Holomorphic actions of PSL(2, C) on compact complex threefolds

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Local holomorphic dynamics, Pisa
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GENERAL PROBLEM

G complex Lie group
$\Gamma \subset G$ discrete subgroup

$$\Gamma \backslash G \curvearrowleft G$$

homogenous space.

Q: does there exist $M \curvearrowleft G$
compact

$$\Gamma \backslash G \hookrightarrow M$$
equivariant?

A: Huckleberry, Oeljeklaus, Lescure
when $M \backslash (\Gamma \backslash G)$ is little

PARTICULAR PROBLEM

$\Gamma \backslash PSL(2,\mathbb{C})$

- T. Nakano '89 $\Big\}$ algebraic setting.
- Kebekus '00

For the action of Γ on $\mathbb{CP}^1$:

Λ - limit set

Ω - discontinuity set

$\Gamma \backslash \Omega$ is a (disconnected curve).

<u>Theorem</u> (G.)

a) $\Gamma \backslash PSL(2,\mathbb{C})$ admits an equivariant compactification if and only if
- finite number of ends
- same number of <u>compact</u> connected components of $\Gamma \backslash \Omega$

b) Any two compactifications are birationally equivalent if Γ is non-elementary

- A. Fujiki: in a) the condition is sufficient

A biequivariant compactification of $PSL(2,\mathbb{C})$.

$$SL(2,\mathbb{C}) \hookrightarrow \mathbb{C}P^4$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow [a:b:c:d:1]$$

the actions (left and right) extend to the closure

$$z_0 z_3 - z_1 z_2 - z_4^2 = 0.$$

- the quotient by the center gives a compactification of $PSL(2,\mathbb{C})$ by adding a doubly ruled surface

Action (to the left) at the boundary:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot (z_1, z_2, z_3)$$

$$\downarrow$$

$$\left(z_1 - \frac{c z_2}{c z_3 + d}, \frac{z_2}{(c z_3 + d)^2}, \frac{a z_3 + \overset{b}{\cancel{d}}}{c z_3 + d} \right)$$

- boundary: $\{ z_2 = 0 \}$
- rulings given by z_1 and z_3
- z_3 is a factor.

- The action of Γ (to the left) on

$$PSL(2,\mathbb{C}) \cup (\mathbb{CP}^1 \times \Omega)$$

is properly discontinuos.

- the quotient is an analytic space with a natural action of $PSL(2,\mathbb{C})$ $\quad M_\Gamma$

- Under the hypotheses, M_Γ is compact

◦ A. Fujiki: same construction

Cond eq

Γ geometrically finite
+ no parabolic elements

The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \longrightarrow \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$$

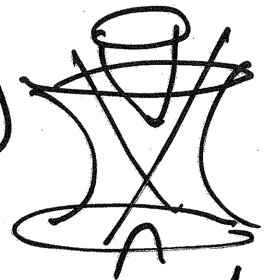
$$\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}$$

$\rightarrow$ vector fields on the manifold

Understand the locus
of linear dependence

1st fact: no fixed points!

Kushnirenko: linearization

$PSL(2,\mathbb{C}) \approx SO(3,\mathbb{C})$  2dimensional orbits

2nd : curves : $\mathbb{C}P^1$ compact

3nd fact : surfaces w/action (no fixed points)

curves
- $\mathbb{C}P^1 \times \Sigma$
- $\mathbb{C}P^1 \times \mathbb{C}P^1$
- $\mathbb{C}P^2$ $\quad (PSL(2,\mathbb{C}) \approx SO(3,\mathbb{C}))$
- Hirzebruck surfaces F_{2n}

no curves
- Elliptic fibrations over $\mathbb{C}P^1$

$T\,\mathbb{C}P^1 \setminus \{0\}$ action

$\mathbb{C} \subset \mathbb{C}^*$

Result:

Classification of germs of maximal, semicomplete Lie algebras of vector fields isom. to $sl(2,\mathbb{C})$

- up to hol. change of coords
- up to conjugacy within the group
- in the neighborhood of a point in a one-dimensional orbit in the closure of a three-d one

maximal: Palais

semicomplete: Rebelo

obstructions: Rebelo, Ghys-Rebelo, Reis, G.

In a neighborhood of curves

Recall: $\left\{ \frac{\partial}{\partial z},\ z\frac{\partial}{\partial z},\ z^2\frac{\partial}{\partial z} \right\}$

$\boxed{2\pi i \mathbb{R}}$

$$0 \to 2\pi i \mathbb{Z} \to \mathbb{C} \to \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} \to 0$$

M^3

$$\frac{\partial}{\partial z_1},\ \left[z_1 + f_1(z_2 z_3)\right]\frac{\partial}{\partial z_1} + f_2(z_2, z_3)\frac{\partial}{\partial z_2} + f_3(\quad)\frac{\partial}{\partial z_3}$$

$$\frac{\partial}{\partial z_1},\ z_1\frac{\partial}{\partial z_1} + \lambda_2 z_2 \frac{\partial}{\partial z_2} + \lambda_3 z_3 \frac{\partial}{\partial z_3}$$

$\lambda_i \in \mathbb{Z}$

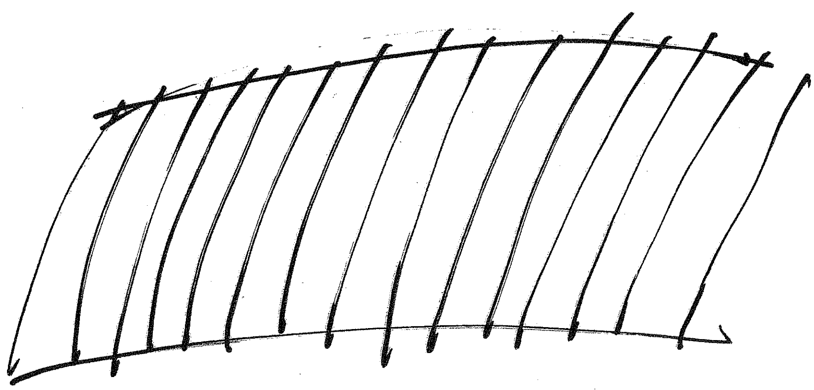
Non-reduced if $\lambda_2\lambda_3 > 0$

Type I – Reduced invariant curves

$$\frac{\partial}{\partial z_1}, \quad z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}$$

$$z_1^2 \frac{\partial}{\partial z_1} + 2 z_1 z_2 \frac{\partial}{\partial z_2} + z_2 \frac{\partial}{\partial z_3}$$

$$\Delta = z_2^2$$



$\leftarrow \mathbb{C}P^1 \times \Sigma$

Type II

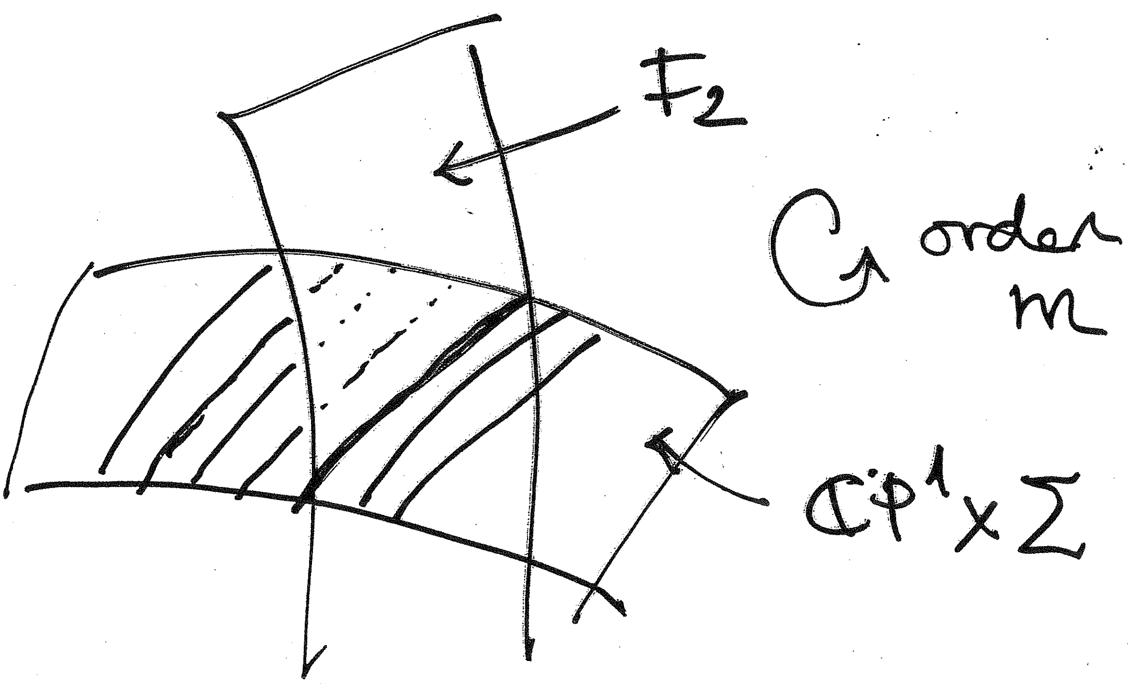
$$\frac{\partial}{\partial z_1}\,,\quad z_1\frac{\partial}{\partial z_1}+z_2\frac{\partial}{\partial z_2}$$

$$\left(z_1^2\,\cancel{z}+\frac{1}{4m^2}\,z_2^2\,z_3^{2q}\right)\frac{\partial}{\partial z_1}$$

$$+\;z_2\left(2z_1-q\,z_2z_3^q\right)\frac{\partial}{\partial z_2}+z_2z_3^{q+1}\frac{\partial}{\partial z_3}$$

$$\Delta = z_2^2 z_3^q$$

- $q>0$
- $m\in\mathbb{Z}\cup\{\infty\}$



Type III

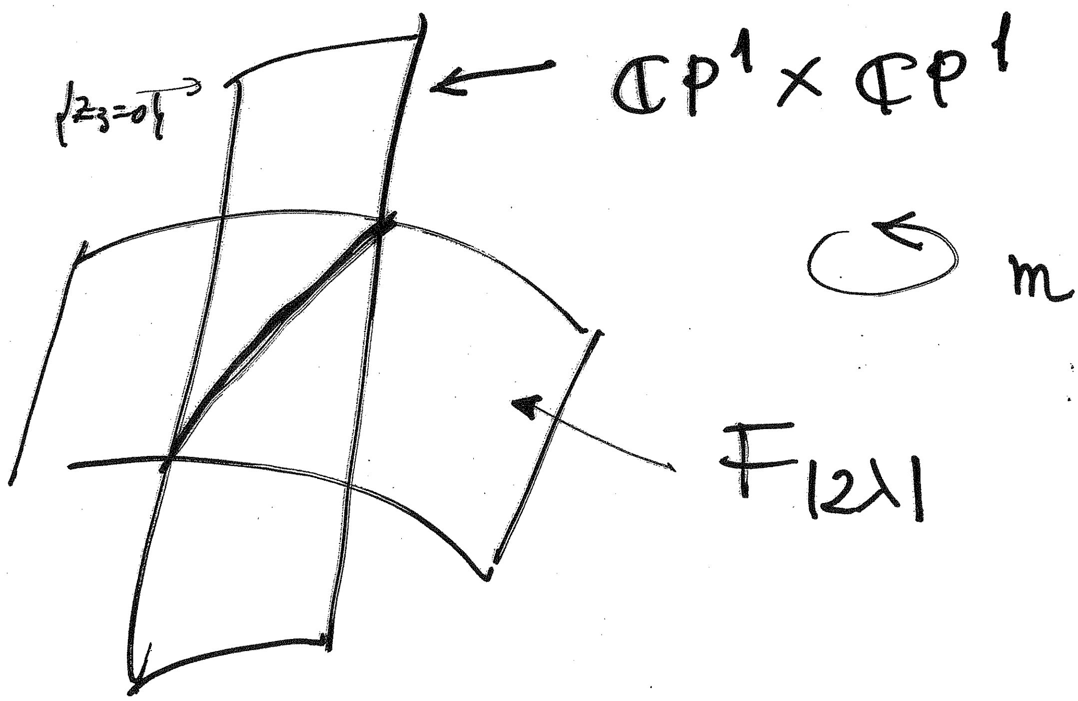
$$\frac{\partial}{\partial z_1}\ , \quad z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + \lambda z_3\frac{\partial}{\partial z_3}\ ,$$

$$\left(z_1^2 + \frac{1}{4m^2}z_2^2\right)\frac{\partial}{\partial z_1} + 2z_1z_2\frac{\partial}{\partial z_2}$$

$$+ z_3(2\lambda z_1 + z_2)\frac{\partial}{\partial z_3}$$

$$\Delta = z_2^2 z_3$$

- $\lambda \in \mathbb{Z},\ \lambda < 0\ ;\quad m \in \mathbb{Z} \cup \{\infty\}$

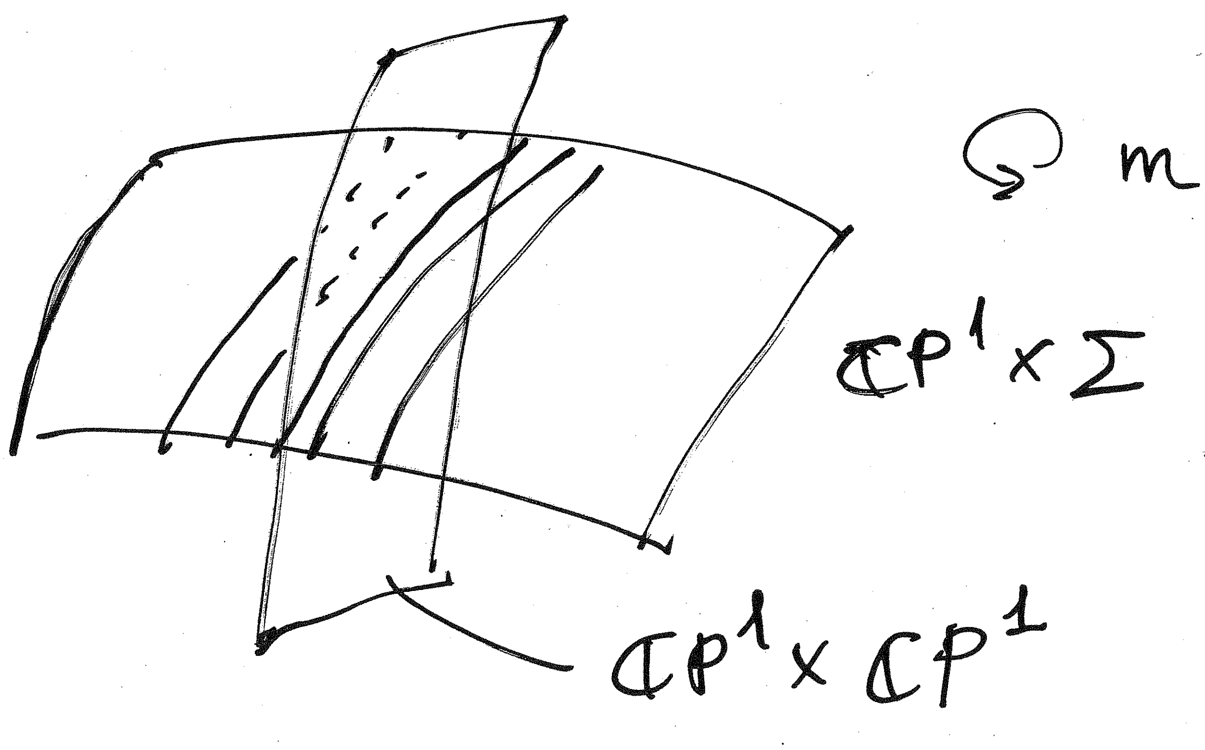


Type IV

$$\frac{\partial}{\partial z_1}, \quad z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}$$

$$\left(z_1^2 + \frac{1}{4m^2} z_2^2\right)\frac{\partial}{\partial z_1} + 2 z_1 z_2 \frac{\partial}{\partial z_2}$$
$$+ z_2 z_3 \frac{\partial}{\partial z_3}$$

- $m \in \mathbb{Z}$ $\qquad \Delta = z_2^2 z_3$



Type V

$$\frac{\partial}{\partial z_1},\ z_1\frac{\partial}{\partial z_1}+\lambda_2 z_2\frac{\partial}{\partial z_2}+\lambda_3 z_3\frac{\partial}{\partial z_3}$$

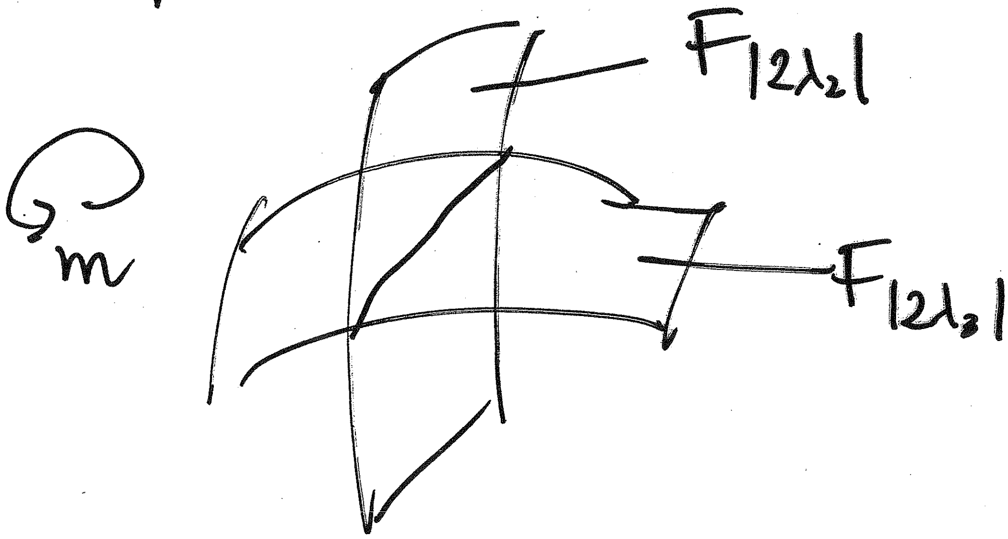
$$\left(z_1+\frac{1}{4m^2}z_2^{2a_2}z_3^{2a_3}\right)\frac{\partial}{\partial z_1}$$

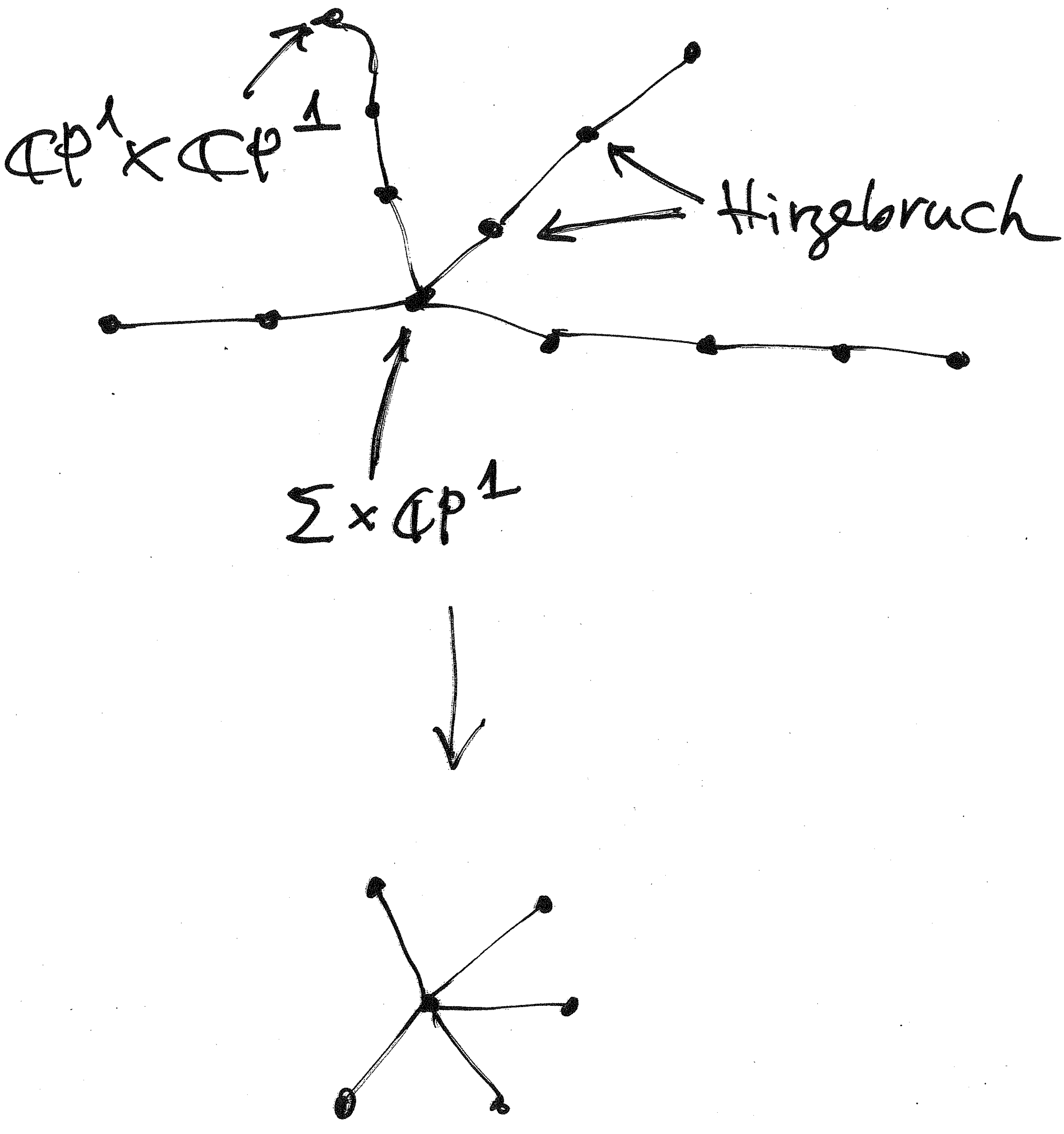
$$+\ z_2\left(2\lambda_2 z_1-a_3 z_2^{a_2}z_3^{a_3}\right)\frac{\partial}{\partial z_2}$$

$$+\ z_3\left(2\lambda_3 z_1+a_2 z_2^{a_2}z_3^{a_3}\right)\frac{\partial}{\partial z_3}$$

$$\Delta = z_2^{a_2+1}z_3^{a_3+1}$$

- $m \in \mathbb{Z}\cup\{\infty\}$
- $\lambda_i, a_i \in \mathbb{Z}$, $\lambda_2, a_i > 0$ $\lambda_3 < 0$
- $a_2\lambda_2 + a_3\lambda_3 = 1$





This is the desingularisation of M_Γ

Ends compactified by
homogenous elliptic fibrations

$\vdots$
$\downarrow$

can change the compactif.
to one of the form

$T \times \mathbb{C}P^1$

$\uparrow$
elliptic curve

$\Downarrow$

the group Γ is elementary.