrow & \text{Infinite determination} \\ \downarrow & & \downarrow \\ \text{Divergence of infinitesimal generator} & \rightarrow & \text{Finite determination} \end{array}$$

Resume

Convergent normal forms

- Determining whether normalizing mappings are analytic
- Exploiting the quasi-analiticity of normalizing mappings
- Determining stable structures

Example of diffeomorphism with no convergent normal form

Based on the rigidity of dicritic structures

Methods to prove divergence

$$\begin{array}{ccc} \text{No convergent normal form} & \rightarrow & \text{Infinite determination} \\ \downarrow & & \downarrow \\ \text{Divergence of infinitesimal generator} & \rightarrow & \text{Finite determination} \\ & & \downarrow \\ & & \text{Finite world} \end{array}$$

Remarks

Theorem

Let $n > 1$. There exists $\varphi \in \mathrm{Diff}_1(\mathbb{C}^n, 0)$ with no convergent normal form.

Remarks

Theorem

Let $n > 1$. There exists $\varphi \in \mathrm{Diff}_1(\mathbb{C}^n, 0)$ with no convergent normal form.

Genericity

The diffeomorphism $\varphi = (x + y(y-x)\Delta, y + y(y-x)) \in \mathrm{Diff}\,(\mathbb{C}^2, 0)$ has no convergent normal form for $\Delta \in \mathbb{C}\{x, y\} \cap (x, y)$ generic.

Remarks

Theorem

Let $n > 1$. There exists $\varphi \in \mathrm{Diff}_1(\mathbb{C}^n, 0)$ with no convergent normal form.

Genericity

The diffeomorphism $\varphi = (x + y(y - x)\Delta, y + y(y - x)) \in \mathrm{Diff}(\mathbb{C}^2, 0)$ has no convergent normal form for $\Delta \in \mathbb{C}\{x, y\} \cap (x, y)$ generic.

Dicritic phenomenon

$\exists \Delta \in \mathbb{C}\{x, y\} \cap (x, y)$ such that $\varphi = (x + y^2(y - x)^2\Delta, y + y^2(y - x)^2)$ has no convergent normal form.