Infinitesimal Automorphisms of Webs

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Local Holomorphic Dynamics
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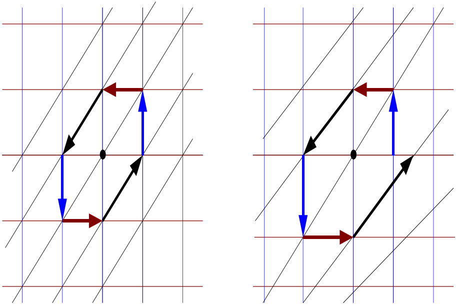
Outline

Holonomy



Following the leaves one obtains germs of diffeomorphisms in one variable whose equivalence class is a local invariant of the web. If all the possible germs are the identity the web is called **hexagonal**.

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$\kappa(\mathcal{W}) = d\gamma$ is canonically attached to $\mathcal{W}$ (the **curvature** of $\mathcal{W}$).

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4. $\mathcal{W}$ is equivalent to the web defined by x, y and $x - y$.

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$$\int \quad \Longrightarrow \quad \sum_{i=1}^{k} g_i(u_i) = 0$$

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$$\dim \frac{\mathcal{A}^j(\mathcal{W})}{\mathcal{A}^{j+1}(\mathcal{W})} \leq k - \dim \left(\mathbb{C} \cdot \ell_1^{j+1} + \cdots + \mathbb{C} \cdot \ell_k^{j+1} \right) = k - \min(j+2, k)$$

Algebraic Webs

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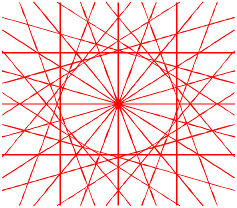

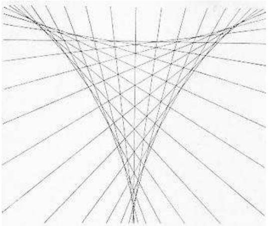

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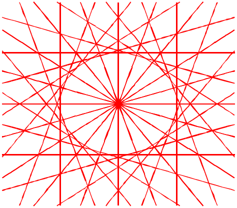

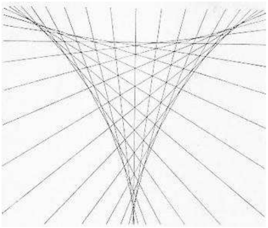

Abel's Theorem $\implies (\Psi_1 \oplus \cdots \oplus \Psi_k)^* \mathrm{H}^0(C, \omega_C) \hookrightarrow \mathcal{A}(\mathcal{W}_C)$

Converse to Abel's Theorem

$L_0 \in \check{\mathbb{P}}^2$ and $U \subset \mathbb{P}^2$ neighborhood of L_0.
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If $\Psi_1^*\omega + \cdots \Psi_k^*\omega = 0$ then there exists a **global** algebraic curve C and a **global** holomorphic 1-form ω on C such that $C_0 \subset C$ and $\omega_0 = \omega_{|C_0}$.

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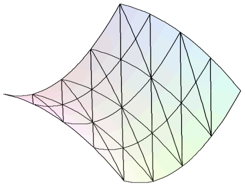
holomorphic = first kind w.r.t. lines = Rosenlicht

Lie's Original Formulation



A **double translation surface** $S \subset \mathbb{R}^3$ that admits two independent parametrizations of the form $(x, y) \mapsto f(x) + g(y)$. S carries a natural 4-web $\mathcal{W}$. The leaves tangents of $\mathcal{W}$ cuts the hyperplane at infinity at 4 germs of curves. Lie's Theorem says that these 4 curves are contained in a degree 4 algebraic curve. Latter generalized by Wirtinger to arbitrary translation manifolds.

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Proof. The **Poincaré Map**

$$\begin{array}{rcl} (\mathbb{C}^2, 0) & \to & \mathbb{P}(\mathcal{A}(\mathcal{W})) \\ p & \mapsto & \{(\eta_1, \ldots, \eta_4) \in \mathcal{A}(\mathcal{W}) \,|\, \eta_i(p) = 0, \forall i\} \end{array}$$

is a local diffeomorphism that linearizes $\mathcal{W}$.

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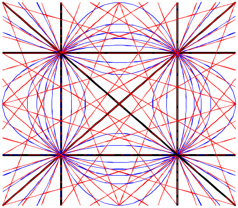

Bol's Exceptional 5-**web** $\mathcal{B}_5$. Four pencil of lines + one pencil of conics. All its 3-subwebs are hexagonal. For almost 70 years it remained the only known example of non-algebrizable 5-web of maximal rank. One of its abelian relations involves the dilogarithm.

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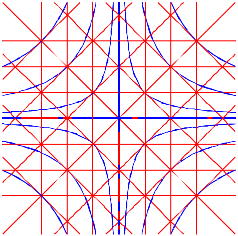
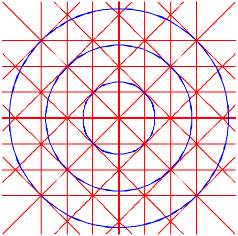
(Appeared already in Buzzano 1939. Not known to be exceptional.)

The simplest Examples





$$u(x,y) = x^2 \pm y^2$$

Polynomial and
Logarithmic
Abelian Relations

Outline

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(joint with David Marin and Luc Pirio)

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Canonical First Integral

$$u_j = \int \frac{\omega_j}{\omega_j(X)}$$

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$$\boxed{P_1(u_1)e^{\lambda_i u_1}du_1 + \cdots + P_k(u_k)e^{\lambda_i u_k}du_k = 0\,,}$$

where the P_j's are polynomials and the λ_i's are eigenvalues of $L_X : \mathcal{A}(\mathcal{W}) \circlearrowleft$.

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$$\boxed{\mathrm{rk}(\mathcal{W} \boxtimes \mathcal{F}_X) = \mathrm{rk}(\mathcal{W}) + (k-1)\,.}$$

$\mathcal{W}$ is of maximal rank if, and only if, $\mathcal{W} \boxtimes \mathcal{F}_X$ is of maximal rank.

Curves invariant by global holomorphic flows

Note that the curves cutted out by

$$x^{\epsilon_1} y^{\epsilon_2} z^{\epsilon_3} \prod_{i=1}^{n} \left(x^a y^b - \lambda_i z^{a+b} \right)$$

where $\epsilon_j \in \{0, 1\}$, $a, b, c \in \mathbb{N}$, $\lambda_i \in \mathbb{C}^*$,

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where $\epsilon_j \in \{0, 1\}$, $a, b, c \in \mathbb{N}$, $\lambda_i \in \mathbb{C}^*$, are left invariant by

$$\begin{aligned} \varphi : \mathbb{P}^2 \times \mathbb{C}^* &\to \mathbb{P}^2 \\ (t, [x : y : z]) &\mapsto [t^{b(a-b)} x : t^{a(a-b)} y : t^{ab} z] \end{aligned}$$

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$deg(C) \geq 4 \implies \mathcal{W}_C \boxtimes \mathcal{F}_X$ is exceptional.

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Remark: If $\mathcal{W}$ is a global web on $\mathbb{P}^2$ then $\kappa(\mathcal{W})$ is rational 2-form with poles on the discriminant of $\mathcal{W}$.

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$\beta(\mathcal{F}, \mathcal{W})$ is the $\mathcal{F}$-barycenter of $\mathcal{W}$.

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When $\deg(\mathcal{F}) = 1$ the web $\mathcal{W}$ is induced by one of the symmetric 1-forms below

Classification

$\mathcal{W} = \mathcal{F} \boxtimes \mathcal{L}_1 \boxtimes \cdots \boxtimes \mathcal{L}_k$
$\mathcal{L}_i$ pencil of *parallel* lines (base point at ∞)
If $k \geq 4$ and $\mathcal{W}$ is exceptional then $\deg(\mathcal{F}) \in \{1, 2, 3\}$.

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5. elementary representation theory to conclude.