The dynamics of maps tangent to the identity and with non vanishing index

Laura Molino
Università di Parma
laura.molino@unipr.it

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OBJECTS

Germs about the origin of holomorphic self-maps of $\mathbb{C}^2$ fixing the origin

$$f \in \mathrm{End}(\mathbb{C}^2, O),$$

such that

$$df_O = id.$$

PURPOSE

Investigate conditions ensuring the existence of parabolic curves, tangent to a given direction $v \in \mathbb{C}^2 \setminus \{O\}$, for $f \in \mathrm{End}(\mathbb{C}^2, O)$, with $df_O = id$.

Let $f \in \mathrm{End}(\mathbb{C}^2, O)$, with $df_O = id$.

Definition. A **parabolic curve** for f at the origin O, is an injective holomorphic map $\varphi\colon \Delta \to \mathbb{C}^2$ satisfying the following properties:

(i) Δ is a simply connected domain in $\mathbb{C}$, with $0 \in \partial\Delta$;

(ii) φ is continuous at the origin, and $\varphi(0) = O$;

(iii) $\varphi(\Delta)$ is invariant under f and

$$(f|_{\varphi(\Delta)})^k \to O, \quad \text{as} \quad k \to \infty.$$

Furthermore, if $[\varphi(\zeta)] \to [v] \in \mathbb{P}^1(\mathbb{C})$ as $\zeta \to 0$, where $[\cdot]$ denotes the canonical projection of $\mathbb{C}^2 \setminus \{O\}$ onto $\mathbb{P}^1(\mathbb{C})$, we say that φ is **tangent** to $[v]$ at the origin.

LITERATURE

Fatou (1924, in $\mathbb{C}^2$)

Écalle (1985, in $\mathbb{C}^n$)

Ueda (1997, in $\mathbb{C}^2$)

Weickert (1998, in $\mathbb{C}^2$)

Hakim (1998, in $\mathbb{C}^n$)

Abate (2001, in $\mathbb{C}^2$)

STRATEGY

We use an **Index** (Abate):

1) **technique** of Hakim;

2) after a finite sequence of **blow-ups**, we study the lifted map in a suitable **corner**.

NECESSARY CONDITION ON $[v]$

$[v] \in \mathbb{P}^1(\mathbb{C})$ must be a **characteristic direction**.

Proposition. [Écalle; Hakim]
Let $f \in \mathrm{End}(\mathbb{C}^n, O)$, with $df_O = id$.
Assume that a point Z, in the domain of f, has an orbit $\{Z_k\}_{k\in\mathbb{N}}$ (where $Z_k := f^k(Z)$) which converges to O, tangently to a complex direction; that is, for some $v \in \mathbb{C}^n \setminus \{O\}$, we have

$$\lim_{k\to\infty} Z_k = O \qquad \text{and} \qquad \lim_{k\to\infty} [Z_k] = [v].$$

Then $[v]$ is a **characteristic direction** for f.

Let $f \in \mathrm{End}(\mathbb{C}^2, O)$, with $df_O = id$. Writing $f = (f_1, f_2)$, let

$$f_j = z_j + P_{j,\nu_j} + P_{j,\nu_j+1} + \cdots$$

be the homogeneous expansion of f_j in series of homogeneous polynomials, where

$$\deg P_{j,k} = k \quad (\text{or} P_{j,k} \equiv 0) \quad \text{and} \quad P_{j,\nu_j} \not\equiv 0$$

with $j = 1, 2$.

Definition. The **order** of f is defined by

$$\nu(f) = \min\{\nu_1, \nu_2\}.$$

Definition. A **characteristic direction** for f is a point $[v] := [v_1 : v_2] \in \mathbb{P}^1(\mathbb{C})$ such that there is $\lambda \in \mathbb{C}$ so that

$$P_{j,\nu(f)}(v_1, v_2) = \lambda v_j, \qquad j = 1, 2.$$

If $\lambda \neq 0$ we say that $[v]$ is **nondegenerate**; if $\lambda = 0$ we say that $[v]$ is **degenerate**.

Definition. The origin $O \in \mathbb{C}^2$ is called **dicritical** for f, if we have

$$z_2 P_{1,\nu(f)}(z_1, z_2) \equiv z_1 P_{2,\nu(f)}(z_1, z_2).$$

Remark: If O is dicritical for $f \Rightarrow$ all points $[v] \in \mathbb{P}^1(\mathbb{C})$ are characteristic directions for f. Proof of the Abate Theorem $\Rightarrow$ infinitely many parabolic curves tangent to all directions (unless a finite number).

We work in **absence** of dicritical fixed points.

SUFFICIENT CONDITION ON $[v]$

Theorem. [Écalle (1985); Hakim (1998)]
Let f be a (germ of) holomorphic self-map of $\mathbb{C}^n$ fixing the origin and tangent to the identity. Then for every nondegenerate characteristic direction $[v]$ of f there are $\nu(f)-1$ parabolic curves tangent to $[v]$ at the origin.

Example:

$$\begin{cases} f_1(z,w) = z + zw + w^2 - z^3 + O(z^2w, zw^2, w^3, z^4), \\ f_2(z,w) = w[1 + z + w + O(z^2, zw, w^2)]. \end{cases}$$

O is a non-dicritical fixed point,
$[1:0] \in \mathbb{P}^1(\mathbb{C})$ is a degenerate characteristic direction.

The line $\{w=0\}$ is f-invariant and the Leau-Fatou flower Theorem $\Rightarrow$
2 attractive petals on $\{w=0\}$ $\Rightarrow$
2 parabolic curves tangent to $[1:0]$.

THE PROOF OF HAKIM (IDEA)

Let $f \in \text{End}(\mathbb{C}^2, O)$, with $df_O = id$, and assume $[v] = [1:0]$ is a nondegenerate characteristic direction for f.

Blowing-up f at the origin and studying the lifted map in a chart centered at $[1:0]$, we get

$$\begin{cases} \tilde{f}_1(u,v) = u - u^{\nu(f)} + O(u^{\nu(f)}v, u^{\nu(f)+1}), \\ \tilde{f}_2(u,v) = v[1 - \lambda u^{\nu(f)-1} + O(u^{\nu(f)}, u^{\nu(f)-1}v)] \\ \qquad\qquad\qquad + O(u^{\nu(f)}), \end{cases}$$

$[v]$ nondegenerate characteristic direction $\Rightarrow$ $\lambda \in \mathbb{C}$ is an invariant.

Changes of variables involved:

$\lambda \notin \mathbb{N}^* \longrightarrow$ polynomials (easy case),

$\lambda \in \mathbb{N}^* \longrightarrow$ polynomials of degree $< \lambda$ and logarithms (hard case).

Example: Assume $\lambda = 1$

$$\begin{cases} \tilde{f}_1(u,v) = u - u^{\nu(f)} + O(u^{\nu(f)}v, u^{\nu(f)+1}), \\ \tilde{f}_2(u,v) = v[1 - u^{\nu(f)-1} + O(u^{\nu(f)}, u^{\nu(f)-1}v)] \\ \qquad\qquad + \alpha_1 u^{\nu(f)} + \alpha_2 u^{\nu(f)+1} + \cdots \\ \qquad\qquad + \alpha_k u^{\nu(f)+k-1} + O(u^{\nu(f)+k}). \end{cases}$$

If we want to kill α_1, α_2, ...,α_k we need to perform

$$\begin{cases} U := u, \\ V := v + \alpha_1 u \log u + u^2 Q_2(\log u) + \cdots + u^k Q_k(\log \end{cases}$$

where Q_j are polynomial solutions of suitable differential equations.

After that, we get

$$\begin{cases} \tilde{f}_1(U,V) = U - U^{\nu(f)} + O(U^{\nu(f)}V, U^{\nu(f)+1}\log U), \\ \tilde{f}_2(U,V) = V[1 - \lambda U^{\nu(f)-1} + O(U^{\nu(f)}\log U) \\ \qquad\qquad + O(U^{\nu(f)-1}V)] + O(U^{\nu(f)+k}(\log U)^{p_k} \end{cases}$$

where p_k is 'related' to $\deg(Q_k)$.

To find parabolic curves for $\tilde{f} \iff$ to find a fixed point for a certain functional operator T.

T is a contraction on a suitable closed subset of a Banach space.

Banach space

$\mathcal{E} := \{V \in \mathrm{Hol}(\Delta, \mathbb{C}) \,|\, V(\zeta) = \zeta^2 V^o(\zeta),\ ||V^o||_\infty < \infty\}$

where $\delta > 0$ and $\Delta := \{\zeta \in \mathbb{C} \,|\, |\zeta^{\nu(f)-1} - \delta| < \delta\}$

Functional operator

$$(TV)(\zeta) := \zeta^\lambda \sum_{j=0}^{\infty} \zeta_j^{-\lambda} H(\zeta_j, V(\zeta_j))$$

with $V \in \mathcal{E}$, where $\zeta_j := (\tilde{f}_1(\zeta, V(\zeta)))^j$ and $H(U,V) := V - \frac{U^\lambda}{U_1^\lambda} V_1$, where $U_1 := \tilde{f}_1(U,V)$ $V_1 := \tilde{f}_2(U,V)$.

For δ small enough and on a suitable subset of

$\{V \in \mathcal{E} \mid V(\zeta) = \zeta^{k+1}(\log \zeta)^{p_k} V^o(\zeta), \, ||V^o||_\infty < \infty\}$

we get
T is a **contraction** $\Rightarrow \exists \, \hat{V}$ **fixed point**

Conclusion: the restriction of

$$\varphi^{\hat{V}}(\zeta) := (\zeta, \hat{V}(\zeta))$$

to any connected component of

$$\Delta := \{\zeta \in \mathbb{C} \mid |\zeta^{\nu(f)-1} - \delta| < \delta\}$$

gives us $\nu(f) - 1$ parabolic curves for $\tilde{f}$.

Theorem. [Abate] Let $f \in \text{End}(\mathbb{C}^2, O)$ with $df_O = id$ and such that the origin $O \in \mathbb{C}^2$ is an isolated fixed point. Then there exist (at least) $\nu(f) - 1$ parabolic curves for f at the origin.

THE PROOF OF ABATE (IDEA)

After a **finite** number of blow-ups we study a more simple map.

After one blow-up $\rightsquigarrow$ $\tilde{f} \in \text{End}(M, S)$,
with $\tilde{f}_{|S} = id_S$, where M is the blow-up of $\mathbb{C}^2$ at the origin and S is the exceptional divisor.

How to single out the points on S on which perform next blow-ups? Answer $\rightsquigarrow$ **singular points**.

Let S be a compact 1-dimensional submanifold of a 2-dimensional complex manifold M, and let $\tilde{f} \in \mathrm{End}(M, S)$ such that $\tilde{f}_{|S} = id_S$.

Proposition. [Abate]
Assume that $\tilde{f}$ is **tangential** to S. Let $p \in S$ be not **singular** and not a corner. Then no infinite orbit of $\tilde{f}$ can stay arbitrarily close to p; that is, there exists a neighbourhood U of p such that for all $q \in U$ either the orbit of q lands on S or $f^{n_0}(q) \notin U$ for some $n_0 \in \mathbb{N}$. In particular, no infinite orbit is converging to p.

At any step of the blowing-up we have a **finite** number of singular points.

After a **finite** number of blow-ups we get a more simple map F:

$$\begin{cases} F_1(u,v) = u - u^{\nu(f)} + O(u^{\nu(f)}v, u^{\nu(f)+1}), \\ F_2(u,v) = v[1 - \lambda u^{\nu(f)-1} + O(u^{\nu(f)}, u^{\nu(f)-1}v)] \\ \qquad\qquad + O(u^{\nu(f)}), \end{cases}$$

with $\lambda \notin \mathbb{N}^*$.

Proof of Hakim Theorem (easy case) $\Rightarrow$ parabolic curves for F. Pushing forward them we get parabolic curves for the original map f.

TOOL

Index (introduced by Abate) allows to get the map F.

At any step of the sequence of blow-ups we can define

$$\mathrm{Ind}(\tilde{f}, S, p),$$

where S is a component of $\mathrm{Fix}(\tilde{f})$, and $p \in S$.

Two types of situations:

Ind$(\tilde{f}, S, p) \equiv \infty$ ($\forall\ p \in S$): $\tilde{f}$ is called **non-tangential** to S;

Ind$(\tilde{f}, S, p) \in \mathbb{C}$: $\tilde{f}$ is called **tangential** to S.

Non-tangential map upstairs $\Leftrightarrow$ dicritical point downstairs.

Proposition. [Abate]
Let $f \in$ End$(\mathbb{C}^2, O)$ be such that $df_O = id$. Let M be the blow-up of $\mathbb{C}^2$ at the origin O, let $S \subset M$ be the exceptional divisor and let $\tilde{f} \in$ End(M, S) be the blow-up of f. Then $\tilde{f}$ is non-tangential to $S \iff$ the origin O is dicritical for f.

Absence of dicritical points $\Leftrightarrow$ we can **avoid** non-tangential situation.

A **combinatorial argument** (inspired by Camacho and Sad) leads to

Corollary. [Abate]
Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ with $df_O = id$ and such that the origin is an isolated fixed point. Let $[v] \in \mathbb{P}^1$ be a characteristic direction for f such that $\mathrm{Ind}(\tilde{f}, \mathbb{P}^1, [v]) \notin \mathbb{Q}^+$ ($\mathbb{P}^1$ is the exceptional divisor of the blow-up of the origin, and $\tilde{f}$ is the blow-up of f). Then there are (at least) $\nu(f) - 1$ parabolic curves for f tangent to $[v]$ at the origin.

Aim of the talk: improve previous Corollary, using a **different method**.

Example

$$\begin{cases} f_1(z,w) = z + zw + O(w^2, z^3, z^2w), \\ f_2(z,w) = w + 2w^2 + bz^3 + z^4 + O(z^5, z^2w, zw^2, w \end{cases}$$

with $b \neq 0$.
$[v] := [1:0]$ is a degenerate characteristic direction for f and $\mathrm{Ind}(\tilde{f}, \mathbb{P}^1, [v]) = 1$.
($\tilde{f}$ is the blow-up of f at the origin).

Let $g \in \mathcal{O}_2$. g has a homogeneous expansion as an infinite sum of homogeneous polynomials

$$g = P_0 + P_1 + \cdots, \text{with} \deg P_j = j (\text{or} P_j \equiv 0);$$

Definition. The **order** of g is defined by $\nu(g) := \min\{j \geq 0 \mid P_j \not\equiv 0\}$.

Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ with $df_O = id$.

$$\begin{cases} f_1 = z + g = z + lg^o, \\ f_2 = w + h = w + lh^o. \end{cases}$$

where $l := \gcd\{g, h\}$, and g, $h \in \mathcal{O}_2$.

Definition. The **pure order** of f at the origin is $\nu_o(f, O) = \min\{\nu(g^o), \nu(h^o)\}$.

Definition. We say that the origin $O \in \mathbb{C}^2$ is a **singular point** for f if $\nu_o(f, O) \geq 1$.

Definition. We say that the origin $O \in \mathbb{C}^2$ is a **corner** for f if $\mathrm{Fix}(f)$ has at least two local components intersecting at O.

After the blow-ups we deal with $\tilde{f} \in \mathrm{End}(M,S)$ such that $\tilde{f}_{|S} = id_S$ and $d\tilde{f}$ acts as the identity on the normal bundle of S in M (S is a 1-dimensional submanifold of a complex 2-manifold M).

How can we extend the previous definitions? By choosing a chart of M, centered at $p \in S$, and considering the local expression of $\tilde{f}$.
In particular $\nu_o(\tilde{f}, p)$ is well-defined.

Definition. Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ with $df_O = id$ and let $[v] \in \mathbb{P}^1$ a characteristic direction for f. We say that f is **regular along** $[v]$ if $\nu_o(\tilde{f}, [v]) = 1$.

RESIDUAL INDEX

Let $\tilde{f} \in \text{End}(M, S)$ such that $\tilde{f}_{|S} = id_S$ and $d\tilde{f}$ acts as the identity on the normal bundle of S in M (S is a 1-dimensional submanifold of a complex 2-manifold M).

In a adapted chart $\{z = 0\}$ centered at $p \in S$ we can introduce

$$k(w) := \lim_{z \to 0} \frac{\tilde{f}_1(z, w)}{z[\tilde{f}_2(z, w) - w]}.$$

Definition.
We say $\tilde{f}$ **non-tangential** to S if $k \equiv \infty$, otherwise we say $\tilde{f}$ **tangential** to S.

Definition.
If $\tilde{f}$ is tangential to S, we call the **residual index** of $\tilde{f}$ at p along S the number

$$\text{Ind}(\tilde{f}, S, p) := \text{Res}_0(k(w)).$$

Theorem. [Molino] Let S be a 1-dimensional submanifold of a complex 2-manifold M and let $F \in \text{End}(M, S)$ be such that $F|_S = id_S$. Assume that dF acts as the identity on the normal bundle of S in M and let F be tangential to S. If $p \in S$ is a singular point of F, not a corner, with $\nu_o(F, p) = 1$ and $\text{Ind}(F, S, p) \neq 0$ then there exist parabolic curves for F in p.

Corollary. [Molino] Let $f \in \text{End}(\mathbb{C}^2, O)$ such that $df_O = id$ and with the origin as a non-dicritical isolated fixed point. Let $[v] \in \mathbb{P}^1$ be a characteristic direction for f and assume f is regular along $[v]$ with $\text{Ind}(\tilde{f}, \mathbb{P}^1, [v]) \neq 0$ ($\mathbb{P}^1$ is the exceptional divisor of the blow-up of the origin, and $\tilde{f}$ is the blow-up of f). Then there exist parabolic curves for f tangent to $[v]$ at the origin.

Theorem $\Rightarrow$ Corollary: put $F := \tilde{f}$.

IDEA OF THE PROOF

In a chart $\{z = 0\}$ adapted to S and centered at $p \in S$ we can write

$$\begin{cases} F_1(z,w) = z + z^r A_1(z,w), \\ F_2(z,w) = w + z^r B_1(z,w), \end{cases}$$

for suitable A_1, $B_1 \in \mathcal{O}_2$ and $r \in \mathbb{N}^*$.

p not a corner $\Rightarrow$ $\gcd(A_1, B_1) = 1$.

$\nu_o(F,p) = 1 \Rightarrow \min\{\nu(A_1), \nu(B_1)\} = 1 \Rightarrow$ $\nu(F) = r + 1$.

F tangential to $S \Rightarrow A_1(z,w) = zA_0(z,w)$, with $\nu(A_0) \geq 0$.

Let

$$\begin{cases} A_0(z,w) = a_{0,0} + a_{1,0}z + a_{0,1}w + a_{2,0}z^2 + \cdots, \\ B_1(z,w) = b_{1,0}z + b_{0,1}w + b_{2,0}z^2 + b_{1,1}zw + \cdots, \end{cases}$$

be the homogeneous expansion of A_0 and B_1.

$\gcd(A_1, B_1) = 1$ and $A_1(z,w) = zA_0(z,w) \Rightarrow z$ does not divide $B_1(z,w) \Rightarrow b_{0,j} \neq 0$ for some $j \geq 1$.

An easy calculation shows that

$$\begin{aligned} \mathrm{Ind}(F,S,p) &= \mathrm{Res}_0 \left(\frac{A_0(0,w)}{B_1(0,w)} \right) \\ &= \mathrm{Res}_0 \left(\frac{a_{0,0} + a_{0,1}w + a_{0,2}w^2 + \cdots}{b_{0,1}w + b_{0,2}w^2 + b_{0,3}w^3 + \cdots} \right). \end{aligned}$$

Set

$$m := \min\{h \in \mathbb{N} \mid a_{0,h} \neq 0\},$$

$$n := \min\{j \in \mathbb{N}^* \mid b_{0,j} \neq 0\}.$$

Remark. $\mathrm{Ind}(F,S,p) \neq 0 \Rightarrow m < n$.

Theorem 1. Assume that either

(a) $m < n - 1$, or

(b) $m = n - 1$ and $\text{Ind}(F, S, p) \neq n$, or

(c) $m = 0$, $n = 1$ and $\text{Ind}(F, S, p) = 1$.

Then there exist (at least) $r+m(r+1)$ parabolic curves for F at the origin.

Theorem 2. Let $n \geq 2$. If $m = n - 1$ and $\text{Ind}(F, S, p) = n$, then there exist $r+1$ parabolic curve for F at the origin.

PROOF THEOREM 1 (IDEA)

case (b) (with $n = 1$) $\Rightarrow$ F has a nondegenerate characteristic direction.

case (a), (b) (with $n > 1$). Studying $\tilde{F}^{[m]}$ in a chart centered at the **corner** $\tau^m(p)$ $\rightsquigarrow$ $\tilde{F}^{[m]}$ has a nondegenerate characteristic direction.

case (c). F has the following form

$$\begin{cases} F_1(z,w) = z - z^{r+1} + O(z^{r+2}, z^{r+1}w), \\ F_2(z,w) = w[1 - z^r + O(z^{r+1}, z^r w)] + O(z^{r+1}). \end{cases}$$

Proof of Hakim Theorem (hard case, with her invariant $\lambda = 1$) $\Rightarrow$ parabolic curves.

PROOF THEOREM 2 (IDEA)

$$\begin{cases} F_1(z,w) = z + a_{0,n-1}z^{r+1}w^{n-1} + a_{1,0}z^{r+2} \\ \qquad\qquad + O(z^{r+3}, z^{r+2}w, z^{r+1}w^n), \\ F_2(z,w) = w[1 + b_{0,n}z^r w^{n-1} + b_{1,1}z^{r+1} \\ \qquad\qquad + O(z^{r+2}, z^{r+1}w, z^r w^n)] \\ \qquad\qquad + b_{1,0}z^{r+1} + O(z^{r+2}), \end{cases}$$

with $r \in \mathbb{N}^*$ and $b_{1,0} \neq 0$, because $\nu_o(F,p) = 1$.

STRATEGY: We use the technique of Hakim.

PROBLEMS:

- find analytic changes of variables to shift the terms $O(z^{r+1})$ in $F_2(z,0)$;

- find a term z^s in F_1, in order to apply the Leau-Fatou result (or something else): we need to know how fast $\zeta_k \to 0$ (as $k \to \infty$), where $\zeta_k := (F_1(\zeta, w(\zeta)))^k$ and with $w(\zeta)$ belonging to a suitable space of functions;

• find the functional operator which realizes the parabolic curves for F as its fixed point. It will be defined in terms of the Index;

• prove that this functional operator, restricted to a closed convex subset of a suitable Banach space, is contracting.

Going back to the proof of Hakim we have, in a chart centered at a nondegenerate characteristic direction $[V] \in \mathbb{P}^1$,

$$\begin{cases} \tilde{f}_1(u,v) = u - u^{\nu(f)} + O(u^{\nu(f)}v, u^{\nu(f)+1}), \\ \tilde{f}_2(u,v) = v[1 - \lambda u^{\nu(f)-1} + O(u^{\nu(f)}, u^{\nu(f)-1}v)] \\ \qquad\qquad + O(u^{\nu(f)}), \end{cases}$$

where $\lambda \in \mathbb{C}$ is the invariant associated to $[V]$.

Remark. If $\lambda \neq 0 \Rightarrow \text{Ind}(\tilde{f}, \mathbb{P}^1, [V]) = \frac{1}{\lambda}$.

IDEA. Since in our setting $\text{Ind}(F, S, p) = n$, the idea is to put $\frac{1}{n}$ where there was λ, in the proof of Hakim.

Changes of variables (Hakim, with $\lambda = 1$)

$$\begin{cases} U := u, \\ V := v + \alpha_1 u \log u \\ \qquad + u^2 Q_2(\log u) + \cdots + u^k Q_k(\log u), \end{cases}$$

where Q_j are polynomial solutions of suitable differential equations.

Changes of variables (Molino)

$$\begin{cases} Z := z, \\ W := w + \alpha_1 z^{\frac{1}{n}} (\log z)^{\frac{1}{n}} \\ \qquad + z^{\frac{2}{n}} Q_2\Big((\log z)^{\frac{1}{n}}\Big) + \cdots + z^{\frac{k}{n}} Q_k\Big((\log z)^{\frac{1}{n}}\Big), \end{cases}$$

where Q_j are holomorphic solutions of suitable differential equations.

Fot the shifted map:

Banach space (Hakim)

$$\mathcal{E} := \Big\{ V \in \text{Hol}(\Delta, \mathbb{C}) \,|\, V(\zeta) = \zeta^{k+1} (\log \zeta)^{p_k} V^o(\zeta), \\ ||V^o||_\infty < \infty \Big\}$$

where $\delta > 0$, $\Delta := \{\zeta \in \mathbb{C} \,|\, |\zeta^r - \delta| < \delta\}$ and p_k is 'related' to the degree of the polynomial Q_k.

Banach space (Molino)
(k:=2n-3 is enough)

$$\mathcal{E} := \Big\{ V \in \text{Hol}(\Delta, \mathbb{C}) \,|\, V(\zeta) = \zeta^2 (\log \zeta)^{\frac{p_k}{n}} V^o(\zeta), \\ ||V^o||_\infty < \infty \Big\}$$

where $\delta > 0$,

$$\Delta := \Big\{ \zeta \in \mathbb{C} \,|\, |\zeta^{r + \frac{n-1}{n}} (\log z)^{\frac{n-1}{n}} - \delta| < \delta \Big\}$$

and p_k is 'related' to the asymptotic expansion of Q_k

Functional operator (Hakim)

$$(TV)(\zeta) := \zeta^{\lambda} \sum_{j=0}^{\infty} \zeta_j^{-\lambda} H(\zeta_j, V(\zeta_j))$$

with $H(z,w) := w - \frac{z^{\lambda}}{z_1^{\lambda}} w_1$.

Functional operator (Molino)

$$(TV)(\zeta) := \zeta^{\frac{1}{n}} \sum_{j=0}^{\infty} \zeta_j^{-\frac{1}{n}} H(\zeta_j, V(\zeta_j))$$

with $H(z,w) := w - \frac{z^{\frac{1}{n}}}{z_1^{\frac{1}{n}}} w_1$.

$z_1 := F_1(z,w)$, $w_1 := F_2(z,w)$
and $\zeta_j := (F_1(\zeta, w(\zeta)))^j$, where F is the shifted map.