Simultaneous reduction to normal forms of commuting singular vector fields with linear parts having Jordan blocks

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Abstract

We study a simultaneous linearizability of d–actions (and the corresponding d-dimensional Lie algebras) defined by commuting singular vector fields in $\mathbb{C}^n$ fixing the origin with a nontrivial Jordan block in the linear parts. We prove the analytic convergence of a formal linearizing transformation under a certain invariant geometric condition for the spectrum of d vector fields generating a Lie algebra. (cf. Example 1.6.) If the condition fails and if we consider the situation where the small denominator occurs, then we show the existence of divergent solutions of an overdetermined system of linearized homological equations. In a smooth category, the situation is completely different. We will show Sternberg's theorem for a commuting system of vector fields with a Jordan block although they do not satisfy the condition.

Key words: singular vector field, linearization, Jordan block, homological equation, Diophantine conditions, Gevrey spaces, decomposition

MSC classification: 37C05, 37F50, 37G05

1 Simultaneous normalization

Let $\mathbb{K}$ be $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, and $B = \infty$, $B = \omega$ or $B = k$ for some $k > 0$. Let $\mathcal{G}_B^n$ denotes a d–dimensional Lie algebra of germs at $0 \in \mathbb{K}^n$ of C^B vector fields vanishing at 0. Let ρ be a germ of singular infinitesimal $\mathbb{K}^d$ $(d \geq 2)$ actions of class C^B

$$\rho : \mathbb{K}^d \longrightarrow \mathcal{G}_B^n. \tag{1.1}$$

*Supported by NATO grant PST.CLG.979347, and GNAMPA-INDAM, Italy.

†Supported by Grant-in-Aid for Scientific Research (No. 14340042), Ministry of Education, Science and Culture, Japan and GNAMPA-INDAM, Italy.

We denote by $Act^B(\mathbb{K}^d : \mathbb{K}^n)$ the set of germs of singular infinitesimal $\mathbb{K}^d$ actions of class C^B in $0 \in \mathbb{K}^n$. By choosing a basis $e_1, \dots, e_d \in \mathbb{K}^n$, the infinitesimal action can be identified with a d–tuple of germs at 0 of commuting vector fields $X^j = \rho(e_j)$, $j = 1, \dots, d$ (cf. [10], [17]). We can define, in view of the commutativity relation, the action

$$\tilde{\rho} : \mathbb{K}^d \times \mathbb{K}^n \longrightarrow \mathbb{K}^n, \tag{1.2}$$

$$\tilde{\rho}(s; z) = X^1_{s_1} \circ \cdots \circ X^d_{s_d}(z) \quad = \quad X^{\sigma_1}_{s_{\sigma_1}} \circ \cdots X^{\sigma_d}_{s_{\sigma_d}}(z), \quad s = (s_1, \dots, s_d), \tag{1.3}$$

for all permutations $\sigma = (\sigma_1, \dots, \sigma_d)$ of $\{1, \dots, d\}$, where X^j_t denotes the flow of X^j. We denote by ρ_{lin} the linear action formed by the linear parts of the vector fields defining ρ.

We shall investigate the necessary and sufficient condition for the linearization of ρ, namely, whether there exists a C^B diffeomorphism g preserving 0 such that g conjugates $\tilde{\rho}$ and $\tilde{\rho_{lin}}$

$$\tilde{\rho}(s; g(z)) \quad = \quad g(\widetilde{\rho_{lin}}(s, z)), \qquad (s, z) \in \mathbb{K}^d \times \mathbb{K}^n. \tag{1.4}$$

We recall that in [10], and [24] the linear parts were supposed to be diagonalizable, while in [29] the existence of $n - d$ anlalytic first integrals was required. (See also [1], [15]). Following Katok's argument in [17], we take a positive integer $m \le n$ such that $\mathbb{K}^n$ is decomposed into a direct sum of m linear subspaces invariant under all $A^\ell = \nabla X_\ell(0)$ $(\ell = 1, \dots, d)$:

$$\begin{aligned} \mathbb{K}^n \quad &= \quad \mathbb{I}^{s_1} + \cdots + \mathbb{I}^{s_m}, \qquad \dim \mathbb{I}^{s_j} = s_j,\ j = 1, \dots, m, \\ &\quad\ \ s_1 + \cdots + s_m = n. \end{aligned} \tag{1.5}$$

The matrices $A^1, \dots, A^d$ can be simultaneously brought in an upper triangular form, and we write again A^ℓ for the matrices,

$$A^\ell \quad = \quad \begin{pmatrix} A^\ell_1 & 0_{s_1 \times s_2} & \dots & 0_{s_1 \times s_m} \\ 0_{s_2 \times s_1} & A^\ell_2 & \dots & 0_{s_2 \times s_m} \\ \vdots & \vdots & \vdots & \vdots \\ 0_{s_m \times s_1} & 0_{s_m \times s_2} & \dots & A^\ell_m \end{pmatrix}, \quad \ell = 1, \dots, d. \tag{1.6}$$

If $\mathbb{K} = \mathbb{C}$, the matrix A^ℓ_j is given by

$$A^\ell_j \quad = \quad \begin{pmatrix} \lambda^\ell_j & A^\ell_{j,12} & \dots & A^\ell_{j,1s_j} \\ 0 & \lambda^\ell_j & \dots & A^\ell_{j,2s_j} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda^\ell_j \end{pmatrix}, \quad \ell = 1, \dots, d,\ j = 1, \dots, m, \tag{1.7}$$

with $\lambda^\ell_j, A^\ell_{j,\nu\mu} \in \mathbb{C}$. On the other hand, if $\mathbb{K} = \mathbb{R}$, then we have, for every $1 \le j \le m$ two possibilities: firstly, all A^ℓ_j $(\ell = 1, \dots, d)$ are given by (1.7) with $\lambda^\ell_j \in \mathbb{R}$. Secondly, $s_j = 2\tilde{s_j}$ is even and A^ℓ_j is a $\tilde{s_j} \times \tilde{s_j}$ square block matrix given by

$$A^\ell_j \quad = \quad \begin{pmatrix} R_2(\lambda^\ell_j, \mu^\ell_j) & A^{12}_{\ell,j} & \dots & A^{1\tilde{s_j}}_{\ell j} \\ 0 & R_2(\lambda^\ell_j, \mu^\ell_j) & \dots & A^{2\tilde{s_j}}_{\ell j} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & R_2(\lambda^\ell_j, \mu^\ell_j) \end{pmatrix}, \ \ell = 1, \dots, d, \tag{1.8}$$

where

$$R_2(\lambda,\mu) := \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix}, \qquad \lambda,\mu \in \mathbb{R}, \tag{1.9}$$

and $A^{rs}_{\ell j}$ are appropriate real matrices.

Following the decomposition (1.7) (respectively, (1.8)) we define $\tilde{\lambda^j}$ by

$$\tilde{\lambda^k} = (\lambda^k_1, \ldots, \lambda^k_m) \in \mathbb{K}^m, \qquad k = 1, \ldots, d. \tag{1.10}$$

Then we assume

$$\tilde{\lambda^1}, \cdots, \tilde{\lambda^d} \text{ are linearly independent in } \mathbb{K}^m. \tag{1.11}$$

One can easily see that (1.11) is invariantly defined.

By (1.6) we define

$$\vec{\lambda_j} = {}^t(\lambda^1_j, \cdots, \lambda^d_j) \in \mathbb{K}^d, \qquad j = 1, \ldots, m, \tag{1.12}$$

and

$$\Lambda_m := \{\vec{\lambda_1}, \ldots, \vec{\lambda_m}\}. \tag{1.13}$$

We define the cone $\Gamma[\Lambda_m]$ by

$$\Gamma[\Lambda_m] = \left\{ \sum_{j=1}^m t_j \vec{\lambda_j} \in \mathbb{K}^d;\ t_j \geq 0, j = 1, \ldots, m, \sum_{j=1}^m t_j \neq 0 \right\}. \tag{1.14}$$

Definition 1.1 *We say that the $\mathbb{K}^d$–action ρ is a Poincaré morphism if there exists a base $\Lambda_m \subset \mathbb{K}^m$ such that $\Gamma[\Lambda_m]$ is a proper cone in $\mathbb{K}^m$, namely it does not contain a straight real line. If the condition is not satisfied, then, we say that the $\mathbb{K}^d$ action is in a Siegel domain.*

Note that the definition is invariant under the choice of the basis Λ_m.

Remark 1.2 *As to the alternative definition of a Poincaré morphism we refer to the definition 6.2.1 of [24].*

Next, we introduce the notion of simultaneous resonances. For $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{K}^m$, $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{K}^m$, we set $\langle \alpha, \beta \rangle = \sum_{\nu=1}^m \alpha_\nu \beta_\nu$. For a positive integer k we define $\mathbb{Z}^m_+(k) = \{\alpha \in \mathbb{Z}^m_+;\ |\alpha| \geq k\}$. Put

$$\omega_j(\alpha) = \sum_{\nu=1}^d |\langle \tilde{\lambda^\nu}, \alpha \rangle - \lambda^\nu_j|, \qquad j = 1, \ldots, m, \tag{1.15}$$

$$\omega(\alpha) = \min\{\omega_1(\alpha), \ldots, \omega_m(\alpha)\}. \tag{1.16}$$

Definition 1.3 *We say that Λ_m is simultaneously nonresonant (or, in short ρ is simultaneously nonresonant), if*

$$\omega(\alpha) \neq 0, \qquad \forall \alpha \in \mathbb{Z}^m_+(2). \tag{1.17}$$

If (1.17) does not hold, then we say that Λ_m is simultaneously resonant.

Clearly, the simultaneously nonresonant condition (1.17) is invariant under the change of the basis Λ_m. We state the first main result of our paper

Theorem 1.4 *Let ρ be a Poincaré morphism. Then ρ is conjugated to a polynomial action by an holomorphic change of variables.*

Remark 1.5 *In case ρ has a semi simple linear part, then Theorem 1.4 is already known. (cf. Theorem 2.1.4 of [24]).*

Example 1.6. We compare our theorem with the results of Stolovitch [24] and Zung [29]. Let ρ be a $\mathbb{R}^2$ action in $\mathbb{R}^n$, $n \geq 4$ with $m = 3$. We choose a basis Λ_2 of $\mathbb{R}^3$ such that

$$\Lambda_2 = \left\{ {}^t(1, 1, \nu), {}^t(0, 1, \mu) \right\}, \qquad \nu, \mu \in \mathbb{R}. \tag{1.18}$$

(cf. [13] for similar and more general reductions of commuting vector fields on the torus).

We will characterize the set of $(\nu, \mu) \in \mathbb{R}^2$ so that the action is a Poincaré morphism, and determine the simultaneous resonances. By (1.14), $\Gamma[\Lambda_2]$ is generated by the set of vectors $\{(1, 0), (1, 1), (\nu, \mu)\}$. Hence the action is a Poincaré morphism if and only if these vectors generate a proper cone, namely (ν, μ) is not in the set $\{(\nu, \mu) \in \mathbb{R}^2; \nu \leq \mu \leq 0\}$. We note that the interesting case is $\mu < \nu \leq 0$, where every generator in (1.18) is in a Siegel domain. Theorem 1.4 can be applied to such a case. In §3 we will show that if the action is not a Poincaré morphism, i.e., $\nu < \mu < 0$, then there exist (ν, μ) with the density of continuum such that the linearized overdetermined system of two homological equations has a divergent solution.

Next we will determine (ν, μ) so that a simultaneous resonance exists. If $\eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{Z}^3_+(2)$ is a simultaneous resonance, we have the following set of equations:

$$\begin{array}{ll} (1) & \eta_1 + \eta_2 + \nu\eta_3 = 1,\ \eta_2 + \mu\eta_3 = 0, \\ (2) & \eta_1 + \eta_2 + \nu\eta_3 = 1,\ \eta_2 + \mu\eta_3 = 1, \\ (3) & \eta_1 + \eta_2 + \nu\eta_3 = \nu,\ \eta_2 + \mu\eta_3 = \mu. \end{array}$$

By elementary computations, in order that one of these equations has a solution η the (ν, μ) satisfies the following:
a) Case $\nu \leq \mu \leq 0$. The resonance exists iff $(\nu, \mu) \in \mathbb{Q}_- \times \mathbb{Q}_-$, where $\mathbb{Q}_-$ is the set of nonpositive rational numbers. The resonance is given by $(1+(\mu-\nu)k, -\mu k, k)$ and $((\mu-\nu)k, 1-k\mu, k)$ where $k \geq 1/(1-\nu)$, $k \in \mathbb{Z}_+$, and $((\nu-\mu)(1-k), \mu(1-k), k)$, where $k \geq (2-\nu)(1-\nu)$, $k \in \mathbb{Z}_+$.
b) Case $\nu > \mu$ and $\mu \leq 0$. The resonance is given by $(0, -\mu/(\nu - \mu), 1/(\nu - \mu))$, where $-\mu/(\nu - \mu) \in \mathbb{Z}_+$, $1/(\nu - \mu) \in \mathbb{Z}_+$ and $2\nu - \mu \leq 1$.
c) Case $\mu > 0$, $\nu \leq \mu$. The resonance is given by $(0, 0, 1/\nu)$, when $\nu = \mu$, $\nu \leq 1/2$, $\nu^{-1} \in \mathbb{Z}_+$, $(0, \nu, 0)$, when $\nu = \mu \geq 2$, $\nu \in \mathbb{Z}_+$, $((\mu - \nu)/\mu, 0, 1/\mu)$, if otherwise, where $(\mu - \nu)/\mu \in \mathbb{Z}_+$, $1/\mu \in \mathbb{Z}_+$ and $\nu + \mu \leq 1$.
d) Case $\nu > \mu$, $\mu \geq 0$. The resonance is given by $(\nu - \mu, \mu, 0)$, where $\nu - \mu \in \mathbb{Z}_+$, $\mu \in \mathbb{Z}_+$ and $\nu \geq 2$.

Let ν be a negative rational number, $\nu = -k_1/k_2$, $k_1, k_2 \in \mathbb{Z}_+$, $k_2 \neq 0$. Let μ be a rational number and satisfy $\mu < \nu$. Assume that the nonlinear part of X^2 is zero. If the nonlinear part of X^1 consists of the resonant terms of X^2, then we have $[X^1, X^2] = 0$. We can easily see that the linearizability of X^1 holds provided $\mu \neq \nu - 1/k_2 = -(k_1 + 1)/k_2$.

2 A Poincaré morphism

We start by showing equivalent forms of a Poincaré morphism.

Proposition 2.1 *The action is a Poincaré morphism if and only if each of the following conditions holds*
i) there exist a positive constant C and an integer k_0 such that

$$\sum_{k=1}^{d} |\sum_{j=1}^{m} \lambda_j^k \alpha_j| \geq C_1|\alpha|, \qquad \forall \alpha \in \mathbb{Z}_+^m(k_0). \tag{2.1}$$

ii) there exists a nonzero vector $c = (c_1, \dots, c_d) \in \mathbb{C}^d$ if $\mathbb{K} = \mathbb{C}$ (respectively, $c = (c_1, \dots, c_d) \in \mathbb{R}^d$ if $\mathbb{K} = \mathbb{R}$) such that

$$c_1\tilde{\lambda}^1 + \cdots + c_d\tilde{\lambda}^d \text{ is in a Poincaré domain,} \tag{2.2}$$

namely, the convex hull of the set $\{\sum_{j=1}^d c_j \lambda_k^j; k = 1, \dots, m\}$ in $\mathbb{C}$ does not contain $0 \in \mathbb{C}$ (respectively,

$$\text{the real parts of } c_1\lambda_j^1 + \cdots + c_d\lambda_j^d,\ j = 1, \dots, m, \text{ are positive.)} \tag{2.3}$$

Proof. First we show (2.1). Suppose that (2.1) does not hold. Then there exists a sequence $\alpha^\ell \in \mathbb{Z}_+^m$, $\ell \in \mathbb{N}$ such that $|\alpha^\ell| \to \infty$ $(\ell \to \infty)$ and

$$\sum_{k=1}^{d} |\sum_{j=1}^{m} \lambda_j^k \alpha_j^\ell| \leq \frac{|\alpha^\ell|}{\ell}, \qquad \ell \in \mathbb{N}. \tag{2.4}$$

By taking a subsequence, if necessary, we may assume that $\alpha^\ell/|\alpha^\ell| \to t^0 = (t_1^0, \dots, t_m^0) \in S_{\ell^1}^1 \bigcap \mathbb{R}_+^m$ when $\ell \to \infty$, where $S_{\ell^1}^1 := \{x \in \mathbb{K}^m;\ \|x\|_{\ell^1} = \sum_{j=1}^m |x_j| = 1\}$ stands for the ℓ^1 unit sphere. By letting $\ell \to \infty$ in (2.4) we get

$$\sum_{k=1}^{d} |\sum_{j=1}^{m} \lambda_j^k t_j^0| = 0.$$

It follows that $\sum_{j=1}^m t_j^0 \vec{\lambda}_j = 0$. Let $J \subset \{1, \dots, m\}$ be such that $\sum_{j \in J} t_j^0 \vec{\lambda}_j \neq 0$. Such a set J exists by (1.11). It follows that

$$0 \neq \sum_{j \in J} t_j^0 \vec{\lambda}_j = - \sum_{j \in \{1,\dots,m\} \setminus J} t_j^0 \vec{\lambda}_j.$$

Hence $\Gamma[\Lambda_m]$ contains a straight line generated by $\sum_{j \in J} t_j^0 \vec{\lambda}_j \neq 0$. This contradicts the assumption that $\Gamma[\Lambda_m]$ is a proper cone.

Conversely, suppose that (2.1) is satisfied. We shall show that $\Gamma[\Lambda_m]$ is proper. Indeed, if otherwise, we can find $t^0 = (t_1^0, \dots, t_m^0) \in S_{\ell^1}^1 \bigcap \mathbb{R}_+^m \setminus 0$ such that

$$\sum_{j=1}^{m} t_j^0 \lambda_j^k = 0, \qquad k = 1, \dots, d. \tag{2.5}$$

Because the set $\{\alpha/|\alpha|; \alpha \in \mathbb{Z}_+^m(2)\}$ is dense in $S_{\ell^1}^1 \bigcap \mathbb{R}_+^m$, there exists a sequence $\alpha^\ell \in \mathbb{Z}_+^m$, $\ell \in \mathbb{N}$ such that $|\alpha^\ell| \to \infty$ $(\ell \to \infty)$ and $\lim_{\ell \to \infty} \alpha^\ell/|\alpha^\ell| = t^0$. Therefore, in view of (2.5), we get

$$\lim_{\ell \to \infty} \left(\frac{1}{|\alpha^\ell|} \sum_{k=1}^{d} |\sum_{j=1}^{m} \lambda_j^k \alpha_j^\ell| \right) = 0,$$

which contradicts (2.1)

Next, we show ii). Suppose that $\Gamma[\Lambda_m]$ be a proper cone in $\mathbb{K}^d$. Then we can find $c = (c_1, \dots, c_d) \in \mathbb{C}^d$ such that $\Gamma[\Lambda_m]$ is contained in the real half–space $P_c := \{z \in \mathbb{K}^d, \operatorname{Re}(\sum_{k=1}^d c_k z_k) > 0\}$. Therefore

$$0 < \operatorname{Re}(\sum_{k=1}^{d} c_k \sum_{j=1}^{m} t_j \lambda_j^k) = \sum_{j=1}^{m} t_j \operatorname{Re}(\sum_{k=1}^{d} c_k \lambda_j^k) \tag{2.6}$$

for all $t \in \mathbb{R}_+^m \setminus 0$, which yields $\operatorname{Re}(\sum_{k=1}^d c_k \lambda_j^k) > 0$ for $j = 1, \dots, m$. We note that, if $\mathbb{K} = \mathbb{R}$, then the use of the real part in the definition of the half–space is superfluous. Finally, we readily see, from (2.2) that, if $\mathbb{K} = \mathbb{C}$ (respectively, (2.3) if $\mathbb{K} = \mathbb{R}$), then the cone $\Gamma[\Lambda_m]$ is contained in P_c. Hence $\Gamma[\Lambda_m]$ is proper. The proof is complete.

Although the following proposition is known, we give an alternative proof for the sake of completeness. (cf. Lemma 3.1 of [25].)

Proposition 2.2 *Let the action ρ be a Poincaré morphism. Then we can find a vector field in the corresponding Lie algebra which has the same resonace as the simultaneous resonance of ρ and is in the Poincaré domain.*

Proof. By ii) of Proposition 2.2 we can find a Poincaré vector field in the Lie algebra as a linear combination of a base corresponding to (2.2). Let c_ν be the numbers in (2.2), and define $\tilde{\lambda}^0 := (\lambda_1^0, \dots, \lambda_m^0) = \sum_{\nu=1}^d c_\nu \tilde{\lambda}^\nu$. Let S be a similtaneous resonance of ρ. Consider

$$\langle \tilde{\lambda}^0, \alpha \rangle - \lambda_j^0 = \sum_{\nu=1}^{d} c_\nu \left(\langle \tilde{\lambda}^\nu, \alpha \rangle - \lambda_j^\nu \right).$$

Because $\sum_{\nu=1}^d |\langle \tilde{\lambda}^\nu, \alpha \rangle - \lambda_j^\nu| \neq 0$ for every $\alpha \in \mathbb{Z}_+^m(2) \setminus S$, it follows that the set $\langle \tilde{\lambda}^0, \alpha \rangle - \lambda_j^0 = 0$ in $c = (c_1, \dots, c_d) \in \mathbb{C}^d$ is a hyperplane if $\alpha \notin S$. It follows that the set

$$\{c = (c_1, \dots, c_d) \in \mathbb{C}^d; \langle \tilde{\lambda}^0, \alpha \rangle - \lambda_j^0 = 0, \exists j, 1 \leq j \leq m, \exists \alpha \in \mathbb{Z}_+^m(2) \setminus S\}$$

is a countable union of nowhere dense closed set. Therefore we can find $c = (c_1, \dots, c_d)$ for which $\sum_{\nu=1}^d c_\nu \tilde{\lambda}^\nu$ satisfies the Poincaré condition and has the resonance S. This proves Proposition 2.2.

We propose a geometric expression of a Poincaré morphism.

Definition 2.3 *Let $r > 0$ and g be a Riemannian metric on $\mathbb{R}^n$. We denote by $\langle \cdot, \cdot \rangle_g$ and $\| \cdot \|_g$ the inner product and the norm with respect to g, respectively. We say that $\mathcal{X}_\nu := \sum_{j=1}^n X_j^\nu(x) \partial_{x_j}$ $(\nu = 1, \dots, d)$ are simultaneously transversal to the sphere $\|x\|_g = r$ if, the vectors $X^\nu := (X_1^\nu, \dots, X_n^\nu)$ $(\nu = 1, \dots, d)$ satisfy*

$$\sum_{\nu=1}^{d} |\langle X^\nu, x \rangle_g| \neq 0, \quad \forall x, \quad \|x\|_g = r. \tag{2.7}$$

Theorem 2.4 *Let $r > 0$. Suppose that $\mathcal{B}_\nu := \sum_{j=1}^n (A^\nu x)_j \partial_{x_j}$ $(\nu = 1, \dots, d)$ be a commuting system of semi-simple linear real vector fields in $\mathbb{R}^n$. Let ρ be the action generated by $\{\mathcal{B}_\nu\}$. We choose a real nonsingular matrix P such that $\Lambda^\nu = P^{-1} A^\nu P$ is a block diagonal matrix given by*
$\Lambda^\nu = diag\ \{R_2(\xi_1^\nu, \eta_1^\nu), \dots, R_2(\xi_{n_1}^\nu, \eta_{n_1}^\nu), \lambda_{n_1+1}^\nu, \dots, \lambda_n^\nu\}$ for some integer
$n_1 \leq n$.. Let g be a Riemannian metric defined by $P^t P$. Then the following conditions are equivalent.

(a) $\mathcal{B}_\nu$ $(\nu = 1, \ldots, d)$ *are simultaneously transversal to the sphere* $\|x\|_g = r$.
(b) ρ *is a Poincaré morphism.*
(c) *There exist real numbers* c_ν $(\nu = 1, \ldots, d)$ *such that* $\sum_{\nu=1}^d c_\nu \mathcal{B}_\nu$ *is transversal to the sphere* $\|x\|_g = r$.

Proof. We note that $\langle x, y\rangle_g = \langle Px, Py\rangle$ and $\|x\|_g = \|Px\|$. By inserting the relation $A^\nu = P\Lambda^\nu P^{-1}$ into (2.7) we can easily see that the simultaneous transversality condition is equivalent to

$$\sum_{\nu=1}^{d} |\langle \Lambda^\nu y, y\rangle| \neq 0, \quad \forall y = (y_1, \ldots, y_n), \quad \|y\| = 1. \tag{2.8}$$

By definition, (2.8) can be written in

$$\sum_{\nu=1}^{d} |\sum_{j=1}^{n_1} \xi_j^\nu (y_{2j-1}^2 + y_{2j}^2) + \sum_{j=n_1+1}^{n} y_j^2 \lambda_j^\nu| \neq 0, \quad \forall y, \quad \|y\| = 1. \tag{2.9}$$

We define $t = (t_1, \ldots, t_n)$, $t \in \mathbb{R}_+^n$, $\sum t_j = 1$ by $t_j = (y_{2j-1}^2 + y_{2j}^2)/2$ if $j \leq n_1$ and $t_j = y_j^2$ if $j > 2n$. Noting that $\xi_j^\nu (y_{2j-1}^2 + y_{2j}^2) = 2t_j \xi_j^\nu = t_j(\xi_j^\nu + i\eta_j^\nu + \xi_j^\nu - i\eta_j^\nu)$ we see that (2.9) is written in $\sum_{\nu=1}^d |\sum_{j=1}^n t_j \lambda_j^\nu| \neq 0$ for every $t \in \mathbb{R}_+^n$ and $\sum t_j = 1$. This is equivalent to (b) by definition. Hence we have proved the equivalence of (a) and (b).

By Proposition 2.2 the condition (b) is equivalent to the existence of real numbers c_ν $(\nu = 1, \ldots, d)$ such that $\sum_{\nu=1}^d c_\nu \mathcal{B}_\nu$ is a Poincaré vector field. By what we have proved in the above $(d = 1)$ this is equivalent to say that $\sum_{\nu=1}^d c_\nu \mathcal{B}_\nu$ is transversal to the sphere $\|x\|_g = r$. Hence we have proved Theorem 2.4.

Proof of Theorem 1.4. By Proposition 2.2 there exists a Poincaré vector field χ_0 in ρ which is in a Poincaré domain and has the same resonance as ρ. If ρ is not resonant, then we have Theorem 1.4. In case there is a resonance of ρ, then it follows from Lemma 3.2 of [25] that, if χ_0 is normalized, then so is ρ. This completes the proof of Theorem 1.4.

3 Divergent solutions of overdetermined systems of linearized homological equations

We now study the action ρ_{lin} which is in a Siegel domain and admits a Jordan block. We assume that the action is formally (simultaneously) linearizable and is not a Poincaré morphism and that the family of linear parts is Diophantine. We shall show that the unique formal solution of a linearized homological equation diverges.

Let $\mathbb{C}_2^n\{x\}$ be the set of n vector functions of convergent power series of x without constant and linear terms. We examine the system of the linearized homology equation

$$L_A v = {}^t(L_1 v, \ldots, L_d v) = f, \quad f := {}^t(f_1, \ldots, f_d) \in (\mathbb{C}_2^n\{x\})^d, \tag{3.1}$$

where L_j is the Lie bracket,

$$L_j v = [A_j x, v] = \langle A_j x, \partial_x\rangle v - A_j v, \quad j = 1, \ldots, d,$$

under the compatibility conditions

$$L_j f_k = L_k f_j, \qquad j, k = 1, \ldots, d. \tag{3.2}$$

First we consider a 2-$\mathbb{C}$ action studied in Example 1.4. We assume that there exists a vector field in the two-dimensional Lie algebra which is not semisimple. In view of Example 1.4 we can choose a base X_1, X_2 with linear parts $A_j \in GL(4;\mathbb{C})$ satisfying $\mathrm{spec}\,(A_1) = \{1, 1, \nu, \nu\}$ and $\mathrm{spec}\,(A_2) = \{0, 1, \mu, \mu\}$, respectively, where $\nu \le \mu \le 0$, $(\nu, \mu) \notin \mathbb{Q} \times \mathbb{Q}$, and

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \nu & \varepsilon \\ 0 & 0 & 0 & \nu \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mu & \varepsilon_0\varepsilon \\ 0 & 0 & 0 & \mu \end{pmatrix}, \tag{3.3}$$

where $\varepsilon \neq 0$ and $\varepsilon_0 \in \mathbb{C}$. We can make $|\varepsilon| > 0$ arbitrarily small by an appropriate linear change of variables.

Let $\omega(\alpha)$ be defined by (1.16). We say that the simultaneous Diophantine order of $\{\mathrm{spec}\,(A_1), \mathrm{spec}\,(A_2)\}$ is τ_0, if, for every $\tau > \tau_0$ there exists $C = C_\tau > 0$ such that

$$\omega(\alpha) \ge C|\alpha|^{-\tau}, \quad \forall \alpha \in \mathbb{Z}_+^4(2), \tag{3.4}$$

while, for every $\tau < \tau_0$ there exists a subsequence $\alpha_\ell \in \mathbb{Z}_+^4(2)$ $(\ell = 1, 2, \ldots)$ such that

$$\omega(\alpha_\ell) \le |\alpha_\ell|^{-\tau}, \qquad \ell \in \mathbb{N}. \tag{3.5}$$

First we note that the conditions (3.4) and (3.5) for $\omega(\alpha)$ are equivalent to the corresponding ones for $\|q\nu\| + \|q\mu\|$ when $q \in \mathbb{N}$, $q \to \infty$, where $\|t\| = \min_{p\in\mathbb{Z}} |p - t|$. Hence the number τ_0 in (3.4) and (3.5) is equal to the speed of the simultaneous approximation of ν and μ, namely $\|q\nu\| + \|q\mu\| \sim Cq^{-\tau_0}$ for some constant $C > 0$ independent of q. Clearly, if (3.4) holds, then we have an upper bound of τ_0. By the result of M. Herman, [16], we have an upper bound $2+\varepsilon$ for every $\varepsilon > 0$ for almost all ν and μ. On the other hand, Moser showed that there exist Liouville numbers ν and μ such that $\|q\nu\| + \|q\mu\| \ge cq^{-\tau}$ for any given $\tau > 2$. (See Theorem 2 of [22]). This implies that for every $\tau > 2$, there exist Liouville numbers ν and μ such that $\tau_0 \le \tau$. We have another upper bound of τ_0 if either ν or μ is an algebraic number. Indeed, by Roth's theorem, for any given $\tau > 1$ there exists $c > 0$ such that by $\|q\nu\| + \|q\mu\| \ge cq^{-\tau}$. Hence we have $\tau_0 \le 1$. Finally, by [Corollary 1B, p.27, 22], if either ν or μ is an irrational number, then we have a lower bound $\tau_0 \ge 1/2$.

We say that α and β are simultaneously Liouville, if (3.5) holds for every $\tau > 0$.

Let $\sigma \ge 1$. We say that a formal power series $f(x) = \sum_\alpha f_\alpha x^\alpha$ is in a Gevrey space $G_2^\sigma(\mathbb{C}^4)$ if $f_\alpha = 0$ for $|\alpha| \le 1$ and, there exist $C > 0$ and $R > 0$ such that

$$|f_\alpha| \le CR^{|\alpha|}|\alpha|!^\sigma, \quad \forall \alpha \in \mathbb{Z}_+^4.$$

We consider the following equation

$$L_A v := {}^t(L_1 v, L_2 v) = f, \quad f = {}^t(f_1, f_2) \in (\mathbb{C}_2^4\{x\})^2,\ x \in \mathbb{C}^4, \tag{3.6}$$

where ${}^t(f_1, f_2)$ satisfies the compatibility condition $L_1 f_2 = L_2 f_1$. Then we have

Theorem 3.1 *Assume that $\varepsilon_0 \neq 0$ is a real number. Then, if $(\nu, \mu) \in \mathbb{Q} \times \mathbb{Q}$ and $\nu < \mu \le 0$, then there exists $f = {}^t(f_1, f_2) \in (\mathbb{C}_2^4\{x\})^2$ such that $L_1 f_2 = L_2 f_1$ and Eq. (3.6) has a formal power series solution $v \notin \bigcup_{1\le\sigma<5/2} G_2^\sigma(\mathbb{C}^4)$. Moreover, there exist $(\nu, \mu) \notin \mathbb{Q}\times\mathbb{Q}$ and $\nu < \mu \le 0$ with the density of continuum such that the same assertion holds.*

Furthermore, suppose that $(\nu, \mu) \notin \mathbb{Q} \times \mathbb{Q}$, (3.4), (3.5) and $\tau_0 < +\infty$ hold. Then (3.6) has a unique solution $v \in \bigcap_{\sigma>3+2\tau_0} G_2^\sigma(\mathbb{C}^4)$ for every ${}^t(f_1, f_2) \in (\mathbb{C}_2^4\{x\})^2$ satisfying $L_1 f_2 = L_2 f_1$.

In order to prove Theorem 3.1, we need a function space $\mathcal{G}$ which is a subspace of a set of holomorphic functions in a neighborhood of the origin. First we give the definition in the case $\varepsilon_0 = 1$, i.e., the nilpotent parts of A_1 and A_2 coincide. We define $\mathcal{G}$ by

$$\mathcal{G} := \left\{ f = {}^t(f_1, f_2, f_3, f_4);\ f_j \equiv f_j(x) = \sum_{\alpha\in C_j} f_\alpha^j x^\alpha,\ j = 1, 2, 3, 4 \right\},$$

where $C_j \subset \mathbb{Z}_+^4(2)$ satisfies the following two conditions.
(1) There exist $c_0 > 0$ and $\tau > \tau_0$ such that for every $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in C_j$, we have $N = \alpha_3 + \alpha_4 \neq 0$, $|\alpha| \geq 2$, and

$$
\begin{aligned}
|\alpha_1 - 1 + (\nu - \mu)N| &< N^{-N(\tau+1)} c_0^N, \quad \forall \alpha \in C_j \ (j = 1, 2), \\
|\alpha_1 + (\nu - \mu)(N - 1)| &< N^{-N(\tau+1)} c_0^N, \quad \forall \alpha \in C_j \ (j = 3, 4).
\end{aligned}
$$

(2) The Diophantine condition for $Spec(A_1)$ holds: namely for every $\tau' < \tau_0 < \tau''$, there exist $c_1 > 0$ and $c_2 > 0$ such that

$$
\begin{aligned}
c_1 N^{-\tau''} < |\alpha_1 + \alpha_2 - 1 + \nu N| < c_2 N^{-\tau'} \quad &\text{if } \alpha \in C_j, j = 1, 2, \\
c_1 N^{-\tau''} < |\alpha_1 + \alpha_2 + \nu N - \nu| < c_2 N^{-\tau'} \quad &\text{if } \alpha \in C_j, j = 3, 4,
\end{aligned}
$$

where $N = \alpha_3 + \alpha_4 \neq 0$.

Remark 3.2 *If $\varepsilon_0 \neq 1$, we replace (1) with the following (1)'.*

(1)' There exist $c_0 > 0$ and $\tau > \tau_0$ such that for every $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in C_j$, we have $N = \alpha_3 + \alpha_4 \neq 0$, and

$$
|\varepsilon_0(\alpha_1 - 1 + \alpha_2 + \nu N) - (\alpha_2 + \mu N)| < N^{-N(\tau+1)} c_0^N, \qquad \text{if } \alpha \in C_1.
$$

In the case $\alpha \in C_2$, we replace α_1 and α_2 in the left-hand side of the above inequality with $\alpha_1 + 1$ and $\alpha_2 - 1$, respectively. Similarly, in the case $\alpha \in C_3$ or $\alpha \in C_4$, we replace α_1 and N in the left-hand side of the above inequality with $\alpha_1 + 1$ and $N - 1$, respectively.

Remark 3.3 *The space $\mathcal{G}$ is a normed space with the norm $\|f\| := \sum_\alpha |f_\alpha|$, where $|f_\alpha| = \sum_j |f_\alpha^j|$ with f being given in the definition of $\mathcal{G}$.*

If the conditions (1) and (2) hold, then we can easily show that the Diophantine condition for Spec(A_2) holds. Hence we have a simultaneous Diophantine condition for Spec(A_1) and Spec(A_2). In the following, we will see that on the support C_j of $\mathcal{G}$, the divergence of the solutions of L_A occurs.

Remark 3.4 *The space $\mathcal{G}$ is not empty for an appropriate choice of ν and μ such that $\nu < \mu < 0$, i.e., the action is not a Poincaré morphism. We first consider the case $\varepsilon_0 = 1$ for the sake of simplicity. If we construct ν and μ so as to satisfy the conditions (1) and (2) for $C_1 = C_2$, then the conditions (1) and (2) for C_j $(j = 3, 4)$ hold if we define $C_3 = C_4$ by replacing α_1 and N in C_1 with $\alpha_1 + 1$ and $N - 1$, respectively. Hence we will consider C_1.*

We can easily construct an irrational number $\nu < 0$ which satisfies (2). In fact, $\alpha_1 + \alpha_2$ and N are given by a continued fraction expansion of ν. Note that α_1 can be taken arbitrarily. Next, by the standard measure theoretic argument, we can show that there exist an irrational number μ with $\nu - \mu < 0$ and the sequence $\{\alpha_1\}$ such that (1) holds. By construction, we can also choose $\mu < 0$ such that $\nu < \mu < 0$. It follows that the action is not a Poincaré morphism. Moreover, we can easily see that the set of ν and μ satisfying (1) and (2) has the density of continuum.

Next we consider the case $\varepsilon_0 \neq 1$. For the sake of simplicity, we give the sketch of the proof for C_1 in the case $0 < \varepsilon_0 < 1$. The other cases can be treated similarly. First we construct ν so as to satisfy (2). Then the sequence of the integers $k \equiv \alpha_1 + \alpha_2 - 1$ and N are also given. In order to show that there exists μ satisfying $(1)'$, we consider the inequality

$$
\left| \frac{\alpha_1 - 1 + (1 - \varepsilon_0^{-1})\alpha_2}{N} - (\varepsilon_0^{-1}\mu - \nu) \right| < N^{-N(\tau+1)-1} c_0^N \varepsilon_0^{-1}.
$$

We consider closed intervals of length $2N^{-N(\tau+1)-1}c_0^N\varepsilon_0^{-1}$ *with the centers at* $\frac{\alpha_1-1+(1-\varepsilon_0^{-1})\alpha_2}{N}$, $(\alpha_1+\alpha_2=k+1)$. *Let* N *and one of these intervals* I_N *are given. Then we can choose* $N'>N$ *and* $I_{N'}$ *such that* I_N *contains* $I_{N'}$*. Hence we can construct a sequence of monotone decreasing intervals. By taking a subsequence, if necessary, we see that there exists* μ *which satisfies* $(1)'$*. By construction the set of* μ *has the density of continuum. We remark that we can take* $\tilde{\nu}:=\varepsilon_0^{-1}\mu-\nu>0$ *if* $0<\varepsilon_0<1$*. Indeed, since* $1-\varepsilon_0^{-1}<0$ *and* $k/N\to-\nu>0$ *as* $k,N\to\infty$*, it follows that one can take the interval* I_N *so that* I_N *is contained in the positive real axis and it is arbitrarily close to the origin. Hence we have* $\mu=\varepsilon_0(\tilde{\nu}+\nu)>\varepsilon_0\nu>\nu$*, which implies that the action is not a Poincaré morphism. Similarly, we can show that there exists* μ *such that the condition does not hold in other cases.*

The proof of Theorem 3.1 follows from the following propositions.

Proposition 3.5 *Assume that* $\varepsilon_0\neq 0$ *is a real number. Then there exist* $\nu<\mu<0$, $(\nu,\mu)\notin\mathbb{Q}\times\mathbb{Q}$ *with the density of continuum, real numbers* c_1, c_2 *and* $k_0>0$ *such that for any* $g\in\mathcal{G}$, $g=\sum_{|\alpha|\geq k_0}g_\alpha x^\alpha$, *there exist* $f_j\in\mathcal{G}$ $(j=1,2)$ *such that*

$$L_1f_2=L_2f_1,\quad g=c_1f_1+c_2f_2.$$

Moreover, $B:=c_1A_1+c_2A_2$ *is nonresonant, and* ω *defined by (1.16) for Spec*(B) *satisfies (3.5).*

Proposition 3.6 *Assume that* $\varepsilon_0\neq 0$ *is a real number. Let* ν, μ, c_1 *and* c_2 *be as in Proposition 3.5. Then there exists* $g\in\mathcal{G}$ *such that the homology equation* $L_Bv=g$ *with* $B:=c_1A_1+c_2A_2$ *has a unique formal power series solution* v *which is not contained in* $\cup_{1\leq\sigma<5/2}G_2^\sigma(\mathbb{C}^4)$.

Remark 3.7 *Our divergence results imply in the case of a single holomorphic vector field, that generically vector fields obtained by nonlinear holomorphic perturbations are nonlinearizable (see R. Pérez Marco [21] for more details). As to the case of smooth* C^∞ *hyperbolic* $\mathbb{R}^2$ *actions we refer [10]. We point out that the divergence in Gevrey classes of formal solutions of overdetermined systems of linear homological equations generalize those for the single vector fields in the presence of nontrivial Jordan blocks (see [15]).*

First we will show Theorem 3.1, assuming Propositions 3.5 and 3.6.
Proof of Theorem 3.1. We will prove the former half. By the result of Example 1.6 (a), we know that if $(\nu,\mu)\in\mathbb{Q}_-\times\mathbb{Q}_-$, then (3.6) has an infinite resonance. It follows that (3.6) with $f=0$ has a formal power series solution $v\notin\bigcup_{1\leq\sigma<5/2}G_2^\sigma(\mathbb{C}^4)$, because (3.6) is a linear equation. Next we assume $(\nu,\mu)\notin\mathbb{Q}\times\mathbb{Q}$. By Proposition 3.6 we can choose $g\in\mathcal{G}$ such that the unique solution v of $L_Bv=g$ with $B=c_1A_1+c_2A_2$ is not contained in $\bigcup_{1\leq\sigma<5/2}G_2^\sigma(\mathbb{C}^4)$. By Proposition 3.5 we choose $f_j\in\mathcal{G}$ $(j=1,2)$ such that $L_1f_2=L_2f_1$ and $g=\sum_{j=1}^2c_jf_j$. Because the solution v of the system of equations $L_Av=f$ is a unique solution of a single equation $L_Bv=g$, we see that v is not contained in $\bigcup_{1\leq\sigma<5/2}G_2^\sigma(\mathbb{C}^4)$. This proves the former half of the theorem.

We will prove the latter half. We consider the system of equations $L_jv=f_j(j=1,2)$, where $L_1f_2=L_2f_1$. Let B denote either A_1 or A_2. For the sake of simplicity, we assume that B is put in a Jordan normal form with the diagonal part $B^0:=\mathrm{diag}\{\lambda_1,\lambda_2,\lambda_3,\lambda_3\}$. The off diagonal element of B is denoted by ε_1. We note that, for the equation $L_1v=f_1$ we have $\lambda_1=\lambda_2=1$, $\lambda_3=\nu$, $\varepsilon_1=\varepsilon$, while for $L_2v=f_2$ we have $\lambda_1=0,\lambda_2=1$, $\lambda_3=\mu$, $\varepsilon_1=\varepsilon_0\varepsilon$. The homology operator corresponding to B is given by

$$L_Bv = \langle B^0x,\partial_x\rangle v+\varepsilon_1R[v]-Bv,\qquad v\in\mathbb{C}_2^4\{x\}, \tag{3.7}$$

$$\langle B^0x,\partial_x\rangle v = \sum_{|\alpha|\geq 2}(\lambda_1\alpha_1+\lambda_2\alpha_2+\lambda_3(\alpha_3+\alpha_4))v_\alpha x^\alpha, \tag{3.8}$$

where $v(x) = \sum_{|\alpha|\ge 2} v_\alpha x^\alpha$ and

$$R[v] \;=\; \sum_{|\alpha|\ge 2} (\alpha_3+1) v_{(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4-1)} x^\alpha. \tag{3.9}$$

For $g(x) = {}^t(g_1, g_2, g_3, g_4) \in \mathbb{C}_2^4\{x\}$ we expand $g_k(x)$ in the Taylor series $g_k(x) = \sum_\alpha g_{\alpha;k} x^\alpha$. For nonnegative integers N, α_1 and α_2 we define V_k and G_k by

$$V_k := {}^t\{v_{(\alpha_1,\alpha_2,N-\ell,\ell);k}\}_{\ell=0}^N, \quad G_k := {}^t\{g_{(\alpha_1,\alpha_2,N-\ell,\ell);k}\}_{\ell=0}^N \quad (k = 1, 2, 3, 4).$$

In view of (3.9), the equation $L_B v = g$ is equivalent to

$$(\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3 N - \lambda_1)V_1 + \varepsilon_1 \mathcal{M}_N V_1 \;=\; G_1, \tag{3.10}$$
$$(\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3 N - \lambda_2)V_2 + \varepsilon_1 \mathcal{M}_N V_2 \;=\; G_2, \tag{3.11}$$
$$(\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3 (N-1))V_3 + \varepsilon_1 \mathcal{M}_N V_3 \;=\; G_3 + \varepsilon_1 V_4, \tag{3.12}$$
$$(\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3 (N-1))V_4 + \varepsilon_1 \mathcal{M}_N V_4 \;=\; G_4, \tag{3.13}$$

where $\mathcal{M}_N$ is given by

$$\mathcal{M}_N = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ N & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & N-1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & N-2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}, \quad N \ge 1, \tag{3.14}$$

and $\mathcal{M}_0 = 0$.

Let $f^j(x) = {}^t(f_1^j(x), \dots, f_4^j(x))$ and let $f_k^j(x) = \sum_\alpha f_{\alpha;k}^j x^\alpha$ $(j = 1, 2; k = 1, \dots, 4)$ be the Taylor expansion of $f_k^j(x)$. We substitute the expansions of v and f^j into the equations $L_j v = f^j$. For every $(\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ and $N \in \mathbb{Z}_+$ such that $\alpha_1 + \alpha_2 + N \ge 2$ we compare the coefficients of x^α $(\alpha_3 + \alpha_4 = N)$ with homogeneous degree $\alpha_1 + \alpha_2 + N$. If we set

$$F^j = {}^t(F_1^j, F_2^j, \dots, F_4^j), \quad F_k^j = {}^t\{f^j_{(\alpha_1,\alpha_2,N-r,r);k}\}_{r=0}^N, \quad j = 1, 2; k = 1, \dots, 4, \tag{3.15}$$

and $V = {}^t(V_1, V_2, \dots, V_4)$, $V_k := {}^t\{v_{(\alpha_1,\alpha_2,N-\ell,\ell);k}\}_{\ell=0}^N$ $(k = 1, 2, 3, 4)$, $H = {}^t(0, 0, V_4, 0)$, then we can write the system of equations $L_j v = f^j$ $(j = 1, 2)$ in the following form

$$\mathcal{A}V = F^1 + \varepsilon H, \quad \mathcal{B}V = F^2 + \varepsilon\varepsilon_0 H, \tag{3.16}$$

where the matrices $\mathcal{A}$ and $\mathcal{B}$ are the block diagonal matrices given by

$$\mathcal{A} := \mathrm{diag}\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\} = \mathrm{diag}\begin{pmatrix} (\alpha_1 + \alpha_2 + \nu N - 1)Id + \varepsilon M_N) \\ (\alpha_1 + \alpha_2 + \nu N - 1)Id + \varepsilon M_N) \\ (\alpha_1 + \alpha_2 + \nu N - \nu)Id + \varepsilon M_N) \\ (\alpha_1 + \alpha_2 + \nu N - \nu)Id + \varepsilon M_N) \end{pmatrix}, \tag{3.17}$$

$$\mathcal{B} := \mathrm{diag}\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4\} = \mathrm{diag}\begin{pmatrix} (\alpha_2 + \mu N)Id + \varepsilon_0\varepsilon M_N) \\ (\alpha_2 + \mu N - 1)Id + \varepsilon_0\varepsilon M_N) \\ (\alpha_2 + \mu N - \mu)Id + \varepsilon_0\varepsilon M_N) \\ (\alpha_2 + \mu N - \mu)Id + \varepsilon_0\varepsilon M_N) \end{pmatrix}. \tag{3.18}$$

We will construct a formal power series solution V from (3.16). Because $(\nu,\mu) \notin \mathbb{Q}\times\mathbb{Q}$ either ν or μ is an irrational number. Suppose that ν is an irrational number. We want to show that for each $k = 1,\dots,4$ either $\mathcal{A}_k$ or $\mathcal{B}_k$ is nonsingular. In order to see this, suppose that $|\alpha| = \alpha_1+\alpha_2+N \ge 2$. If $N \ne 0,1$, then by the irrationality of ν, the matrices $\mathcal{A}_k$ $(k = 1,\dots,4)$ are nonsingular. If $N = 0$ or $N = 1$, then by the condition $\alpha_1+\alpha_2+N \ge 2$, $\mathcal{A}_k$ $(k = 1,\dots,4)$ are nonsingular. Similarly, if μ is an irrational number, then we can show that either $\mathcal{A}_k$ or $\mathcal{B}_k$ is nonsingular for each $k = 1,\dots,4$.

First we will determine V_4. By inductive arguments and $L_1 f^2 = L_2 f^1$ we get

$$\begin{aligned} v_{(\alpha_1,\alpha_2,N-\ell,\ell);4} &= \sum_{r=0}^{\ell} \frac{(-\varepsilon_1)^r}{(\lambda_1\alpha_1+\lambda_2\alpha_2+\lambda_3(N-1))^{r+1}} \\ &\times \frac{(N-\ell+r)!}{(N-\ell)!} g_{(\alpha_1,\alpha_2,N-\ell+r,\ell-r);4} \end{aligned} \tag{3.19}$$

for $\ell = 0,1,\dots,N$, provided $\lambda_1\alpha_1+\lambda_2\alpha_2+\lambda_3(N-1)\ne 0$. Note that, if $\mathcal{A}_4$ is nonsingular, then (3.19) is valid for $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \nu$, $\varepsilon_1 = \varepsilon$, $g_{(\alpha_1,\alpha_2,N-\ell+r,\ell-r);4} = f^1_{(\alpha_1,\alpha_2,N-\ell+r,\ell-r);4}$, while if $\mathcal{B}_4$ is nonsingular, then (3.19) is valied for $\lambda_1 = 0, \lambda_2 = 1$, $\lambda_3 = \mu$, $\varepsilon_1 = \varepsilon_0\varepsilon$, $g_{(\alpha_1,\alpha_2,N-\ell+r,\ell-r);4} = f^2_{(\alpha_1,\alpha_2,N-\ell+r,\ell-r);4}$. Similar explicit formulas are derived for $v_{\alpha_1,\alpha_2,N-\ell,\ell;k}$, $k = 1,2$. As to the term $v_{(\alpha_1,\alpha_2,N-\ell,\ell);3}$, there appears the term $\varepsilon_1 V_4^N$ in the right-hand side of (3.12).

By (3.4) we have

$$|\alpha_1+\alpha_2+\nu N-\nu| + |\alpha_2+\mu N-\mu| \ge C|\alpha_1+\alpha_2+N|^{-\tau} \tag{3.20}$$

for some $C > 0$. It follows that either $|\alpha_1+\alpha_2+\nu N-\nu| \ge C|\alpha_1+\alpha_2+N|^{-\tau}/2$ or $|\alpha_2+\mu N-\mu| \ge C|\alpha_1+\alpha_2+N|^{-\tau}/2$ holds. Suppose that the former estimate holds. We have the same estimate in case the latter inequality holds. Without loss of generality we may assume that $C < 2$. Let τ be such that $\tau > \tau_0$. Then we have

$$\begin{aligned} |\alpha_1+\alpha_2+\nu(N-1)|^{r+1} &\ge (C/2)^{r+1}|\alpha_1+\alpha_2+N|^{-\tau(r+1)} \\ &\ge (C/2)^{N+1}|\alpha_1+\alpha_2+N|^{-\tau(N+1)}. \end{aligned} \tag{3.21}$$

Noting that $(N-\ell+r)!/(N-\ell)! \le N!$, we see from (3.19) that if $g_{(\alpha_1,\alpha_2,N-\ell+r,\ell-r);4}$ has a G^s estimate, namely, $g_{(\alpha_1,\alpha_2,N-\ell+r,\ell-r);4} = O((\alpha_1+\alpha_2+N)!^{s-1})$ modulo exponential factors, then $v_{(\alpha_1,\alpha_2,N-\ell,\ell);4} = O((\alpha_1+\alpha_2+N)!^{s+\tau})$. Especially, if $s = 1$, then we have $v_{(\alpha_1,\alpha_2,N-\ell,\ell);4} = O((\alpha_1+\alpha_2+N)!^{\tau+1})$. Similarly, we can easily see that $v_{(\alpha_1,\alpha_2,N-\ell,\ell);j}$ $(j = 1,2,4)$ have the estimate $v_{(\alpha_1,\alpha_2,N-\ell,\ell);j} = O((\alpha_1+\alpha_2+N)!^{\tau+1})$.

Next we determine $v_{(\alpha_1,\alpha_2,N-\ell,\ell);3}$ by a similar relation like (3.19). We can easily see that there appears $v_{(\alpha_1,\alpha_2,N-\ell,\ell);4}$ in the right-hand side of the recurrence relation. Hence the right-hand side is $O((\alpha_1+\alpha_2+N)!^{\tau+1})$. It follows that $v_{(\alpha_1,\alpha_2,N-\ell,\ell);3} = O((\alpha_1+\alpha_2+N)!^{2\tau+2})$. Since $\tau > \tau_0$ is arbitray, $v_{(\alpha_1,\alpha_2,N-\ell,\ell);3} = O((\alpha_1+\alpha_2+N)!^{\sigma})$ for $\sigma > 2+2\tau_0$. This ends the proof of Theorem 3.1.

Proof of Proposition 3.5. The eigenvalues of $B := c_1A_1+c_2A_2$ is given by $c_1, c_1+c_2, c_1\nu+c_2\mu$ with multiplicity. We shall show that there exists a set $E \subset \mathbb{R}^2$ with Lebesgue measure zero such that if $(c_1,c_2) \notin E$, then B is nonresonant. For every $\alpha = (\alpha_1,\dots,\alpha_4) \in \mathbb{Z}_+^4$, $|\alpha| \ge 2$, the resonance relations are given by

$$c_1\alpha_1 + (c_1+c_2)\alpha_2 + (c_1\nu+c_2\mu)(\alpha_3+\alpha_4) = c_1, \tag{3.22}$$

and the ones with c_1 in the right-hand side replaced by c_1+c_2 and $c_1\nu+c_2\mu$, respectively. Because the argument is similar, we consider the first relation. It follows from (3.22) that

$$c_1(\alpha_1+\alpha_2+\nu(\alpha_3+\alpha_4)-1) + c_2(\alpha_2+\mu(\alpha_3+\alpha_4)) = 0.$$

Because $(\nu,\mu)\notin\mathbb{Q}\times\mathbb{Q}$ and $|\alpha|\geq 2$, we can easily see that either $\alpha_1+\alpha_2+\nu(\alpha_3+\alpha_4)-1\neq 0$ or $\alpha_2+\mu(\alpha_3+\alpha_4)\neq 0$ holds. Hence the set of $(c_1,c_2)\in\mathbb{R}^2$ satisfying (3.22) is a straight line. Therefore the set E of all (c_1,c_2) satisfying a resonance relations has Lebesgue measure zero.

In order to see that $Spec(B)$ satisfies (3.5), let $\tilde{\omega}_j(\alpha)$ ($\alpha\in\mathbb{Z}_+^4$) be defined by (1.15) for B. Then there exists $K>0$ such that $\tilde{\omega}_j(\alpha)\leq K\omega_j(\alpha)$ for $j=1,\ldots,4$ and all $\alpha\in\mathbb{Z}_+^4$. It follows that (3.5) holds for $\tilde{\omega}_j(\alpha)$.

Let $(c_1,c_2)\notin E$ and $g\in\mathcal{G}$ be given. We want to solve the system of equations

$$L_1f^2=L_2f^1,\qquad c_1f^1+c_2f^2=g. \tag{3.23}$$

By expanding $f^j(x)={}^t(f_1^j,f_2^j,\ldots,f_4^j)$ into the Taylor series we define F^j by (3.15). We similarly define $G={}^t(G_1,G_2,\ldots,G_4)$, $G_k=\{g_{(\alpha_1,\alpha_2,N-r,r),k}\}_{r=0}^N$, where $g(x)={}^t(g_1,g_2,\ldots,g_4)$, $g_k(x)=\sum_\alpha g_{\alpha,k}x^\alpha$. We set $H^1:={}^t(0,0,F_4^1,0)$ and $H^2:={}^t(0,0,F_4^2,0)$. We substitute the expansions of f^j and g into (3.23). For every $(\alpha_1,\alpha_2)\in\mathbb{Z}_+^2$ and $N\in\mathbb{Z}_+$ such that $\alpha_1+\alpha_2+N\geq 2$ we compare the coefficients of x^α of homogeneous degree $\alpha_1+\alpha_2+N$. Then we can write (3.23) in the following form

$$\mathcal{A}F^2-\mathcal{B}F^1-\varepsilon H^2+\varepsilon\varepsilon_0H^1=0,\qquad c_1F^1+c_2F^2=G, \tag{3.24}$$

where $\mathcal{A}$ and $\mathcal{B}$ are given by (3.17) and (3.18).

First we will construct a formal power series solution F^j ($j=1,2$) of (3.24) for a given G. Because we know that (cf. the proof of Theorem 3.1) either $\mathcal{A}_k$ or $\mathcal{B}_k$ is nonsingular for each $k=1,2,\ldots,4$, it follows from (3.24) that

$$\mathcal{A}_kF_k^2-\mathcal{B}_kF_k^1=0,\quad c_1F_k^1+c_2F_k^2=G_k,\quad k=1,2,4.$$

Assuming that $\mathcal{A}_k$ is nonsingular we obtain $F_k^2=\mathcal{A}_k^{-1}\mathcal{B}_kF_k^1$, and hence $c_1F_k^1+c_2\mathcal{A}_k^{-1}\mathcal{B}_kF_k^1=G_k$. It follows that

$$F_k^1=(c_1+c_2\mathcal{A}_k^{-1}\mathcal{B}_k)^{-1}G_k=(c_1\mathcal{A}_k+c_2\mathcal{B}_k)^{-1}\mathcal{A}_kG_k, \tag{3.25}$$

if $c_1\mathcal{A}_k+c_2\mathcal{B}_k$ is nonsingular. The last condition holds if (c_1,c_2) is not contained in a set of Lebesgue measure zero in $\mathbb{R}^2$, which may depend on α_1,α_2,N. We have similar relations if $\mathcal{B}_k$ is nonsingular.

In case $k=3$, we obtain $\mathcal{A}_3F_3^2-\mathcal{B}_3F_3^1=\varepsilon(F_4^2-\varepsilon_0F_4^1)$ instead of $\mathcal{A}_kF_k^2-\mathcal{B}_kF_k^1=0$. A simple computation yields that

$$F_3^1=(c_1\mathcal{A}_3+c_2\mathcal{B}_3)^{-1}\mathcal{A}_3G_3-\varepsilon c_2(c_1\mathcal{A}_3+c_2\mathcal{B}_3)^{-1}(F_4^2-\varepsilon_0F_4^1).$$

By taking the union of all exceptional sets of (c_1,c_2) with α_1,α_2 and N in the set of nonnegative integers such that $\alpha_1+\alpha_2+N\geq 2$, we see that there exists a unique formal power series solution $f^j(x)$ ($j=1,2$) of (3.23), provided (c_1,c_2) is not in an exceptional set of Lebesgue measure zero.

We will show the convergence of $f^j(x)$ ($j=1,2$). It is sufficient to prove the convergence of $f^1(x)$ since we may take $c_1\neq 0$ in view of the choice of c_1 in the above argument. By the definition of $\mathcal{G}$ and (2), we can easily see that L_1^{-1} exists on $\mathcal{G}$, namely

$$L_1^{-1}L_1=L_1L_1^{-1}=Id\quad\text{on}\quad\mathcal{G}.$$

Let $g\in\mathcal{G}$. Then it follows from (3.23) and the relation $L_1f_2=L_2f_1$ that $L_1g=c_1L_1f_1+c_2L_2f_1$. Hence we have

$$g=c_1f_1+c_2L_1^{-1}L_2f_1.$$

Now we have

$$L_1^{-1}L_2 = L_2L_1^{-1} = (L_2 - \varepsilon_0 L_1)L_1^{-1} + \varepsilon_0 Id, \quad \text{on } \mathcal{G}.$$

By definition, we have

$$L_2 - \varepsilon_0 L_1 = \langle A_2 x, \partial_x\rangle - \varepsilon_0\langle A_1 x, \partial_x\rangle + \varepsilon_0 A_1 - A_2.$$

Hence, $L_2 - \varepsilon_0 L_1$ is semi-simple. By the condition (2) and the proof of the latter half of Theorem 3.1, it follows that the absolute value of the coefficient of x^α of $L_1^{-1}g$ ($g = \sum_\alpha g_\alpha x^\alpha$) is bounded by $N^{(\tau''+1)N}C^N|g_\alpha|$ for some $C > 0$, where $\tau'' > \tau_0$ can be taken arbitrarily close to τ_0. On the other hand, the operator $(L_2 - \varepsilon_0 L_1)$ is the one which multiplies the coefficients of x^α with $(\alpha_2 + \mu N - \varepsilon_0(\alpha_1 + \alpha_2 - 1 + \nu N))$ for the first component. We have similar expressions for other components. By the condition $(1)'$, the absolute value of the term is bounded by $N^{-N(\tau+1)}c_0^n$ for some $\tau > \tau_0$. Because $\tau'' > \tau_0$ can be taken arbitrarily close to τ_0, the growth $N^{N(\tau''+1)}C^n$ which comes from L_1^{-1} is absorbed by the term $N^{-N(\tau+1)}c_0^n$. Therefore, the operator $(L_2 - \varepsilon_0 L_1)L_1^{-1}$ maps $\mathcal{G}$ to $\mathcal{G}$. By taking k_0 sufficiently large, the norm of $(L_2 - \varepsilon_0 L_1)L_1^{-1}$ on the space $\mathcal{G} \cap \{g = \sum_\alpha g_\alpha x^\alpha; |\alpha| > k_0\}$ can be made arbitrarily small.

In view of the construction of c_1 and c_2 we may assume that $c_1 + c_2\varepsilon_0 \neq 0$. Writing

$$g = c_1 f_1 + c_2 L_1^{-1}L_2 f_1 = (c_1 + c_2\varepsilon_0 + R)f_1,$$

where $R = (\varepsilon_0 L_1 - L_2)L_1^{-1}$, and by noting that R preserves homogeneous polynomials, we see that $(c_1 + c_2\varepsilon_0 + R)^{-1}$ exists as a map from $\mathcal{G}$ to $\mathcal{G}$. Therefore we have $f_1 \in \mathcal{G}$. This completes the proof of Proposition 3.5.

Proof of Proposition 3.6 : Let g be the convergent power series defined by $g_{\alpha;k} = 0$ for $k = 1, 2, 3$ and all $\alpha \in \mathbb{Z}_+^4(2)$; $g_{\alpha_1,\alpha_2,\alpha_3,\alpha_4;4} = 0$ if $\alpha_4 \geq 1$; $g_{(\alpha_1,\alpha_2,N,0);4} = 1$ for $(\alpha_1, \alpha_2, N, 0) \in C_4$, where C_4 is given in the definition of $\mathcal{G}$. We want to solve $L_B v = g$. Let λ_j be the eigenvalues of B. By the same argument as in the proof of Theorem 3.1 we have the formula (3.19). Then we have

$$v_{(\alpha_1,\alpha_2,0,N);4} = (-\varepsilon_1)^N(\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3(N-1))^{-N-1}N!, \tag{3.26}$$

for all $(\alpha_1, \alpha_2, N) \in C_4$. We can easily see from the conditions (1) and (2) of the definition of $\mathcal{G}$ that $Spec(A_2)$ also satisfies (2). It follows that $Spec(B) = Spec(c_1A_1 + c_2A_2)$ also satisfies the following estimate: for every $\tau' < \tau_0$ we can find a constant $C > 0$ and a subsequence $\{(\alpha_{1,k}, \alpha_{2,k}, N_k)\}_{k=1}^\infty$ such that

$$|(\lambda_1\alpha_{1,k} + \lambda_2\alpha_{2,k} + \lambda_3(N_k - 1))^{-1}| \geq CN_k^{\tau'}, \quad \forall k \in \mathbb{N}.$$

Therefore, by (3.26)

$$|v_{(\alpha_{1,k},\alpha_{2,k},0,N_k);4}| \geq (C|\varepsilon_1|)^{N_k}N_k^{(N_k+1)\tau'}N_k!, \qquad k \in \mathbb{N}, \alpha_1 \in \mathbb{Z}_+(2). \tag{3.27}$$

Because $\varepsilon_1 \neq 0$, $\tau' < \tau_0$ and $1/2 \leq \tau_0$, (3.27) and Stirling's formula, $N! \geq C^N N^N, \forall N \in \mathbb{Z}_+$ lead to the assertion. This ends the proof of Proposition 3.6.

Example 3.8 *We give an example of a formal Gevrey linearization. (cf. Theorem 3.1.) We consider*

$$L_\Lambda u = R(x+u), \qquad \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\tau & -1 \\ 0 & 0 & -\tau \end{pmatrix}, \tag{3.28}$$

where $\tau > 0$ is an irrational number. For $C \gg 1$, let f be an analytic function $f(x_1, x_2) = \sum_\alpha f_\alpha x_1^{\alpha_1}x_2^{\alpha_2}$, where the summation with respect to α is taken for $\alpha \in \mathbb{Z}_+^2(2)$ such that $1 <$

$\alpha_1-\tau\alpha_2<C$, $\alpha_1+\alpha_2\to\infty$. We define $R(x)={}^t(x_1^2,x_2f(x_1,x_2),0)$. Then we shall show that the unique solution of (3.28) is in G^2.

Indeed, we may look for the solution of the equation in the form $u={}^t(x_1^2,x_2w(x),0)$. We can easily see that w satisfies the equation

$$(x_1\partial_{x_1}-\tau x_2\partial_{x_2}-\tau x_3\partial_{x_3}+x_3\partial_{x_2})w=(1+w)f(x_1+x_1^2,x_2(1+w))\equiv g(x). \tag{3.29}$$

We substitue the expansion $w(x)=\sum_\alpha w_\alpha x^\alpha$ into (3.29). We can easily see that the sum of the expansion of $w(x)$ can be taken for α such that $\alpha_1-\tau(\alpha_2+\alpha_3)>1$, because the support of f satisfies the property and the left-hand side operator of (3.29) maps functions with support in $\alpha_1-\tau(\alpha_2+\alpha_3)>1$ to those with the same property. If we expand $g(x)=\sum_\alpha g_\alpha x^\alpha$, then by the same calculations as in (3.19) we obtain

$$w_{(\alpha_1,N-\ell,\ell)}=\sum_{r=0}^{\ell}\frac{1}{(\alpha_1-\tau N)^{r+1}}\frac{(N-\ell+r)!}{(N-\ell)!}g_{(\alpha_1,N-\ell+r,\ell-r)},\quad r=0,1,\dots,\ell. \tag{3.30}$$

If we can show that $g_{(\alpha_1,N-\ell+r,\ell-r)}=O((\ell-r)!)$ modulo terms of order K^{α_1+N} $(K>0)$, then we can easily see that $w_{(\alpha_1,N-\ell,\ell)}=O(\ell!)$. This proves that the solution u of (3.29) is in G^2.

If $\alpha_1+N=2$, then no term from w appears in $g_{(\alpha_1,N-\ell+r,\ell-r)}$ in (3.30). Hence, by the analyticity assumption of f, we obtain the desired estimate for w_α with $\alpha_1+N=2$, $\alpha_2+\alpha_3=N$. Suppose that we have $w_{(\alpha_1,N-\ell,\ell)}=O(\ell!)$ up to $\alpha_1+N<\nu$ for some $\nu>2$. Then by the definition of $g(x)=(1+w)f(x_1+x_1^2,x_2(1+w))$ and simple computations of the substitution of a Gevrey power series into an analytic functions, we see that $g_{(\alpha_1,N-\ell,\ell)}=O(\ell!)$. Hence, by the inductive argument we obtain the desired estimate, $w_{(\alpha_1,N-\ell,\ell)}=O(\ell!)$, $\alpha_1+N=\nu$. This completes the proof.

We will briefly mention the general case of $d-$ *actions*. We suppose that there exist j, $1\le j\le m$ and ℓ_0, $1\le\ell_0\le d$ such that $A_j^{\ell_0}$ in (1.7) admits only one dimensional eigenspace, i.e., the geometric multiplicity of λ_j^ℓ is one. For a positive integer r we define the r square nilpotent matrix N_r by

$$N_r=\begin{pmatrix}0&1&0&\dots&0\\0&0&1&\dots&0\\\vdots&\vdots&\ddots&\ddots&\vdots\\0&0&0\dots&0&1\\0&0&0\dots&0&0\end{pmatrix}. \tag{3.31}$$

By assumption we have

$$A_j^{\ell_0}=\lambda_j^{\ell_0}Id+\varepsilon N_{s_j},\quad \varepsilon\neq0. \tag{3.32}$$

By the explicit description of the centralizers of matrices (cf. [14]) all other matrices have the following form

$$A_j^\ell=\lambda_j^\ell Id+\sum_{k=1}^{s_j-1}\varepsilon_k^{\ell j}(N_{s_j})^k\quad \varepsilon_k^{\ell j}\in\mathbb{C},\ k=1,\dots,s_j-1. \tag{3.33}$$

We have

Theorem 3.9 *Assume (3.32). Then there exist $\varepsilon_k^{\ell j}$ in (3.33), λ_j^ℓ, $(\ell=1,2,\dots,d;j=1,2,\dots,n)$ with the density of continuum such that the followings hold:*
(i) The simultaneously nonresonant condition (1.17) and the following condition hold.

There exists a sequence $\alpha^\ell\in\mathbb{Z}_+^n(2)$, $\ell\in\mathbb{N}$ and a positive number c_0 such that $|\alpha^\ell|\to\infty$ $(\ell\to\infty)$ and

$$0<\omega(\alpha^\ell)\le c_0,\quad \ell\in\mathbb{N}. \tag{3.34}$$

(ii) There exists an $f:={}^t(f_1,f_2,\dots,f_d)\in(\mathbb{C}_2^\sigma\{x\})^d$ satisfying (3.2) such that $v=L_A^{-1}f$ is not contained in the set $\bigcup_{1\le\sigma<2}G_2^\sigma(\mathbb{C}^n)$.

4 Sternberg's theorem for commuting vector fields

The results in section 2 imply that the simultaneous linearization of a Poincaré morphism with a Jordan block is reduced essentially to the Poincaré–Dulac theorem for a single vector field in an analytic category. On the other hand, in view of the results in section 3, the reduction seems impossible if the action is not a Poincaré morphism.

In this section we shall illustrate that the situtation is completely different in a smooth category. We consider two commuting vector field in $\mathbb{R}^4$ which are in a Siegle domain and only one of the two has a linear part with nontrivial Jordan block. Obviously, the action is not a Poincaré morphism. We will show that they are simultaneously linearizable in C^k for every $k \geq 1$.

Let $X(y)$ and $Y(y)$ be commuting C^∞ vector fields with the common singular point at the origin $0 \in R^4$. Suppose that $\nabla X(0) = A$, $\nabla Y(0) = B$, where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\nu & 0 \\ 0 & 0 & 0 & -\nu \end{pmatrix}, \tag{4.1}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\mu & \varepsilon \\ 0 & 0 & 0 & -\mu \end{pmatrix}, \quad \varepsilon \neq 0. \tag{4.2}$$

We assume that the action is not a Poincaré morphism, namely, (cf. Example 4.1)

$$\nu > \mu > 0, \; \nu \in \mathbb{R} \setminus \mathbb{Q}. \tag{4.3}$$

We also note that the irrationality of ν implies that X, and hence the pair (X, Y) is nonresonant. Then we have

Theorem 4.1 *Suppose that the conditions (4.1), (4.2) and (4.3) are verified. Let $m \geq 1$ be an integer. Then there exists a C^m change of the variables $y = u(x) = x + v(x)$, $v(0) = 0$, $\nabla v(0) = 0$ near the origin which transforms both X and Y to their linear parts.*

We need to prepare lemmas in order to prove our theorem. In view of Sternberg's theorem we assume, without loss of generality, that X is linear, i.e.

$$Xv(y) = \langle \nabla v(y), Ay \rangle. \tag{4.4}$$

Let $R(y) = (R_1(y), R_2(y), R_3(y), R_4(y))$ be the nonlinear part of Y

$$Yf(y) = \langle \nabla f(y), By + R(y) \rangle. \tag{4.5}$$

Suppose that the change of variables $y = u(x) = x + v(x)$, $v(0) = 0$, $\nabla v(0) = 0$ linearizes both the vector fields X and Y. Then we can easily see that $v(x)$ satisfies the system of homology equations

$$\langle \nabla v(x), Bx \rangle - Bv = R(x + v(x)), \tag{4.6}$$

and

$$\langle \nabla v(x), Ax \rangle - Av = 0. \tag{4.7}$$

We write $x = (x_1, x_2, x'')$ and $z = (z_1, z')$. Let $c_1 > 0$ and $0 < c_2 \leq 1$ be constants. Then we define

$$\Omega = \{x' = (x_2, x_3, x_4) = (x_2, x'') \in \mathbb{R}^3; |x_2| < c_1,\ |x''| < c_2\}, \tag{4.8}$$

$$\Omega_1 = \{x_1 \in \mathbb{R}; |x_1| < 1\} \times \Omega. \tag{4.9}$$

Then we have

Lemma 4.2 *Let $k = \infty$ or $k \geq 1$ be an integer. Let L be given by*

$$L = \sum_{j=1}^{2} x_j \partial_{x_j} - \nu \sum_{k=3}^{4} x_k \partial_{x_k}.$$

Then a C^k solution of

$$Lf(x) - f(x) \ = \ 0, \quad x = (x_1, x_2, x_3, x_4) \in \Omega_1, \tag{4.10}$$

(respectively,

$$Lw(x) + \nu w(x) \ = \ 0 \quad x = (x_1, x_2, x_3, x_4) \in \Omega_1) \tag{4.11}$$

is given by

$$f(x) \ = \ x_1 \varphi_\pm(\frac{x_2}{x_1}, x_3|x_1|^\nu, x_4|x_1|^\nu), \quad \textit{for } \pm x_1 > 0, \tag{4.12}$$

or

$$f(x) \ = \ x_2 \varphi_\pm(\frac{x_2}{x_1}, x_3|x_1|^\nu, x_4|x_1|^\nu), \quad \textit{for } \pm x_1 > 0, \tag{4.13}$$

(respectively, by

$$w(x) \ = \ |x_1|^{-\nu} \psi_\pm(\frac{x_2}{x_1}, x_3|x_1|^\nu, x_4|x_1|^\nu), \quad \textit{for } \pm x_1 > 0\ ), \tag{4.14}$$

where $\varphi_\pm(z) \in C^k(\Omega)$ (respectively $\psi_\pm(z) \in C^k(\Omega)$.)

Proof. Let L be the operator given in the lemma. We want to solve (4.10) and (4.11). First we solve (4.10) in the region $x_1 > 0$. If we set $f(x) = x_1\varphi(x)$ (resp. $f(x) = x_2\psi(x)$), then we have that

$$L\varphi(x) = 0, \qquad (\textit{resp. } L\psi(x) = 0). \tag{4.15}$$

By the theorem in page 61 of [2], the solutions of (4.15) are given by the first integral of the corresponding characteristic equation. For the sake of simplicity, we consider the equation $L\varphi(x) = 0$. The characteristic equation is given by

$$\frac{dx_1}{x_1} = \frac{dx_2}{x_2} = -\frac{dx_3}{\nu x_3} = -\frac{dx_4}{\nu x_4}. \tag{4.16}$$

If we integrate (4.16) by taking x_1 as an independent variable, then we obtain

$$x_2 = x_1 x_2^0,\ x_3 = x_1^{-\nu} x_3^0,\ x_4 = x_1^{-\nu} x_4^0, \tag{4.17}$$

where x_2^0, x_3^0, x_4^0 are certain constants. It follows that the first integral $\varphi_+(x)$ is given by

$$\varphi_+(x) \equiv \tilde{\varphi}_+ \left(\frac{x_2}{x_1}, x_3 x_1^\nu, x_4 x_1^\nu \right) = \tilde{\varphi}_+(x_2^0, x_3^0, x_4^0), \tag{4.18}$$

for some differentiable function $\tilde{\varphi}_+$. Hence, the general solution of (4.10) in $x_1 > 0$ is given by $f(x) = x_1\varphi_+(x)$ (resp. $f(x) = x_2\varphi_+(x)$ for possibly different φ_+).

In case $x_1 < 0$ we make the same argument by replacing x_1 with $-x_1$. We see that there exists $\varphi_-(x)$ such that $f(x) = x_1\varphi_-(x)$ (resp. $f(x) = x_2\varphi_-(x)$ for possibly different φ_-.)

Next we consider the equation (4.11). We set $w(x) = |x_1|^{-\nu}\psi(x)$. For the sake of simplicity we consider the case $x_1 > 0$. The case $x_1 < 0$ can be treated similarly if we replace x_1 with $-x_1$. We can easily see that ψ satisfies $L\psi = 0$. Hence it follows from the above argument that

$$w(x) = x_1^{-\nu}\psi_+(x) = x_1^{-\nu}\tilde{\psi}_+\left(\frac{x_2}{x_1}, x_3x_1^\nu, x_4x_1^\nu\right). \tag{4.19}$$

This ends the proof. □

By the commutativity we see that every component of $v = R(x) = (R_1, \ldots, R_4)$ satisfies either (4.10) or (4.11). Hence, by Lemma 4.1 we have

$$\begin{aligned} R_j(x) &= x_j\Psi^j_\pm(\frac{x_2}{x_1}, x_3|x_1|^\nu, x_4|x_1|^\nu), \quad \text{for } \pm x_1 > 0,\ j = 1, 2, & (4.20)\\ R_j(x) &= |x_1|^{-\nu}\Psi^j_\pm(\frac{x_2}{x_1}, x_3|x_1|^\nu, x_4|x_1|^\nu), \quad \text{for } \pm x_1 > 0,\ j = 3, 4 & (4.21) \end{aligned}$$

for some functions $\Psi^j_\pm$. In the following we will cut off $R_j(x)$ with a smooth function being identically equal to 1 in some neighborhood of the origin and with support contained in a small neighborhhood of the origin, which we give in the proof of Theorem 4.1. For the sake of simplicity, we denote the modified $R_j(x)$ with the same letter. We set

$$z_1 = x_2/x_1,\ z_2 = x_3|x_1|^\nu,\ z_3 = x_4|x_1|^\nu. \tag{4.22}$$

For every $x_1 \neq 0$, we define $\Psi^j_\pm(z)$ by (4.20) and (4.21), namely

$$\begin{aligned} \Psi^j_\pm(z) &= x_j^{-1}R_j(x_1, x_1z_1, |x_1|^{-\nu}z_2, |x_1|^{-\nu}z_3), \quad \text{for } \pm x_1 > 0,\ j = 1, 2, & (4.23)\\ \Psi^j_\pm(z) &= |x_1|^\nu R_j(x_1, x_1z_1, |x_1|^{-\nu}z_2, |x_1|^{-\nu}z_3), \quad \text{for } \pm x_1 > 0,\ j = 3, 4. & (4.24) \end{aligned}$$

We can easily see that $\Psi^j_\pm \in C^\infty(\mathbb{R}^3_z)$ $(j = 1, 2, 3, 4)$.

By (4.7) and simple computations we see that every component of $v(x) = (v_1(x), \ldots, v_4(x))$ satisfies either (4.10) or (4.11). It follows from Lemma 4.1 that every component of v has an expression

$$v_j(x) = x_j\varphi^j_\pm(\frac{x_2}{x_1}, x_3|x_1|^\nu, x_4|x_1|^\nu), \quad \text{for } \pm x_1 > 0,\ j = 1, 2, \tag{4.25}$$

and

$$v_j(x) = |x_1|^{-\nu}\varphi^j_\pm(\frac{x_2}{x_1}, x_3|x_1|^\nu, x_4|x_1|^\nu), \quad \text{for } \pm x_1 > 0,\ j = 3, 4, \tag{4.26}$$

for some $\varphi^j_\pm$.

We substitute the transformation (4.22) and (4.25), (4.26) into (4.6), and we rewrite (4.6) as an equation of z for the unknown functions $\varphi^j_\pm(z)$ with a parameter x_1. Recalling that $v_j = x_j\varphi^j_\pm$ and $v_j = |x_1|^{-\nu}\varphi^j_\pm$ we obtain

$$\begin{aligned} x_2\partial_{x_2}v_1 &= x_1z_1\partial_{z_1}\varphi^1_\pm(z), \quad x_3\partial_{x_3}v_1 = x_1z_2\partial_{z_2}\varphi^1_\pm(z), & (4.27)\\ x_4\partial_{x_4}v_1 &= x_1z_3\partial_{z_3}\varphi^1_\pm(z), \quad x_4\partial_{x_3}v_1 = x_1z_3\partial_{z_2}\varphi^1_\pm(z), & (4.28) \end{aligned}$$

and we have similar relations for $v_2 = x_2\varphi_\pm^2(x)$ and $v_j = |x_1|^{-\nu}\varphi_\pm^j(x)$. In fact we have

$$\langle \nabla v_1(x), Bx\rangle = x_1\mathcal{L}\varphi_\pm^1(z), \quad \text{for } \pm x_1 > 0, \tag{4.29}$$
$$\langle \nabla v_2(x), Bx\rangle - v_2(x) = x_2\mathcal{L}\varphi_\pm^2(z), \quad \text{for } \pm x_1 > 0, \tag{4.30}$$
$$\langle \nabla v_j(x), Bx\rangle = |x_1|^{-\nu}\mathcal{L}\varphi_\pm^j(z), \quad \text{for } \pm x_1 > 0,\ j = 3, 4, \tag{4.31}$$

where

$$\mathcal{L}f(z) \;=\; z_1\partial_{z_1}f(z) - (\mu z_2 - \varepsilon z_3)\partial_{z_2}f(z) - \mu z_3\partial_{z_3}f(z). \tag{4.32}$$

We define $\varphi_\pm(z) = {}^{tr}(\varphi_\pm^1(z), \varphi_\pm^2(z), \varphi_\pm^3(z), \varphi_\pm^4(z))$.

Lemma 4.3 *We have the expression*

$$R_j(x + v(x)) = x_jE_\pm^j(z, \varphi_\pm(z)), \quad \textit{for } \pm x_1 > 0,\ j = 1, 2, \tag{4.33}$$

where $E_\pm^j(z, w)$ is given by

$$E_\pm^j(z, w) = (1 + w_j)\Psi_\pm^j\left(z_1\frac{1 + w_2}{1 + w_1}, (z_2 + w_3)|1 + w_1|^\nu, (z_3 + w_4)|1 + w_1|^\nu\right) \tag{4.34}$$

and

$$R_j(x + v(x)) = |x_1|^{-\nu}E_\pm^j(z, \varphi_\pm(z)) \quad \textit{for } \pm x_1 > 0,\ j = 3, 4, \tag{4.35}$$

with

$$E_\pm^j(z, w) = |1 + w_1|^{-\nu}\Psi_\pm^j\left(z_1\frac{1 + w_2}{1 + w_1}, (z_2 + w_3)|1 + w_1|^\nu, (z_3 + w_4)|1 + w_1|^\nu\right). \tag{4.36}$$

Proof. We have

$$\begin{aligned}
\frac{x_2 + v_2(x)}{x_1 + v_1(x)} &= \frac{x_2(1 + \varphi_\pm^2(z))}{x_1(1 + \varphi_\pm^1(z))} \\
&= \frac{x_2}{x_1}\frac{1 + \varphi_\pm^2(z)}{1 + \varphi_\pm^1(z)} = z_1\frac{1 + \varphi_\pm^2(z)}{1 + \varphi_\pm^1(z))}. && (4.37)\\
(x_3 + v_3(x))|x_1 + v_1|^\nu &= (x_3 + |x_1|^{-\nu}\varphi_\pm^3(z))|x_1|^\nu|1 + \varphi_\pm^1(z)|^\nu \\
&= (x_3|x_1|^\nu + \varphi_\pm^3(z))|1 + \varphi_\pm^1(z)|^\nu \\
&= (z_2 + \varphi_\pm^3(z))|1 + \varphi_\pm^1(z)|^\nu. && (4.38)\\
(x_4 + v_4(x))|x_1 + v_1|^\nu &= (x_4 + |x_1|^{-\nu}\varphi_\pm^4(z))|x_1|^\nu|1 + \varphi_\pm^1(z)|^\nu \\
&= (x_4|x_1|^\nu + \varphi_\pm^4(z))|1 + \varphi_\pm^1(z)|^\nu \\
&= (z_3 + \varphi_\pm^4(z))|1 + \varphi_\pm^1(z)|^\nu. && (4.39)
\end{aligned}$$

Hence, if $j = 1, 2$, we get

$$\begin{aligned}
& R_j(x + v(x)) = (x_j + v_j(x)) \\
\times\;& \Psi_\pm^j(\frac{x_2 + v_2(x)}{x_1 + v_1(x)}, (x_3 + v_3(x))|x_1 + v_1|^\nu, (x_4 + v_4)|x_1 + v_1(x)|^\nu) \\
=\;& x_j(1 + \varphi_\pm^j)\Psi_\pm^j\left(z_1\frac{1 + \varphi_\pm^2}{1 + \varphi_\pm^1}, (z_2 + \varphi_\pm^3)|1 + \varphi_\pm^1|^\nu, (z_3 + \varphi_\pm^4)|1 + \varphi_\pm^1|^\nu\right),
\end{aligned} \tag{4.40}$$

which yields (4.33). Similarly, we can readily prove (4.35). The proof of the lemma is complete. □

Now we are ready to write explicitly the reduction of the overdetermined system for v: $(X_A - A)v = 0$, $(X_b - B)v = R(x + v(x))$ into a 4×4 system of equations for $\varphi_{\pm}(z)$ in $z \in \Omega$ with a parameter x_1. Then the new system of semilinear homological equations for $\varphi_{\pm}$ is written as follows

$$(\mathcal{L} - \tilde{B})(\varphi_{\pm}) \;=\; E_{\pm}(z, \varphi_{\pm}(z)), \quad E_{\pm}(z, w) = (E^1_{\pm}(z, w), \ldots, E^4_{\pm}(z, w)), \tag{4.41}$$

where $E^j_{\pm}(z, w)$ are given by (4.34) and (4.36) and

$$\tilde{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu & \varepsilon \\ 0 & 0 & 0 & -\mu \end{pmatrix}. \tag{4.42}$$

We prepare a lemma.

Lemma 4.4 *Let $\nu > 0$ be an irrational number. Let $f(x)$ and $w(x)$ be smooth solutions of (4.10) and (4.11) in Ω_1, respectively satisfying that*

$$f(0) \;=\; w(0) = 0 \tag{4.43}$$
$$\nabla f(0) \;=\; \nabla w(0) = 0. \tag{4.44}$$

We cut off $f(x)$ and $w(x)$ with a smooth function being identically equal to 1 in some neighborhood of the origin and with support contained in a small neighborhhood of the origin. For the sake of simplicity we denote the modified functions with the same letter. Let $\varphi_{\pm}(z)$ and $\psi_{\pm}(z)$ be defined by (4.12), (4.13) and (4.14), respectively by the same way as (4.23) and (4.24). Then, for every $\alpha \in \mathbb{Z}^3_+$, we have

$$\partial^{\alpha}_z \Theta(z_1, 0) = 0, \qquad \forall z = (z_1, 0) \in \Omega, \tag{4.45}$$

with $\Theta = \varphi_{\pm}$ and $\Theta = \psi_{\pm}$.

Proof. Because ν is an irrational number we can easily see, from (4.10) and (4.11) that every $f(x)$ and $w(x)$ satisfying (4.43) and (4.44) are flat at the origin, namely all derivatives $\partial^{\alpha}_x f(x)$, $\partial^{\alpha}_x w(x)$ ($\alpha \in \mathbb{N}^4$) vanish at the origin $x = 0$. Let $\Theta(z) = \varphi_{\pm}(z)$, and set $f(x) = x_1 \varphi_{\pm}(x_2/x_1, x_3|x_1|^{\nu}, x_4|x_1|^{\nu})$, $x_1 \neq 0$. Then we have

$$\partial^{\alpha'}_x \left(x_1^{-1} f(x)\right) = \partial^{\alpha'}_x \varphi_{\pm}(x_2/x_1, x_3|x_1|^{\nu}, x_4|x_1|^{\nu}) \tag{4.46}$$
$$= \;\; x_1^{-\alpha_2} |x_1|^{\nu(\alpha_3+\alpha_4)} \partial^{\alpha_2}_{z_1} \partial^{\alpha_3}_{z_2} \partial^{\alpha_4}_{z_3} \varphi_{\pm}(z)\big|_{z_1 = x_2/x_1, z_2 = x_3|x_1|^{\nu}, z_3 = x_4|x_1|^{\nu}}. \tag{4.47}$$

We let x tend to zero so as to satisfy $x_2/x_1 = z_1$, $z_2 = x_3|x_1|^{\nu} = 0$ and $z_3 = x_4|x_1|^{\nu} = 0$. Then we have

$$\partial^{\alpha_2}_{z_1} \partial^{\alpha_3}_{z_2} \partial^{\alpha_4}_{z_3} \varphi_{\pm}(z_1, 0, 0) = \lim_{x \to 0} x_1^{\alpha_2} |x_1|^{-\nu(\alpha_3+\alpha_4)} \partial^{\alpha'}_x \left(x_1^{-1} f(x_1, x_2, 0, 0)\right) = 0, \tag{4.48}$$

because $f(x)$ is flat at the origin. The other cases will be proved similarly. □

Remark. Let $\varphi_{\pm}(z) \in C^k(\Omega)$ be given. Assume that (4.45) is satisfied for $\Theta = \varphi_{\pm}$ up to some finite $|\alpha|$. Then the function $f(x)$ defined by (4.12) gives a finitely smooth solution of

(4.10) if ν is an irrational number. Indeed, the finite smoothness at $x_1 = 0$ follows from the argument of Lemma 4.4.

In order to solve (4.41) we introduce a function space. Let $N \geq 1$ and $k \leq N$ be integers. Let $0 < c_2' < c_2 \leq 1$ be a constant. Then we define

$$\begin{aligned} \|V\|_{k;N} &= \sup_{z\in\mathbb{R}^3, 0<|z'|\leq c_2'} \sum_{|\alpha|\leq k} |z'|^{|\alpha|} \left|\partial_z^\alpha \left(|z'|^{-N} V(z)\right)\right|, \\ |V(z)| &= (\sum_{j=1}^3 |V_j(z)|^2)^{1/2}, \quad V(z) = (V_1(z), V_2(z), V_3(z)). \end{aligned} \tag{4.49}$$

The set of all C^k functions $V(z)$ such that $\|V\|_{k,N} < \infty$ is a Banach space $B_{k;N}$ with the norm $\|\cdot\|_{k;N}$. Then we have

Lemma 4.5 *i) For any integers $k \geq 0$ and $0 \leq \ell \leq N$, there exists a constant $C_{k,N} > 0$ such that*

$$\|u\|_{k;\ell} \leq C_{k,N} \|u\|_{k;N}, \quad \forall u \in B_{k;N}. \tag{4.50}$$

ii) For every $f, g \in B_{k;N}$ we have $fg \in B_{k;N}$ and there exists a constant $C_{k,N} > 0$ such that

$$\|fg\|_{k;N} \leq C_{k,N} \|f\|_{k;N} \|g\|_{k;N}, \quad \forall f, g \in B_{k;N}. \tag{4.51}$$

Proof. Because $|z'| \leq 1$, we have, for $|\alpha| \leq k$

$$\begin{aligned} & |z'|^{|\alpha|} \partial^\alpha (|z'|^{-\ell} u(z)) = |z'|^{|\alpha|} \partial^\alpha (|z'|^{N-\ell} |z'|^{-N} u(z)) \\ = \; & |z'|^{|\alpha|} \sum_{\beta+\gamma=\alpha} \partial^\beta |z'|^{N-\ell} \partial^\gamma (|z'|^{-N} u(z)) \leq C_1 \sup |z'|^{|\gamma|} |\partial^\gamma (|z'|^{-N} u(z))| \end{aligned}$$

for some $C_1 > 0$. This proves i).

In order to prove ii) we have, for $|\alpha| \leq k$

$$\begin{aligned} & |z'|^{|\alpha|} |\partial^\alpha (|z'|^{-N} fg)| \leq \sum_{\beta+\gamma=\alpha} |z'|^{|\beta|} |\partial^\beta (|z'|^{-N} f)| |z'|^{|\gamma|} |\partial^\gamma g| \\ \leq \; & C_2 \|f\|_{k;N} \|g\|_{k,0} \leq C_3 \|f\|_{k;N} \|g\|_{k,N}. \end{aligned} \tag{4.52}$$

Here $C_2 > 0$ and $C_3 >$ are constants. This proves ii) . $\square$

Let C be given by

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\mu & \varepsilon \\ 0 & 0 & -\mu \end{pmatrix}. \tag{4.53}$$

Then we define the operator Q by

$$QV = -\int_0^\infty e^{-t\tilde{B}} V(e^{tC} z) dt, \quad V = (V_1, \ldots, V_4) = (\varphi_\pm^1, \varphi_\pm^2, \varphi_\pm^3, \varphi_\pm^4). \tag{4.54}$$

We can easily see that $U = QV$ gives the solution of $(\mathcal{L} - \tilde{B})U = V$. Then we have

Lemma 4.6 *Let the integers k and N satisfy that $0 \leq k < N - \mu$ and $\mu(k+1-N) + k < 0$. Then there exists $C_{k,N}(\Omega) > 0$ such that*

$$\|QV\|_{k;N} \leq C_{k,N}(\Omega) \|V\|_{k;N}, \quad \forall V \in B_{k;N}. \tag{4.55}$$

Proof. First we note that

$$e^{tC}z = (e^t z_1, e^{-\mu t}z_2 + e^{-\mu t}\varepsilon t z_3, e^{-\mu t}z_3), \tag{4.56}$$

$$e^{-t\tilde{B}}V = (V_1, V_2, e^{\mu t}(V_3 - \varepsilon t V_4), e^{\mu t}V_4). \tag{4.57}$$

Hence we have

$$\begin{aligned} V(e^{tC}z) &= V(e^t z_1, e^{-\mu t}(z_2 + \varepsilon t z_3), e^{-\mu t}z_3) \\ &= e^{-\mu N t}((z_2 + \varepsilon t z_3)^2 + z_3^2)^{N/2}\tilde{V}(e^{tC}z), \end{aligned} \tag{4.58}$$

where $\tilde{V}(\zeta) = V(\zeta)/|\zeta'|^N$. It follows that the right-hand side integral of (4.54) converges, because the growing term $e^{\mu t}$ in e^{-tB} can be absorbed by $e^{-\mu N t}$, $(\mu > 0)$. First we consider the case $k = 0$. By (4.57) and (4.58) we have

$$\begin{aligned} \|QV\|_{0;N} &= \sup_{z\in\mathbb{R}^3, 0<|z'|\le c_2'} \left(\frac{1}{|z'|^N}\int_0^\infty \left|e^{-t\tilde{B}}V(e^{tC}z)\right| dt\right) \\ &\le \sup_{z\in\mathbb{R}^3, 0<|z'|\le c_2'} \left(\frac{1}{|z'|^N}\int_0^\infty (1+|\varepsilon|t)e^{\mu t}\left|V(e^{tC}z)\right| dt\right) \\ &\le \sup\left(\frac{1}{|z'|^N}\int_0^\infty (1+|\varepsilon|t)e^{\mu(1-N)t}((z_2+|\varepsilon|tz_3)^2+z_3^2)^{N/2}\left|\tilde{V}(e^{tC}z)\right| dt\right). \end{aligned} \tag{4.59}$$

On the other hand we note that

$$|z'|^{-N}((z_2 + |\varepsilon|tz_3)^2 + z_3^2)^{N/2} \le |z'|^{-N}(|z'| + |\varepsilon|t|z_3|)^N \le (1 + |\varepsilon|t)^N. \tag{4.60}$$

In order to estimate $\tilde{V}(e^{tC}z)$ we note the following inequality

$$e^{-\mu t}(|z_2 + \varepsilon t z_3|^2 + z_3^2)^{1/2} \le |z'|(1 + |\varepsilon|t)e^{-\mu t} \le |z'| \le c_2', \tag{4.61}$$

because we have $|\varepsilon| < \mu$. It follows that

$$|\tilde{V}(e^{tC}z)| \le \sup_{z\in\mathbb{R}^3, 0<|z'|\le c_2'} |\tilde{V}(z)|. \tag{4.62}$$

It follows that the right-hand side of (4.59) is estimated in the following way

$$\le \sup_{z\in\mathbb{R}^3, 0<|z'|\le c_2'} |\tilde{V}(z)| \int_0^\infty (1 + |\varepsilon|t)^{N+1}e^{\mu(1-N)t}dt \le C\|V\|_{0;N} \tag{4.63}$$

for some $C > 0$ independent of V. It follows that $\|QV\|_{0;N} \le C\|V\|_{0;N}$ for some $C > 0$.

Next we will estimate the derivative $|z'|^{|\alpha|}\partial_z^\alpha(|z'|^{-N}QV)$. By Leibnitz rule it is sufficient to estimate the term $|z'|^{|\alpha|}\partial^\gamma|z'|^{-N}\partial^{\alpha-\gamma}(QV)$, where $\alpha \ge \gamma$. By simple computations, we have $|z'|^{|\alpha|}\partial^\gamma|z'|^{-N} \le C_1|z'|^{-N+|\alpha|-|\gamma|}$ for some $C_1 >$ independent of z'. On the other hand, we have

$$\begin{aligned} \partial^{\alpha-\gamma}(QV) &= -\partial^{\alpha-\gamma}\int_0^\infty e^{-t\tilde{B}}((z_2+\varepsilon t z_3)^2+z_3^2)^{N/2}e^{-\mu N t}\tilde{V}(e^{tC}z)dt \\ &= -\sum_{\beta\le\alpha-\gamma}\int e^{-t\tilde{B}-\mu N t}\partial_z^\beta((z_2+\varepsilon t z_3)^2+z_3^2)^{N/2}\partial^{\alpha-\gamma-\beta}\tilde{V}(e^{tC}z)dt. \end{aligned} \tag{4.64}$$

We can easily see

$$\left|\partial_z^\beta((z_2 + \varepsilon t z_3)^2 + z_3^2)^{N/2}\right| \le C_2(1 + |\varepsilon|t)^N|z'|^{N-|\beta|} \tag{4.65}$$

for some $C_2 > 0$. If we set $\alpha - \beta - \gamma = \delta$, $\delta = (\delta_1, \delta_2, \delta_3)$, then we have

$$\partial^{\alpha-\beta-\gamma}\tilde{V}(e^{tC}z) = e^{t\delta_1-\mu(\delta_2+\delta_3)t}(\partial_1^{\delta_1}\partial_2^{\delta_2}(\varepsilon t\partial_2 + \partial_3)^{\delta_3}\tilde{V})(e^{tC}z). \tag{4.66}$$

It follows that

$$\begin{aligned}
& |z'|^{|\alpha|}\partial^{\gamma}|z'|^{-N}|\partial^{\alpha-\gamma}(QV)| \\
\leq\ & C_3|z'|^{-N+|\alpha|-|\gamma|}\sum_{\beta}\int_0^\infty e^{\mu t-\mu Nt}|\partial_z^{\beta}((z_2+\varepsilon tz_3)^2+z_3^2)^{N/2}| \\
\times\ & |\partial^{\alpha-\gamma-\beta}\tilde{V}(e^{tC}z)|dt \\
\leq\ & C_3|z'|^{-N+|\alpha|-|\gamma|}\sum_{\beta}\int_0^\infty e^{\mu t-\mu Nt}(1+|\varepsilon|t)^{N+1}|z'|^{N-|\beta|}|\partial^{\alpha-\gamma-\beta}\tilde{V}(e^{tC}z)|dt \\
\leq\ & C_4\int_0^\infty \sum_{|\xi|=|\alpha-\beta-\gamma|\leq k}|z'|^{|\xi|}|(\partial_z^{\xi}\tilde{V})(e^{tC}z)|(1+|\varepsilon|t)^{N+1+|\xi|}e^{\mu t-\mu Nt+|\alpha|t}dt.
\end{aligned} \tag{4.67}$$

In order to estimate $|z'|^{|\xi|}|(\partial_z^{\xi}\tilde{V})(e^{tC}z)|$, we set $\zeta = e^{tC}z$. Then we have

$$\begin{aligned}
& |z'|^{|\xi|}|(\partial_z^{\xi}\tilde{V})(e^{tC}z)| = |(e^{-tC}\zeta)'|^{|\xi|}|(\partial^{\xi}\tilde{V})(\zeta)| \\
\leq\ & e^{\mu|\xi|t}|(\partial^{\xi}\tilde{V})(\zeta)|((\zeta_2+\varepsilon t\zeta_3)^2+\zeta_3^2)^{|\xi|/2} \\
\leq\ & e^{\mu kt}(1+|\varepsilon|t)^k|\zeta'|^{|\xi|}|(\partial^{\xi}\tilde{V})(\zeta)| \leq \|V\|_{k;N}e^{\mu kt}(1+|\varepsilon|t)^k.
\end{aligned} \tag{4.68}$$

By assumption we have $(1+k-N)\mu + |\alpha| \leq (1+k-N)\mu + k < 0$. Hence the right-hand side integral in (4.67) converges. Therefore we see that the right-hand side of (4.67) can be estimated by $C_5\|V\|_{k;N}$. □

Proof of Theorem 4.1. By setting $\varphi_\pm = QV$, (4.41) is equivalent to

$$V \;=\; E_\pm(z, QV). \tag{4.69}$$

We define the sequence $V_\pm^j$ $(j = 0, 1, 2, \ldots)$ by

$$V_\pm^0 = E_\pm(z, 0), \quad V_\pm^1 = E_\pm(z, QV_\pm^0) - E_\pm(z, 0), \tag{4.70}$$

and

$$V_\pm^{j+1} = E_\pm(z, V_\pm^0 + \cdots + V_\pm^j) - E_\pm(z, V_\pm^0 + \cdots + V_\pm^{j-1}), \quad j = 1, 2, \ldots \tag{4.71}$$

Let the integers k and N satisfy that $0 \leq k < N - \mu$ and $\mu(k+1-N) + k < 0$. We will show the convergence of $\sum_{j=0}^\infty V_\pm^j$. By definition we have $V_\pm^0 = E_\pm(z, 0) = \Psi_\pm(z)$. Next we have

$$V_\pm^1 \;=\; E_\pm(z, QV_\pm^0) - E_\pm(z, 0) = QV_\pm^0\int_0^1 \nabla_w E_\pm(z, \tau QV_\pm^0)d\tau. \tag{4.72}$$

Let $\varepsilon' > 0$ be a small constant chosen later, and suppose that

$$\|\Psi_\pm\|_{k;N} < \varepsilon', \quad \|\nabla\Psi_\pm\|_{k;N} < \varepsilon'. \tag{4.73}$$

Then, by Lemma 4.6 and the definition of $V_\pm^0$ we have

$$\|\tau QV_\pm^0\|_{k;N} \leq c_1\|V_\pm^0\|_{k;N} = c_1\|\Psi_\pm\|_{k;N} < c_1\varepsilon' \tag{4.74}$$

for some $c_1 > 0$ independent of $\Psi_\pm$. Here we recall from (4.70) that $V_\pm^0 = E_\pm(z,0)$ and $E_\pm(z,0) = \Psi_\pm(z)$ by (4.34) and (4.36).

In order to estimate $\|\nabla_w E_\pm(\cdot, \tau Q V_\pm^0)\|_{k;N}$, we set $w = (w_1, \dots, w_4) = \tau Q V_\pm^0$ and

$$\zeta = (\zeta_1, \zeta') = \left(z_1 \frac{1+w_2}{1+w_1}, (z_2 + w_3)|1+w_1|^\nu, (z_3+w_4)|1+w_1|^\nu \right).$$

The differentiation $\partial_z^\alpha(\nabla_w E_\pm(z, \tau Q V_\pm^0))$ consists of terms which are product of $\partial^\beta \nabla \Psi_\pm(\zeta)$ $(\alpha \geq \beta)$ and the differentiations of w. First, the product of differentiations of w is bounded by a constant in view of (4.74). On the other hand, in order to estimate

$$|z'|^{|\beta|}|\partial^\beta \nabla \Psi_\pm(\zeta)| \leq |z'|^{|\beta|}|\zeta'|^{-|\beta|}|\zeta'|^{|\beta|}|\partial^\beta \nabla \Psi_\pm(\zeta)| \leq |z'|^{|\beta|}|\zeta'|^{-|\beta|}\|\nabla \Psi_\pm\|_{k;N},$$

we consider $|z'|^{|\beta|}|\zeta'|^{-|\beta|}$. By Lemma 4.2 we see that w_3 and w_4 can be divisable by $z_2^2 + z_3^2$, respectively. By the smallness of w, the term $|z'|^{|\beta|}|\zeta'|^{-|\beta|}$ can be bounded by a constant. Hence, if $\varepsilon' > 0$ is sufficiently small, then we obtain, by the definition of $E_\pm(z,w)$ in (4.41), (4.34) and (4.36),

$$\|\nabla_w E_\pm(\cdot, \tau Q V_\pm^0)\|_{k;N} \leq c_2 \|\nabla \Psi_\pm\|_{k;N} < c_2 \varepsilon' \tag{4.75}$$

for some $c_2 > 0$ independent of ε' and $\Psi_\pm$.

It follows from (4.72) that

$$\|V_\pm^1\|_{k;N} \leq \|Q V_\pm^0\|_{k;N} \int_0^1 \|\nabla_w E_\pm(z, \tau Q V_\pm^0)\|_{k;N} d\tau \leq c_1 c_2 \varepsilon'^2.$$

In order to show the general case, we assume that $\|V_\pm^j\|_{k;N} \leq c_1^j c_2^j \varepsilon'^{j+1}$ for $j = 0, 1, 2, \dots, k$. Then we have

$$\|\sum_{j=0}^k V_\pm^j\|_{k;N} \leq \sum_{j=0}^k c_1^j c_2^j \varepsilon'^{j+1} \leq \frac{\varepsilon'}{1 - c_1 c_2 \varepsilon'}. \tag{4.76}$$

By definition we have

$$\begin{aligned} V_\pm^{k+1} &= E_\pm(z, Q(V_\pm^0 + \cdots + V_\pm^k) - E_\pm(z, Q(V_\pm^0 + \cdots + V_\pm^{k-1}) \\ &= Q V_\pm^k \int_0^1 \nabla_w E_\pm(z, Q(V_\pm^0 + \cdots + V_\pm^{k-1}) + \tau Q V_\pm^k) d\tau. \end{aligned} \tag{4.77}$$

By the apriori estimate (4.76) and Lemma 4.4 the substitution in the right-hand side of (4.77) is well defined. Moreover, by the same argument as in the proof of (4.75) we see that

$$\|\nabla_w E_\pm(z, Q(V_\pm^0 + \cdots + V_\pm^{k-1}) + \tau Q V_\pm^k)\|_{k;N} \leq c_2 \varepsilon'.$$

It follows from (4.77) that

$$\|V_\pm^{k+1}\|_{k;N} \leq \|Q V_\pm^k\|_{k;N} c_2 \varepsilon' \int_0^1 d\tau \leq c_1^{k+1} c_2^{k+1} \varepsilon'^{k+2}.$$

Hence we have the estimate of $V_\pm^j$ for $j = k+1$. It follows that the series $V_\pm := \sum_{j=0}^\infty V_\pm^j$ converges in $B_{k;N}$ and $V_\pm$ is a solution of (4.69). We note that, by (4.76) $V_\pm$ satisfies the estimate $\|V_\pm\|_{k;N} \leq \varepsilon'(1 - c_1 c_2 \varepsilon')^{-1}$, and $V_\pm$ is divisable by $|z'|^2$.

Next we verify the smallness assumption (4.73) uniformly with respect to $x_1 \neq 0$ in some neighborhood of $x_1 = 0$. Because the argument is similar we consider the condition $\|\Psi_\pm\|_{k;N} < \varepsilon'$. In view of the definition of $\Psi_\pm$ in (4.23) and (4.24), we estimate

$x_j^{-1}R_j(x_1, x_1z_1, |x_1|^{-\nu}z_2, |x_1|^{-\nu}z_3)$, $(j = 1, 2)$ and $|x_1|^{\nu}R_j(x_1, x_1z_1, |x_1|^{-\nu}z_2, |x_1|^{-\nu})$, $(j = 3, 4)$ with $x_1 \neq 0$ close to 0. Because the argument is similar, we consider the case $j = 1$. We have

$$|z'|^{|\alpha|}\left|\partial_z^{\alpha}(|z'|^{-N}\Psi_{\pm}^1(z))\right| = x_1^{-1}|z'|^{|\alpha|}\left|\partial_z^{\alpha}(|z'|^{-N}R_1(x_1, x_1z_1, |x_1|^{-\nu}z_2, |x_1|^{-\nu}z_3))\right|. \tag{4.78}$$

By Lemma 4.4 we have that, for every positive integer p, the term

$$R_1(x_1, x_1z_1, |x_1|^{-\nu}z_2, |x_1|^{-\nu}z_3)|z'|^{-p}$$

is smooth at $z = 0$. Because

$$|z'|^p = (|x_1|^{\nu}|x_1|^{-\nu}|z'|)^p = (|x_1|^{\nu}|x''|)^p, \quad x'' = (x_3, x_4),$$

and $|x''|$ is bounded by the support condition of R_j, the negative power $|z'|^{-N}$ in the right-hand side of (4.78) is absorbed by $|z'|^p$ if p is sufficiently large. On the other hand, if the differentiation ∂_z^{α} is applied to $R_1(x_1, x_1z_1, |x_1|^{-\nu}z_2, |x_1|^{-\nu}z_3)$, then the negative power of $|x_1|$ appears. These terms are also uniformly bounded when $x_1 \to 0$, because there appears positive power of $|x_1|$ from $|z'|^p$. Because all derivatives of $R(x)$ at the origin vanish, we see that the right-hand side of (4.78) can be made arbtrarily small if we cut off $R_(x)$ in a sufficiently small neighborhood of the origin. This proves that we have (4.73).

We set $\varphi_{\pm} = QV_{\pm} \in B_{k;N}$, and $\varphi_{\pm}(z) = (\varphi_{\pm}^1(z), \varphi_{\pm}^2(z), \varphi_{\pm}^3(z), \varphi_{\pm}^4(z))$. The function $\varphi_{\pm}$ is a solution of (4.41). Then we define $v^j(x)$ $(j = 1, 2, 3, 4)$ by (4.25) and (4.26). For a given integer m, we can easily see that $v^j(x)$ is a C^m function if we take k and N in $B_{k;N}$ sufficiently large. If we rewrite (4.41) with the variable x, then we see that v is a solution of (4.6), where the nonlinear part R is modified by a cutoff function. In order to show that v is a solution of the original (4.6) we will show the apriori estimate of v. Indeed, if $|x + v| < \varepsilon''$ for sufficiently small ε'', then v is a solution of (4.6). By Lemma 4.6 and the uniform estimate of $V_{\pm}$ in x_1 we know that $\phi_{\pm}^1(z)$ is uniformly bounded in z and x_1. It follows that $v_1(x) = x_1\phi_{\pm}^1$ is arbitrarily small if x_1 is sufficiently small. Similarly we can show that $v_2(x) = x_2\phi^2$ is small by the estimate of $V_{\pm}$. On the other hand, we have $x_3 + v_3(x) = x_3 + |x_1|^{-\nu}\phi_{\pm}^3(z)$. Because $\phi_{\pm}^3$ is divisable by $|z'|^2$ and $|z'| = |x_1|^{\nu}|x''|$, by Lemma 4.4 we see that $|x_3 + v_3(x)| < \varepsilon''$ uniformly in x_1. Similarly we can show the same estimate for $x_4 + v_4$. Therefore we see that v is a solution of (4.6). This completes the proof.

Acknowledgements. This work was done when the first author stayed in RIMS, Kyoto University and the second author stayed in the University of Cagliari. The first author thanks Prof. T. Kawai for the invitation to RIMS, Kyoto University and for the excellent conditions for the research on the subject of the paper. The second author expresses the sincere gratitude to the University of Cagliari for the support in the beginning of this research. The authors also acknowledge useful discussions on problems related to the paper with S. Marmi, R. Perez Marco and L. Stolovitch. The authors also would like to express their gratitudes to an anonymous referee who suggested a substantial improvement of Theorem 1.4 and other critical remarks.

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