# Semi-complete vector fields of saddle-node type in dimension 3

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# Complex Differential Equations

Given a holomorphic vector field $X$, we consider the differential equation

$$\dot{x} = X(x)$$

It defines a 1-dimensional foliation.
The existence Theorem of solutions guarantees a local solution for the differential equation for each initial condition.

**Case** $T \in \mathbb{R}$ - It is possible to define a maximal solution: constructed by analytic continuation.
$U_x$ - maximal domain of definition of the solution
$\Omega = \{(T, x) : T \in U_x\}$
Flow of $X$:

$$\begin{array}{rccc} \Phi : & \Omega & \rightarrow & \mathbb{C}^n \\ & (t, x) & \mapsto & (\phi_1^x(t), \ldots, \phi_n^x(t)) \end{array}$$

**Case** $T \in \mathbb{C}$ - In general, this is not possible

# Semi-complete vector fields

## Definition

*$M$ - complex manifold*
*$X$ - holomorphic vector field defined on $U$, $U \subseteq M$.*
*We say that $X$ is semi-complete in $U$ if there exists*

$$\Phi : \Omega \subseteq \mathbb{C} \times U \to U$$

*holomorphic, $\{0\} \times U \subseteq \Omega$, such that*

a) $\Phi(0, x) = x \qquad \forall x \in M$

b) $X(x) = \dfrac{d}{dT}|_{T=0}\Phi(T, x)$

c) $\Phi(T_1 + T_2, x) = \Phi(T_2, \Phi(T_1, x))$, *when both members are defined*

d) $(T_i, x) \in \Omega$ *e* $(T_i, x) \to \partial\Omega \quad \Rightarrow \quad \Phi(T_i, x)$ *escapes from any compact subset of $U$*

## Proposition (Rebelo)

*M - complex manifold*
*X - complete holomorphic vector field defined on M*
*The restriction of X to any connected open set U ($U \subseteq M$) is semi-complete in U.*

$$\forall L \text{ regular leaf }, \exists dT_L \text{ 1-form such that} dT_L.X \equiv 1$$

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  $\Rightarrow \int_c dT_L \neq 0 \ \forall c$ *embedded*

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  $\Rightarrow \int_c dT_L \neq 0 \ \forall c$ *embedded*
- *X holomorphic vector field on U*
  $\forall c : [0,1] \to L$, *L regular leaf,* $c(0) \neq c(1)$, $\int_c dT_L \neq 0$
  $\Rightarrow$ *X is semi-complete in U*

## Example

- $X = x\frac{\partial}{\partial x}$ *is complete (semi-complete)*
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- $X = x^3\frac{\partial}{\partial x}$ *is not semi-complete*
  $c(t) = e^{\pi i t},\ t \in [0,1],$
  $\int_c dT = \int_c \frac{dx}{x^3} = \left[-\frac{1}{x^2}\right]_1^{-1} = 0$

## Lemma (Rebelo)

*$X$ 1-dimensional meromorphic vector field $X = f(x)\partial/\partial x$*
*$X$ semi-complete $\Rightarrow$ $X$ admits a holomorphic extension to the origin and $J_0^2 X \not\equiv 0$.*
*$f(0) = f'(0) = 0 \Rightarrow X$ analytically conjugated to $x^2\partial/\partial x$*

## Theorem (Rebelo)

*$X$ holomorphic vector field on $\mathbb{C}^2$*
*$p$ isolated singularity of $X$*
*$X$ semi-complete $\Rightarrow J_p^2 X \not\equiv 0$*

**Camacho/Sad**'s Theorem allows us to reduce the problem in dimension 2 to the 1-dimensional case.

## Conjecture (Ghys)

*$X$ holomorphic vector field on* $(\mathbb{C}^3, 0)$
$0 \in \mathbb{C}^3$ *isolated singularity*
*$X$ semi-complete*
$\Rightarrow J_0^2 X \neq 0$

The semi-complete vector fields in dimension 2, with an isolated singularity, were completely classified by **Ghys** and **Rebelo**.

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- The characterization in dimension 2 uses the Seidenberg's Theorem: In dimension 3 we have some results of dessingularization (Cano, Panazzolo), but not so powerful like the Seidenberg one for dimension 2.
- In dimension 2: the exceptional divisor is an algebraic invariant curve In dimension 3: the exceptional divisor is $\mathbb{CP}(2)$ not admitting, in general, an algebraic invariant curve
- Irreducible cases:
  - exactly 3 eigenvalues different from zero
  - exactly 2 eigenvalues different from zero
  - exactly 1 eigenvalues different from zero

  The are no acceptable normal forms for the last case.

# Semi-complete vector fields in dimension 3

**Saddle-node**

All results presented here are valid for higher dimension since $(\lambda_1 \ldots, \lambda_{n-1})$ belongs to the Poincaré domain.

$\mathcal{F}$ - saddle-node foliation

normal form

$$X : \begin{cases} \dot{x} = x^{p+1} \\ \dot{y} = y\lambda_1 + xA(x,y,z) \\ \dot{z} = z\lambda_2 + xB(x,y,z) \end{cases}$$

formal normal form

$$Y : \begin{cases} \dot{x} = x^{p+1} \\ \dot{y} = y(\lambda_1 + \alpha_1 x) \\ \dot{z} = z(\lambda_2 + \alpha_2 x) \end{cases}$$

## Proposition

*$\mathcal{F}$ admits a semi-complete representative* $\Rightarrow p = 1$

## Proof.

Let $L$ be a regular leaf, $L \not\subseteq \{x = 0\}$

$$dT = \frac{dx}{x^{p+1}}, \quad c(t) = (re^{2\pi it/p}, 0, 0) \Rightarrow \int_{c_L} dT = 0$$

where $c_L$ is the lift of $c$ to $L$.
$p \geq 2 \Rightarrow c_L$ is an embedded curve $\Rightarrow X$ cannot be semi-complete.

For $fX$, the time form is given by $dT = \frac{dx}{x^{p+1}f(x,y,z)}$. It is sufficient to restrict to a sector of amplitude greater than $2\pi/p$ but less than $2\pi$. The problem is reduced to the 1-dimensional case since in such sector $y = y(x)$ and $z = z(x)$.

□

### Theorem

*Let $\mathcal{F}$ be a saddle-node foliation on $(\mathbb{C}^3, 0)$, 0 isolated singularity. $\mathcal{F}$ is associated to a semi-complete vector field iff it admits*

$$\begin{cases} \dot{x} = x^2 \\ \dot{y} = y(\lambda_1 + \alpha_1 x) \\ \dot{z} = z(\lambda_2 + \alpha_2 x) \end{cases}$$

*as normal form, with $(\alpha_1, \alpha_2) \in \mathbb{Z}^{n-1}$.*

The proof is divided in two steps:

**Proposition**

*X semi-complete, then*

1. *X has holomorphic central manifold (hcm)*
2. *the holonomy relative to the hcm is trivial*

The reciprocal is also valid.

**Proposition**

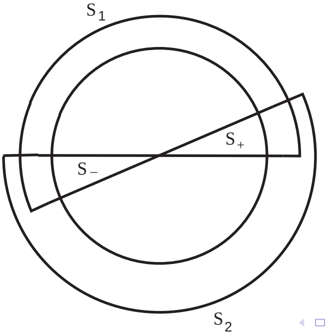
*X has hcm and the holonomy is trivial iff X is analytically conjugated to its formal normal form and* $(\alpha_1, \alpha_2) \in \mathbb{Z}^2$.

## Theorem (Theorem of Malmquist)

*Let $\hat{H}$ be the unique formal transformation conjugating $X$ and its formal normal form. There exist holomorphic transformations $H_{1,2}$ defined on sectors $S_{1,2} \times (\mathbb{C}^{n-1}, 0)$, covering a neighborhood of the origin, such that*

a) *$H_{1,2}$ is a holomorphic conjugation between $X$ and its formal normal form*

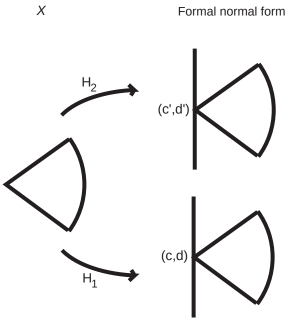
b) *$H_{1,2} \tilde{\rightarrow} \hat{H}$ in $S_{1,2}$, as $x \rightarrow 0$*



Solution of the formal normal form:

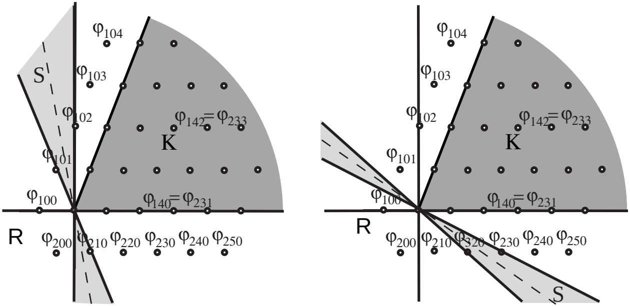
$$\begin{cases} y(x) = cx^{\alpha_1} e^{-\frac{\lambda_1}{x}} \\ z(x) = dx^{\alpha_2} e^{-\frac{\lambda_2}{x}} \end{cases}$$

$(c, d)$ works like a parametrization of the leaf




Let $g_+ = (H_2 \circ H_1^{-1})|_{S_+}$ and $g_- = (H_2 \circ H_1^{-1})|_{S_-}$

Let $\varphi_{i,Q} = (Q, \lambda) - \lambda_i$




on the left case

$$g_+ : \{(c,d) \mapsto (c + a_{100} + a_{101} d, d + a_{200})\}$$

while on the right one

$$g_+ : \{(c,d) \mapsto (c + a_{100}, d + a_{300} + a_{210} c + a_{220} c^2)\}.$$

In the first case

$$g_- : \{(c,d) \mapsto (c + \sum_{\substack{Q\neq(1,0)\\Q\neq(0,0)\\Q\neq(0,1)}} a_{1ij}c^i d^j, d + \sum_{\substack{Q\neq(0,1)\\Q\neq(0,0)}} a_{2ij}c^i d^j\}.$$

**Proposition**

*$\mathcal{F}$ admits a hcm iff $a_{100} = a_{200} = 0$.*

## Lemma

*$X$, as above, is semi-complete in a neighbourhood of the origin $\Rightarrow$ there is no translation, i.e., $a_{100} = a_{200} = 0$.*

## Proof.

$\{y = 0, z = 0\}$ - hcm for the formal normal form
$L \supset H_1^{-1}(\{y = 0, z = 0\})$
$c(t) = (re^{2\pi it}, 0, 0)$, $t \in [0, 1]$, $c_L$ - the lift of $c$ to $L$

$$\int_{c_L} dT_L = \int_{re^{2\pi it}} \frac{dx}{x^2} = 0$$

$X$ is semi-complete $\Rightarrow$ $c_L$ is closed $\Rightarrow$ $(a_{100}, a_{200}) = (0, 0) \Rightarrow X$ has hcm

□

## Lemma

*$X$ semi-complete $\Rightarrow$ holonomy relative to the hcm is trivial.*

## Proof.

$L_0$ - hcm
$c_0$ - lift of $c$ to $L_0$, which is closed
$L$ regular leaf, $L \not\subseteq \{x = 0\}$
$c_L$ - lift of $c$ to $L$

$$\int_{c_L} dT_L = \int_{e^{2\pi it}} \frac{dx}{x^2} = 0$$

$X$ semi-complete $\Rightarrow$ $c_L$ is closed for all $L$ $\Rightarrow$ the holonomy is trivial

□

Other representatives must be considered, but conclusions are the same.

- Assume there is no holomorphic central manifold/ holonomy is not trivial
- The curve $c_L$ cannot be closed
- It is possible to continue the curve. The new curve is embedded and the integral over the new curve vanishes.

**Proposition**

*$X$ has hcm and the holonomy is trivial iff $X$ is analytically conjugated to its formal normal form and $(\alpha_1, \alpha_2) \in \mathbb{Z}^2$.*

The idea of the proof is to use induction over $|Q|$:

$$a_{iQ} = 0, \quad \forall Q : |Q| \leq q : a_{iQ} \text{ in } g_+$$

$$\Rightarrow a_{iQ} = 0, \quad \forall Q : |Q| = q + 1 : a_{iQ} \text{ in } g_+$$

The hypothesis is verified for $|Q| = 0$

$g_+$ is polynomial $\Rightarrow$ the induction stops in a finite number of steps.

By the appearence of the terms expressions $x^{\alpha_i}$ in the solutions of the formal normal form

$$\text{trivial holonomy} \Rightarrow (\alpha_1, \alpha_2) \in \mathbb{Z}^2$$

In the 3-dimensional case

$$g_+ : \{(c, d) \mapsto (c + a_{100} + a_{101} d, d + a_{200})\}$$

$$g_- \circ g_+ = id \quad \Leftrightarrow \begin{cases} ((c + a_{101} d) + \sum_{\substack{Q \neq (1,0) \\ Q \neq (0,0) \\ Q \neq (0,1)}} a_{1ij} (c + a_{101} d)^i d^j) e^{2\pi i \alpha_1} = c \\ ((d + \sum_{\substack{Q \neq (0,1) \\ Q \neq (0,0)}} a_{2ij} (c + a_{101} d)^i d^j) e^{2\pi i \alpha_2} = d \end{cases}$$

$$g_- \circ g_+ = id \quad \leftrightarrow \quad n-1 \text{ equations}$$

Assume $a_{iQ} = 0, \forall Q : |Q| \leq q$ such that $a_{iQ} \in g_+$

Fix $Q_0$ such that $|Q_0| = q+1$ and $a_{iQ_0} \in g_+$ for some $i$

Assume $\{i : (Q_0, \gamma) - \gamma_i \in R\} = \{1, \ldots, k\}$
$\Rightarrow c^{Q_0}$ can appear only on the $i^{th}$ equation when $Q = e_j$ is allowed in the $i^{th}$ component of $g_-$, for some $j = 1, \ldots, k-1$.

Thus we get the system

$$\begin{pmatrix} 1 & a_{1e_2} & a_{1e_3} & \ldots & a_{1e_k} \\ a_{2e_1} & 1 & a_{2e_3} & \ldots & a_{2e_k} \\ a_{3e_1} & a_{3e_2} & 1 & \ldots & a_{3e_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{ke_1} & a_{ke_2} & a_{ke_3} & \ldots & 1 \end{pmatrix} . \begin{pmatrix} a_{1Q_0} \\ a_{2Q_0} \\ a_{3Q_0} \\ \vdots \\ a_{kQ_0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

### Property

$a_{iQ}$ *in* $g_{-} \Leftrightarrow (Q,\lambda) - \lambda_i \notin R$

$$a_{ie_j}: \quad (e_j,\lambda) - \lambda_i = \lambda_j - \lambda_i$$
$$a_{je_i}: \quad (e_i,\lambda) - \lambda_j = \lambda_i - \lambda_j$$

$$\lambda_j - \lambda_i \notin R \Rightarrow \lambda_i - \lambda_j \in R$$

It is possible to rearrange the variable in order that

$$A = \begin{pmatrix} 1 & a_{1e_2} & a_{1e_3} & \dots & a_{1e_k} \\ 0 & 1 & a_{2e_3} & \dots & a_{2e_k} \\ 0 & 0 & 1 & \dots & a_{3e_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Thus, by induction $g_+ = id \Rightarrow g_- = id$ also.

### Proposition

*Up to analytic conjugation, two vector fields of saddle-node type, with hcm, are analytically equivalent iff the holonomies relatively to the hcm are analytically conjugated.*