Local Structure of Singular Foliations

Pisa, 23 / Jan / 2007

<u>Disclaimer</u>: I know nothing about complex dynamics

<u>Main Reference</u>: Chapter 6 of a small monographe entitled "Poisson Structures and their normal forms", Progress in Math., Vol 242, 2005. <u>Joint Work with J-P DUFOUR</u>

<u>1st Question</u>: What is a <u>linear</u> singular foliation?

* For a vector field X with a singular point O : $X(O) = 0$, one can talk about the linear part $X^{(1)}$ of X.

If $X = X^{(1)}$ $(X \cong X^{(1)})$ then one says that X is linear (linearizable)

* Now consider a singular foliation $\mathcal{F}$ and assume that O is a sing. point of $\mathcal{F}$. What does it mean for $\mathcal{F}$ to be <u>linear</u> in a neighborhood of O?

Think about it $\dashrightarrow$ (at least) 2 non-equivalent possible answers:

<u>Answer 1</u> $\mathcal{F}$ is linear if it is generated by linear vector fields. In other words, $\mathcal{F}$ is generated by a linear representation of a Lie algebra.

<u>Answer 2</u> $\mathcal{F}$ is linear if it is generated by a linear <u>integrable q-vector field</u> or a <u>linear integrable diff. p-form</u>
(q = dim, p = codim of $\mathcal{F}$)

Now, what's a <u>quadratic</u> singular foliation?

<u>Answer 1'</u> $\rightsquigarrow$ problems, because if X, Y are quadratic vector fields then $[X, Y]$ is cubic! (Quadratic vector fields don't form a Lie alg.; "quadratic action" does not make much sense in general)

__Answer 2'__ $\longrightarrow$ no problems, one can talk about (quasi) homogeneous integrable q-vector fields / p-forms.

__Integrable q-vector fields__ = q-dim generalization of vector fields

- vector field $\approx$ singular 1-dim foliation + contravariant leaf-wise volume forme
- integrable q-vector field $\approx$ singular q-dim fol. + contravariant leaf-wise volume forme.

__Def__ A q-vector field Λ is called __integrable__ if $\forall x$ s.t. $\Lambda(x) \neq 0$ $\exists$ local coord. syst. $(z_1, \dots, z_n)$ in which

$$\Lambda = f(z) \frac{\partial}{\partial z_1} \wedge \dots \wedge \frac{\partial}{\partial z_q}$$

[locally, the leaves are spanned by $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_q}$]

Remarks 1) Λ integrable $\leadsto$ $\mathcal{F}_\Lambda$ - associated (singular) q-dim foliation

2) $\forall$ sing. foliation can be essentially obtained from an integrable q-vector field ($\mathcal{F}$ & $\mathcal{F}'$ are essentially the same if corresponding distributions coincide almost everywhere)

3) One can talk about commuting integrable multi-vector fields

Λ - integrable k-vector field

Π - ——— l-vector field

$$[\Lambda, \Pi] = 0 \quad \text{(Schouten bracket)}$$

$\leadsto$ notion of commuting foliations (to be studied?)

__Lem__ Λ, Π commute and $\Lambda \wedge \Pi(0) \neq 0$ then $\exists$ local coord. syst. $z_1, \dots, z_n$ in which

$$\Lambda = \frac{\partial}{\partial z_1} \wedge \dots \wedge \frac{\partial}{\partial z_k}, \quad \Pi = \frac{\partial}{\partial z_{k+1}} \wedge \dots \wedge \frac{\partial}{\partial z_{k+l}}$$

(simultaneous rectification)

__Cor__ Λ, Π commute $\Rightarrow$ $\Lambda \wedge \Pi$ also integrable.

4) Integrable q-vector fields $(q \neq 2)$ are also called __Nambu structure__ (this name is invented by Takhtajan in 1994).

Nambu (1970s):

volume forme in $\mathbb{R}^3$ + 2 functions $\Rightarrow$ vector fied

(symplectic form + 1 function $\Rightarrow$ vector field)

Integrable p-forms

Idea: Annulator should be integrable distribution of corank p.
(gives rise to foliation of codim p)

Def A p-form ω is called <u>integrable</u> iff

$$\begin{cases} \text{(i)} \quad i_A \omega \wedge \omega = 0 \\ \text{(ii)} \quad i_A \omega \wedge d\omega = 0 \end{cases} \quad \forall \ (p-1)\text{-vector } A$$

<u>Remarks</u> 1) When $p = 1$ then (i) is empty and (ii) means

$$\omega \wedge d\omega = 0$$

2) $\Omega = dz_1 \wedge \dots \wedge dz_n$ volume form in a local coord system, Λ is a q-vector field, $w = \Lambda \lrcorner \Omega$ a p-form ($p + q = n$). Then

$$\Lambda \text{ is integrable} \iff \omega \text{ is integrable}$$

Linear Foliations Λ (or ω) is called linear if its coeff. (in a given coord. syst.) are linear.

$$\boxed{\begin{array}{c} \Lambda \lrcorner (dz_1 \wedge \dots \wedge dz_n) = \omega \\ \Lambda \text{ linear} \iff \omega \text{ linear} \end{array}}$$

Thm (Dufour-Z 1999; de Medeiros 2000) Classification of linear integrable p-forms / q-vector fields: They are of 2 types:

$n = p+q$

$q \geq r \geq -1$

$q-r \geq s \geq 0$

Type I $\omega = dz_1 \wedge \ldots \wedge d\_{p-1} \wedge dQ$

where $$Q = \sum_{j=p}^{p+r} \pm z_j^2/2 + \sum_{i=1}^{s} z_i z_{p+r+i}$$

(Λ is such that $\Lambda \lrcorner (dz_1 \wedge \ldots \wedge dz_n) = \omega$)

Type II $$\Lambda = \frac{\partial}{\partial z_1} \wedge \ldots \wedge \frac{\partial}{\partial z_{q-1}} \wedge \underbrace{\left(\sum_{i,j=q}^{n} a_{ij} z_i \frac{\partial}{\partial z_j} \right)}_{X}$$

$$\omega = \Lambda \lrcorner (dz_1 \wedge \ldots \wedge dz_n)$$

a_{ij} – constant

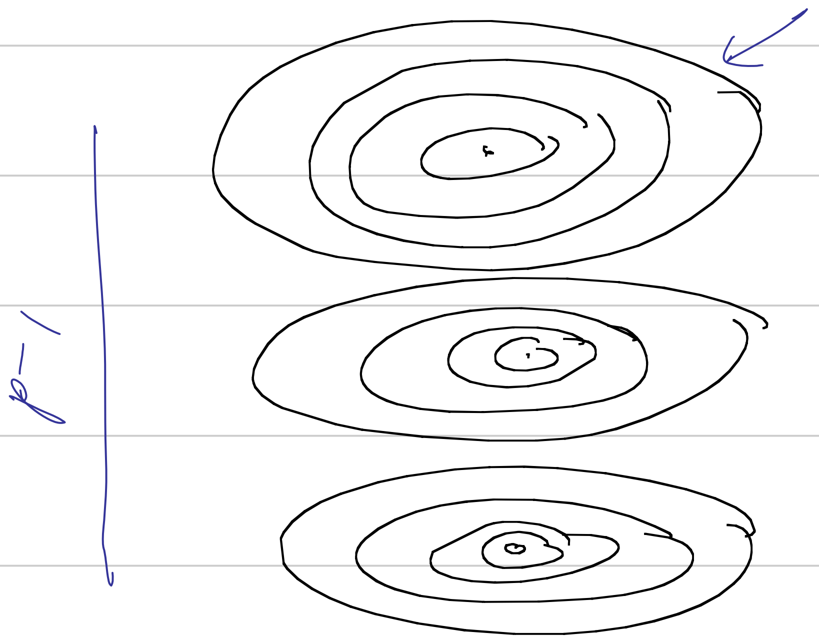
X - linear v.f.

Remark A curious duality between Type I & Type II

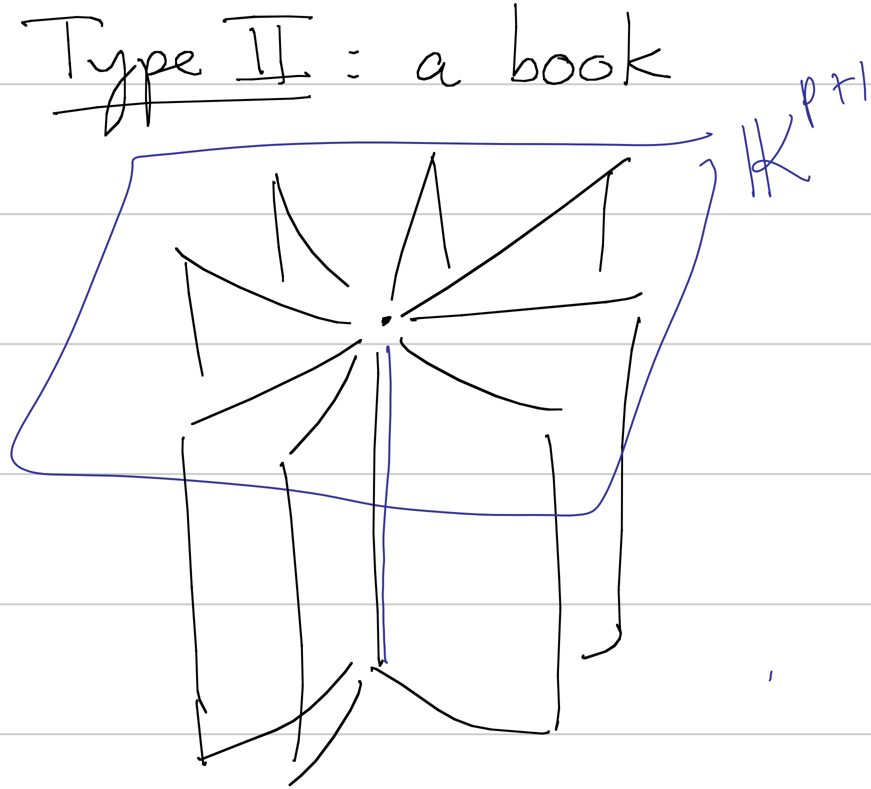
Proof Linear algebra.

Corresponding foliations:

<u>Type I</u> : a pile of cabbage



<u>Type II</u> : a book



(1st integals : $z_1, \dots, z_{p-1}, Q$)

(q commuting vector fields : $\partial/\partial z_1, \dots, \partial/\partial z_{q-1}, X$)

they are "dual", "transversal" to each other ($p \longleftrightarrow q$)

<u>Joke</u> Eat cabbage <u>& read books</u> !

Linearization of Singular Foliations

Observation If ω is an integrable p-form, $\omega(0) = 0$, write

$$\omega = \underbrace{\omega^{(1)}} + \omega^{(2)} + \dots$$

Then $\omega^{(1)}$ is integrable. [Proof: Consider Taylor exp. of the conditions $i_A \omega \wedge \omega = 0$ and $i_A \omega \wedge d\omega = 0$, where A is constant]

The same with $\Lambda = \Lambda^{(1)} + \Lambda^{(2)} + \dots$

$\rightsquigarrow$ it makes sense to talk about the linear part & the linearization problem.

Type I Case

$$\omega = \omega^{(1)} + \dots$$
$$\omega^{(1)} = dz_1 \wedge \dots \wedge dz_{p-1} \wedge dQ$$

We say that Λ or $\omega = \Lambda \lrcorner (dz_1 \wedge \dots \wedge dz_n)$ is <u>nondegenerate</u> if $Q = \sum_{i=p}^{n} \pm z_i^2/2$. (In the real case, we say that it's an <u>elliptic singularity</u> if Q is definite positive)

<u>Thm</u> (Dufour-Z) Suppose that $\omega^{(1)}$ is of Type I nondegen. Then ω is linearizable (formally if ω is formal, analytically if ω analytic, smoothly if ω is smooth elliptic)

up to multiplication by a non-vanishing function

(ω isom. to $\boxed{f\, dz_1 \wedge \dots \wedge dz_{q-1} \wedge dQ}$, $f(0) \neq 0$)

Idea of Proof

- Step 1) de Rham lemma $\longrightarrow$ ω is decomposable:

$$\omega = \gamma_1 \wedge \ldots \wedge \gamma_{p-1} \wedge \alpha$$

where $\gamma_i = dz_i + \ldots$

- Step 2) Godbillon - Vey algorithm $\longrightarrow$ formal linearization up to multiplication by a function
- Step 3) — Analytic case : Malgrange's Frobenius with sing.
  $\rightsquigarrow$ formal 1st integrals can be choosen analytic
  — smooth case : blow-up

Remark Cannot kill the multiplicative factor f in general, because ω is not closed in general.

In terms of Λ:

__Thm'__ Suppose Λ type I nondegenerate. Then it is formally linearizable in the formal case, and smoothly linearizable in the smooth elliptic case.

__Remarks:__ 1) No need to multiply Λ by a non-vanishing function!

2) Conjecture: it's true also in the analytic case

__Proof__ Use the previous thm $\Rightarrow$ can write

$$\Lambda = f \Lambda^{(1)}$$

Then try to kill f!

Type II case

Thm [Dufour-Z]

$$\Lambda^{(1)} = \frac{\partial}{\partial z_1} \wedge \dots \wedge \frac{\partial}{\partial z_{q-1}} \wedge \underbrace{\left(\sum_{i,j=q}^{n} b_j^i z_i \frac{\partial}{\partial z_j} \right)}_{X}.$$

If $\operatorname{trace}(b_j^i) := \sum_{i=q}^{n} b_i^i \neq 0$, then $\exists$ local (smooth / analytic / holom.) coord. syst. $(x_1, \dots, x_n)$ in which

$$\Lambda = \frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_{q-1}} \wedge Y$$

with $Y = \sum_{i=q}^{n} c_i(x_q, \dots, x_n) \frac{\partial}{\partial x_i}$ (Y does not depend on $x_1, \dots, x_{q-1}$)

$\Longrightarrow$ Reduce the problem to the linearization problem for Y.

(apply Sternberg, Bruno, etc.)

Idea of proof: use generalized "Kupka's phenomenon"

Kupka phenomenon ⟨ Reduction of Dimension ⟩

Kupka's Thm ω - integrable 1-form s.t. $d\omega(0) \neq 0$.

Then ω can be written as

$$\boxed{\omega = a(z_1, z_2)\, dz_1 + b(z_1, z_2)\, dz_2}$$

(ω = pull back of an 1-form on $\mathbb{K}^2$ by a submersion)

In terms of multi-vector fields

$$\begin{array}{ccc} \Lambda & \longleftrightarrow & \omega \\ \downarrow & & \downarrow \\ D_\Omega \Lambda & \longleftrightarrow & d\omega \end{array}$$

$\Omega = dz_1 \wedge \cdots \wedge dz_n$

$\omega = \Lambda \lrcorner \Omega$

$D_\Omega \Lambda \lrcorner \Omega = d\omega$

D_Ω is called the <u>modular operator</u> (or <u>curl operator</u>) w.r.t Ω

<u>Thm (Dufour-Z)</u> Let Λ be an integrable q-vector field s.t. $D_\Omega \Lambda(0) \neq 0$ w.r.t. some volume form Ω. Then $\exists$ local coord. syst. $(x_1, \dots, x_{q-1}, y_1, \dots, y_{p+1})$ in which

$$\boxed{\Lambda = \frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_{q-1}} \wedge X}$$

where X is a vector field which depends only on $y_1, \dots, y_{p+1}$

(X may be viewed as a vector field in $\mathbb{K}^{p+1}$)

<u>Remark</u> Linearization Thm for Type II singularities is a consequence of the above thm.

The proof is quite simple:

Λ integrable $\Rightarrow$ $D_\Omega \Lambda$ also integrable $+$ $D_\Omega \Lambda(0) \neq 0$

$$\Rightarrow D_\Omega \Lambda = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_{q-1}}$$

$$\rightsquigarrow \Lambda = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_{q-1}} \wedge X$$

Translating the above Thm back to integrable p-forms $\Rightarrow$

<u>Thm (de Medeiros)</u> ω integrable p-form s.t. $d\omega(0) \neq 0$.

Then ω = pull back of a p-form on $\mathbb{K}^{p+1}$ by a submersion.

Further generalization of Kupka phenomenon:

Def Λ is called of **Type II r** if it can be written (locally) as
$$\Lambda = \frac{\partial}{\partial z_1} \wedge \dots \wedge \frac{\partial}{\partial z_r} \wedge \Pi$$
where Π is a $(q-r)$-vector field in $\mathbb{K}^{n-r}$ (it does not depend on $z_1, \dots, z_r$)

Thm (Dufour-Z) Let Λ be an integrable multi-vector field & Ω a volume form. If $D_\Omega \Lambda$ is of type II r then so is Λ.

Thm' ω - integrable p-form on $\mathbb{K}^n$, $s \leq n$. If $d\omega \not\equiv 0$ and depends on only s coordinates (in a neighborhood of 0) then the same holds true for ω.

Thm (de Medeiros) Let ω be an integrable holomorphic p-form in $(\mathbb{C}^n, 0)$ with coordinates $(x_1, \dots, x_n)$, $2 \leq p \leq n-1$

i) If codim $\{ i_\pi \omega = 0 \} \geq 3$ where $\pi = \frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_{p-1}}$

then ω can be written as

$$\omega = f \, dg \wedge dy_1 \wedge \dots \wedge dy_{p-1} \qquad f(0) \neq 0$$

ii) If $\text{codim}_V \{ \omega|_V = 0 \} \geq 3$ where $V = \{ x_{p+2} = \dots = x_n = 0 \} \subset \mathbb{C}^n$

(dim $V = p+1$) then the dual integrable q-vector field Λ

$(\omega = \Lambda \lrcorner (dx_1 \wedge \dots \wedge dx_n))$ is of type $\underline{\text{II}}\,(q-1)$ at 0, i.e

$$\Lambda = \frac{\partial}{\partial y_1} \wedge \dots \wedge \frac{\partial}{\partial y_{q-1}} \wedge Y \qquad p+q = n$$

Remark : Proof becomes simpler when using multi-vector fields.

Corollary of the 2nd part of the above Thm :

<u>Thm (Camacho - Lins Neto, de Medeiros)</u> Assume that ω is an integrable p-form s.t. $1 \leq p \leq n-2$

$$\operatorname{codim}_W \{ d\omega|_W = 0 \} \geq 3$$

where $W = \{ x_{p+3} = \cdots = x_n = 0 \}$. Then ω = pull-back of a p-form on $\mathbb{C}^{p+2}$ by a submersion.

A few open question ?

- Classify / study homogeneous singular foliations ?
  (quadratic, cubic, ...)
- A more general version of Frobenius with singularities ?
  (for integrable k-vector fields / p-forms : $\exists$? local 1st integrals)
- Study commuting foliations ?
- "Complete integrability" ?