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Normal forms of commuting singular vector fields with linear parts having Jordan blocks

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Abstract

The seminar will focus on the simultaneous linearizability of d–actions (and the corresponding d-dimensional Lie algebras) defined by commuting singular vector fields in $\mathbb{C}^n$ fixing the origin with nontrivial Jordan blocks in the linear parts.

One of the principal motivation comes from recent results of L. Stolovitch and N.T. Zung on simultaneous reductions to normal forms of commuting vector fields.

We prove the analytic convergence of formal linearizing transformations under an invariant condition (cone condition) for the spectrum of d vector fields generating a Lie algebra.

We show that the presence of simultaneous Diophantine phenomena and nontrivial Jordan blocks leads, in contrast to the semi-simple case, of the existence of divergent solutions of an overdetermined systems of linearized homological equations even if the simultaneous Bruno type condition (introduced by L. Stolovitch) holds.

The Gevrey character of the divergent solutions is investigated and the role of symmetries leading to convergent normal forms.

Some geometrical aspects of the intersections of flows of non-diagonalizable holomorphic vector fields with spheres in $\mathbb{C}^n$ will be outlined.

The results are obtained in collaboration with M. Yoshino (Hiroshima University), see

[YoGr2006] T. Gramchev and M. Yoshino, *Simultaneous reduction to normal forms of commuting singular vector fields with linear parts having Jordan blocks*, preprint, Dipartimento di Matematica e Informatica, Università di Cagliari, May 2006.

Plan of the talk

1. The mathematical set-up of the problems.

2. Motivations and previous results.

3. The Bruno condition (A) revisited.

4. Cone condition, Poincaré morphism (L. Stolovitch) and convergence.

5. Results for nonsolvability of a class of overdetermined systems of linear homological equations.

6. Some examples of divergent solutions in the presence of resonances.

7. Intersection of holomorphic flows with spheres: cycles and singular points in the presence of nilpotent parts.

8. Concluding remarks (divergent solutions in the presence of higher order resonances, simultaneous reduction to resonant normal forms, holomorphic solutions for generalizations of harmonic oscillators).

1. The mathematical set-up of the problems

Let $\mathbb{K}$ be $\mathbb{K}=\mathbb{C}$ or $\mathbb{K}=\mathbb{R}$, and $B=\infty$, $B=\omega$ or $B=k$ for some $k>0$. Let $\mathcal{G}_B^n$ denotes a d–dimensional Lie algebra of germs at $0\in\mathbb{K}^n$ of C^B vector fields vanishing at 0. Let ρ be a germ of singular infinitesimal $\mathbb{K}^d$ ($d\geq 2$) actions of class C^B

$$\rho:\mathbb{K}^d\longrightarrow\mathcal{G}_B^n. \tag{1}$$

We denote by $Act^B(\mathbb{K}^d:\mathbb{K}^n)$ the set of germs of singular infinitesimal $\mathbb{K}^d$ actions of class C^B in $0\in\mathbb{K}^n$. By choosing a basis $e_1,\dots,e_d\in\mathbb{K}^n$, the infinitesimal action can be identified with a d–tuple of germs at 0 of commuting vector fields $X^j=\rho(e_j)$, $j=1,\dots,d$ (cf. F.Dumortier and R. Roussarie [DuRo1980], L. Stolovitch [Sto2000], [Sto2005], see also A. Katok and S. Katok [KaKa2000] for actions on $\mathbb{T}^n$).

We can define, in view of the commutativity relation, the action

$$\begin{array}{rcll}\tilde\rho:\mathbb{K}^d\times\mathbb{K}^n & \longrightarrow & \mathbb{K}^n, & (2)\\ \tilde\rho(s;z) & = & X^1_{s_1}\circ\cdots\circ X^d_{s_d}(z) & \\ & = & X^{\sigma_1}_{s_{\sigma_1}}\circ\cdots X^{\sigma_d}_{s_{\sigma_d}}(z),\quad s=(s_1,\dots,s_d) & (3)\end{array}$$

for all permutations $\sigma=(\sigma_1,\dots,\sigma_d)$ of $\{1,\dots,d\}$, where X^j_t denotes the flow of X^j. We denote by ρ_{lin} the linear action formed by the linear parts of the vector fields defining ρ.

We shall investigate the necessary and sufficient conditions for the linearization of ρ, namely, whether there exists a C^B diffeomorphism g preserving 0 such that g conjugates $\tilde\rho$ and $\tilde{\rho_{lin}}$

$$\tilde\rho(s;g(z)) \;=\; g(\widetilde{\rho_{lin}}(s,z)),\qquad (s,z)\in\mathbb{K}^d\times\mathbb{K}^n. \tag{4}$$

We recall that in [DuRo1980], and [Sto2000], [Sto2005] the linear parts were supposed to be diagonalizable, while in the paper of T.N. Zung [Zung2002] the existence of $n-d$ anlalytic first integrals was required (see also M. Abate [Ab2000], D. Delatte and T. Gramchev [DeLGr2002] for nondiagonalizable holomorphic maps and T. Gramchev [Gr2003] for nondiagonalizable holomorphic vector fields).

Following the arguments in [KaKa2000], we can choose a positive integer $m \leq n$ such that $\mathbb{K}^n$ is decomposed into a direct sum of m linear subspaces invariant under all $A^\ell = \nabla X_\ell(0)$ ($\ell = 1, \ldots, d$):

$$\begin{aligned} \mathbb{K}^n &= \mathbb{I}^{s_1} + \cdots + \mathbb{I}^{s_m}, \qquad \dim \mathbb{I}^{s_j} = s_j,\ j = 1, \ldots, m, \\ & \quad s_1 + \cdots + s_m = n. \end{aligned} \tag{5}$$

The matrices $A^1, \ldots, A^d$ can be simultaneously brought in an upper triangular form, and we write again A^ℓ for the matrices,

$$A^\ell = \begin{pmatrix} A_1^\ell & 0_{s_1 \times s_2} & \ldots & 0_{s_1 \times s_m} \\ 0_{s_2 \times s_1} & A_2^\ell & \ldots & 0_{s_2 \times s_m} \\ \vdots & \vdots & \vdots & \vdots \\ 0_{s_m \times s_1} & 0_{s_m \times s_2} & \ldots & A_m^\ell \end{pmatrix}, \tag{6}$$

for $\ell = 1, \ldots, d$. If $\mathbb{K} = \mathbb{C}$, the matrix A_j^ℓ is given by

$$A_j^\ell = \begin{pmatrix} \lambda_j^\ell & A_{j,12}^\ell & \ldots & A_{j,1s_j}^\ell \\ 0 & \lambda_j^\ell & \ldots & A_{j,2s_j}^\ell \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_j^\ell \end{pmatrix}, \tag{7}$$

for $\ell = 1, \ldots, d$, $j = 1, \ldots, m$, with $\lambda_j^\ell, A_{j,\nu\mu}^\ell \in \mathbb{C}$.

On the other hand, if $\mathbb{K} = \mathbb{R}$, then we have, for every $1 \leq j \leq m$ two possibilities: firstly, all A_j^ℓ ($\ell = 1, \ldots, d$)

are given by (7) with $\lambda_j^\ell \in \mathbb{R}$. Secondly, $s_j = 2\tilde{s}_j$ is even and A_j^ℓ is a $\tilde{s}_j \times \tilde{s}_j$ square block matrix given by

$$A_j^\ell \;=\; \begin{pmatrix} R_2(\lambda_j^\ell, \mu_j^\ell) & A_{\ell,j}^{12} & \dots & A_{\ell j}^{1\tilde{s}_j} \\ 0 & R_2(\lambda_j^\ell, \mu_j^\ell) & \dots & A_{\ell j}^{2\tilde{s}_j} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & R_2(\lambda_j^\ell, \mu_j^\ell) \end{pmatrix}, \tag{8}$$

for $\ell = 1, \dots, d$, where

$$R_2(\lambda, \mu) := \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix}, \qquad \lambda, \mu \in \mathbb{R}, \tag{9}$$

and $A_{\ell j}^{rs}$ are appropriate real matrices.

Following the decomposition (7) (respectively, (8)) we define $\tilde{\lambda^j}$ by

$$\tilde{\lambda^k} = (\lambda_1^k, \dots, \lambda_m^k) \in \mathbb{K}^m, \qquad k = 1, \dots, d. \tag{10}$$

Then we assume

$$\tilde{\lambda^1}, \cdots, \tilde{\lambda^d} \ \text{ are linearly independent in } \mathbb{K}^m. \tag{11}$$

One can easily see that (11) is invariantly defined.

Broadly speaking, we dwell upon two issues:

Problem 1: What happens if we drop the diagonalizablity requirement on the linear parts of the vector fields. Is the simultaneous Bruno condition sufficient in the Jordan block case?

Problem 2. To propose examples of nonlinearizable commuting vector fields which satisfy some but not all of the hypotheses for simultaneous linearizability.

2. Motivations and previous results

Broadly speaking, the typical results for holomorphic normla forms in the presence of small divisors are shown under the hypotheses "the linear part is diagonalizable".

A part from the celebrated works of A. Bruno (the so called (A) condition), few results are available on convergent normal forms for nondiagonalizable normal forms.

2. The condition (A) of Bruno

The fundamental work of A. Bruno contains two crucial hypotheses (conditions): the arithmetic-Diophantine one, called (ω) condition, and a second restriction, called (A) condition (which splits into ($A1$) or ($A2$) conditions. In fact, the Jordan blocks are allowed, but in that case the (A) condition means that there exist a line such that there are no eigenvalues staying on different semi–planes and the eigenvalues with nontrivial Jordan blocks are not on the line (a condition in [DeLGr2002] for biholomoprphic maps is for the unit circle).

3. The cone condition

We introduce a cone condition. By (6) we define

$$\vec{\lambda_j} = {}^t(\lambda_j^1, \cdots, \lambda_j^d) \in \mathbb{K}^d, \qquad j = 1, \ldots, m, \tag{12}$$

and

$$\Lambda_m := \{\vec{\lambda_1}, \ldots, \vec{\lambda_m}\}. \tag{13}$$

We define the cone $\Gamma[\Lambda_m]$ by

$$\Gamma[\Lambda_m] = \left\{ \sum_{j=1}^{m} t_j \vec{\lambda_j} \in \mathbb{K}^d;\ t_j \geq 0, j = 1, \ldots, m, \sum_{j=1}^{m} t_j \neq 0 \right\}. \tag{14}$$

Definition. We say that the $\mathbb{K}^d$–action ρ satisfies a cone condition if there exists a base $\Lambda_m \subset \mathbb{K}^m$ such that $\Gamma[\Lambda_m]$ is a proper cone in $\mathbb{K}^m$, namely it does not contain a straight real line. If the condition is not satisfied, then, we say that the $\mathbb{K}^d$ action is in a Siegel domain.

Note that the definition is invariant under the choice of the basis Λ_m.

Remark We can show that the above definition of the cone condition is equivalent to that ρ is a Poincaré morphism. (cf. Stolovitch, [**?**]).

Next, we introduce the notion of simultaneous resonances. For $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{K}^m$, $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{K}^m$, we set $\langle \alpha, \beta \rangle = \sum_{\nu=1}^{m} \alpha_\nu \beta_\nu$. For a positive integer k we define $\mathbb{Z}_+^m(k) = \{\alpha \in \mathbb{Z}_+^m;\ |\alpha| \geq k\}$. Put

$$\omega_j(\alpha) = \sum_{\nu=1}^{d} |\langle \tilde{\lambda}^\nu, \alpha \rangle - \lambda_j^\nu|, \qquad j = 1, \ldots, m, \tag{15}$$

$$\omega(\alpha) = \min\{\omega_1(\alpha), \ldots, \omega_m(\alpha)\}. \tag{16}$$

Definition. We say that Λ_m is simultaneously non-resonant (or, in short ρ is simultaneously non-resonant), if

$$\omega(\alpha) \neq 0, \qquad \forall \alpha \in \mathbb{Z}_+^m(2). \tag{17}$$

If (17) does not hold, then we say that Λ_m is simultaneously resonant.

Clearly, the simultaneously non-resonant condition (17) is invariant under the change of the basis Λ_m.

We state the first main result.

Theorem (T.G & M. Yoshino, 2006) Let ρ be a $\mathbb{K}^d$ analytic action which satisfies the cone condition. Then ρ is conjugated to a polynomial action by a holomorphic change of variables. In addition, if ρ is simultaneously non-resonant then ρ is linearizable by a uniquely determined holomorphic change of the variables $x \to y$ of the form $x = u(y) = y + v(y)$, $v(y)$ being at least quadratic.

Remark. In case ρ has a semi simple linear part, then Theorem 1.4 is known. (cf. Theorem 2.1.4 of [Sto2000]).

Theorem 1.4′. *Let ρ be a $\mathbb{K}^d$ action which satisfies a cone condition. Then ρ is conjugated to a polynomial action by a holomorphic change of variables.*

We start by showing equivalent forms of the cone condition.

Proposition. The cone condition is equivalent to each of the following conditions
i) there exist a positive constant C and an integer k_0 such that

$$\sum_{k=1}^{d} |\sum_{j=1}^{m} \lambda_j^k \alpha_j| \geq C_1|\alpha|, \qquad \forall \alpha \in \mathbb{Z}_+^m(k_0). \tag{18}$$

ii) there exists a nonzero vector $c = (c_1, \ldots, c_d) \in \mathbb{C}^d$ if $\mathbb{K} = \mathbb{C}$ (respectively, $c = (c_1, \ldots, c_d) \in \mathbb{R}^d$ if $\mathbb{K} = \mathbb{R}$) such that

$$c_1\tilde{\lambda}^1 + \cdots + c_d\tilde{\lambda}^d \text{ is in a Poincaré domain,} \tag{19}$$

namely, the convex hull of the set $\{\sum_{j=1}^{d} c_j\lambda_k^j; k = 1, \ldots, m\}$ in $\mathbb{C}$ does not contain $0 \in \mathbb{C}$ (respectively,

$$\text{the real parts of } c_1\lambda_j^1 + \cdots + c_d\lambda_j^d,\ j = 1, \ldots, m, \text{ are positive.)} \tag{20}$$

Proof. First we show (18). Suppose that (18) does not hold. Then there exists a sequence $\alpha^\ell \in \mathbb{Z}_+^m$, $\ell \in \mathbb{N}$ such that $|\alpha^\ell| \to \infty$ $(\ell \to \infty)$ and

$$\sum_{k=1}^{d} |\sum_{j=1}^{m} \lambda_j^k \alpha_j^\ell| \leq \frac{|\alpha^\ell|}{\ell}, \qquad \ell \in \mathbb{N}. \tag{21}$$

By taking a subsequence, if necessary, we may assume that $\alpha^\ell/|\alpha^\ell| \to t^0 = (t_1^0, \ldots, t_m^0) \in S_{\ell^1}^1 \bigcap \mathbb{R}_+^m$ when $\ell \to \infty$, where $S_{\ell^1}^1 := \{x \in \mathbb{K}^m; \|x\|_{\ell^1} = \sum_{j=1}^{m} |x_j| = 1\}$ stands for the ℓ^1 unit sphere. By letting $\ell \to \infty$ in (21) we get

$$\sum_{k=1}^{d} |\sum_{j=1}^{m} \lambda_j^k t_j^0| = 0.$$

It follows that $\sum_{j=1}^m t_j^0 \vec{\lambda}_j = 0$. Let $J \subset \{1, \dots, m\}$ be such that $\sum_{j \in J} t_j^0 \vec{\lambda}_j \neq 0$. Such a set J exists by (11). It follows that

$$0 \neq \sum_{j \in J} t_j^0 \vec{\lambda}_j = - \sum_{j \in \{1,\dots,m\} \setminus J} t_j^0 \vec{\lambda}_j.$$

Hence $\Gamma[\Lambda_m]$ contains a straight line generated by $\sum_{j \in J} t_j^0 \vec{\lambda}_j \neq 0$. This contradicts the assumption that $\Gamma[\Lambda_m]$ is a proper cone.

Conversely, suppose that (18) is satisfied. We shall show that $\Gamma[\Lambda_m]$ is proper. Indeed, if otherwise, we can find $t^0 = (t_1^0, \dots, t_m^0) \in S_{\ell^1}^1 \bigcap \mathbb{R}_+^m \setminus 0$ such that

$$\sum_{j=1}^m t_j^0 \lambda_j^k = 0, \qquad k = 1, \dots, d. \tag{22}$$

Because the set $\{\alpha/|\alpha|; \alpha \in \mathbb{Z}_+^m(2)\}$ is dense in $S_{\ell^1}^1 \bigcap \mathbb{R}_+^m$, there exists a sequence $\alpha^\ell \in \mathbb{Z}_+^m$, $\ell \in \mathbb{N}$ such that $|\alpha^\ell| \to \infty$ $(\ell \to \infty)$ and $\lim_{\ell \to \infty} \alpha^\ell/|\alpha^\ell| = t^0$. Therefore, in view of (22), we get

$$\lim_{\ell \to \infty} \left(\frac{1}{|\alpha^\ell|} \sum_{k=1}^d |\sum_{j=1}^m \lambda_j^k \alpha_j^\ell| \right) = 0,$$

which contradicts (18)

Next, we show ii). Suppose that $\Gamma[\Lambda_m]$ be a proper cone in $\mathbb{K}^d$. Then we can find $c = (c_1, \dots, c_d) \in \mathbb{C}^d$ such that $\Gamma[\Lambda_m]$ is contained in the real half–space $P_c := \{z \in \mathbb{K}^d, \Re(\sum_{k=1}^d c_k z_k) > 0\}$. Therefore

$$0 < \Re(\sum_{k=1}^d c_k \sum_{j=1}^m t_j \lambda_j^k) = \sum_{j=1}^m t_j \Re(\sum_{k=1}^d c_k \lambda_j^k) \tag{23}$$

for all $t \in \mathbb{R}^m_+ \setminus 0$, which yields $\Re(\sum_{k=1}^d c_k \lambda_j^k) > 0$ for $j = 1, \ldots, m$. We note that, if $\mathbb{K} = \mathbb{R}$, then the use of the real part in the definition of the half–space is superfluous. Finally, we readily see, from (19) that, if $\mathbb{K} = \mathbb{C}$ (respectively, (20) if $\mathbb{K} = \mathbb{R}$), then the cone $\Gamma[\Lambda_m]$ is contained in P_c. Hence $\Gamma[\Lambda_m]$ is proper. The proof is complete.

Next, we give the proof of Theorem 1.4. Fist we deal with the last part of the theorem. The assertion for simultaneously non-resonant actions satisfying the cone condition follows from

Proposition. Let the action ρ satisfy the cone condition. Then we can find a vector field in the corresponding Lie algebra which is non-resonant and is in the Poincaré domain.

Proof. By ii) of Proposition 2.1 we can find a Poincaré vector field in the Lie algebra as a linear combination of a base corresponding to (19).

Let c_ν be the number in (19), and define $\tilde{\lambda}^0 := (\lambda_1^0, \ldots, \lambda_m^0) = \sum_{\nu=1}^d c_\nu \tilde{\lambda}^\nu$.

Consider

$$\langle \tilde{\lambda}^0, \alpha \rangle - \lambda_j^0 = \sum_{\nu=1}^d c_\nu \left(\langle \tilde{\lambda}^\nu, \alpha \rangle - \lambda_j^\nu \right).$$

Because $\sum_{\nu=1}^d |\langle \tilde{\lambda}^\nu, \alpha \rangle - \lambda_j^\nu| \neq 0$ for $\forall \alpha \in \mathbb{Z}^m_+(2)$ by the simultaneous non-resonant condition it follows that the set $\langle \tilde{\lambda}^0, \alpha \rangle - \lambda_j^0 = 0$ in $c = (c_1, \ldots, c_d) \in \mathbb{C}^d$ is a hyperplane. It follows that the set

$\{c = (c_1, \ldots, c_d) \in \mathbb{C}^d; \langle \tilde{\lambda}^0, \alpha \rangle - \lambda_j^0 = 0, \exists j, 1 \leq j \leq m, \exists \alpha \in \mathbb{Z}^m_+(2)\}$

is a countable union of nowhere dense closed set. Therefore we can find $c = (c_1, \ldots, c_d)$ for which $\sum_{\nu=1}^{d} c_\nu \tilde{\lambda}^\nu$ satisfies a non-resonant condition and the cone condition. This proves Proposition 2.2.

Next, we show the assertion in the presence of simultaneous resonances. By Lemma 3.1 of L. Stolovitch [Sto2005] (applied to the semisimple linear parts of the action ρ) there exists a linear vector field $s_0 = J^1(\rho(e_0))$ in the linear span of the image of the 1-jet of ρ such that s_0 is in the Poincare domain and has the same resonances as the 1-jet of ρ. By lemma 3.2 of the same article, if $\rho(e_0)$ is normalized, then so is ρ. We note that the presence of nontrivial Jordan blocks may cancel some resonant monomials due to the commuting properties.

Next, we propose a geometric expression of the cone condition

Definition Let $r > 0$ and g be a Riemannian metric on $\mathbb{R}^n$. We denote by $\langle \cdot, \cdot \rangle_g$ and $\| \cdot \|_g$ the inner product and the norm with respect to g, respectively. We say that $\mathcal{X}_\nu := \sum_{j=1}^{n} X_j^\nu(x) \partial_{x_j}$ $(\nu = 1, \ldots, d)$ are simultaneously transversal to the sphere $\|x\|_g = r$ if, the vectors $X^\nu := (X_1^\nu, \ldots, X_n^\nu)$ $(\nu = 1, \ldots, d)$ satisfy

$$\sum_{\nu=1}^{d} |\langle X^\nu, x \rangle_g| \neq 0, \quad \forall x, \quad \|x\|_g = r. \tag{24}$$

Theorem Let $r > 0$. Suppose that $\mathcal{B}_\nu := \sum_{j=1}^{n} (xA^\nu)_j \partial_{x_j}$ $(\nu = 1, \ldots, d)$ be a commuting system of semi-simple linear real vector fields in $\mathbb{R}^n$. We choose a real nonsingular matrix P such that $\Lambda^\nu = P^{-1} A^\nu P$ is a block diagonal matrix given by

$$\Lambda^\nu = \text{diag}\ \{R_2(\xi_1^\nu, \eta_1^\nu), \ldots, R_2(\xi_{n_1}^\nu, \eta_{n_1}^\nu), \lambda_{n_1+1}^\nu, \ldots, \lambda_n^\nu\}$$

for some integer $n_1 \leq n$. Let g be a Riemannian metric defined by P^tP. Then the following conditions are equivalent.
(a) $\mathcal{B}_\nu$ $(\nu = 1, \dots, d)$ are simultaneously transversal to the sphere $\|x\|_g = r$.
(b) $\mathcal{B}_\nu$ $(\nu = 1, \dots, d)$ satisfy a cone condition.
(c) There exist real numbers c_ν $(\nu = 1, \dots, d)$ such that $\sum_{\nu=1}^{d} c_\nu \mathcal{B}_\nu$ is transversal to the sphere $\|x\|_g = r$.

Proof. We note that $\langle x, y\rangle_g = \langle Px, Py\rangle$ and $\|x\|_g = \|Px\|$. By inserting the relation $A^\nu = P\Lambda^\nu P^{-1}$ into (24) we can easily see that the simultaneous transversality condition is equivalent to

$$\sum_{\nu=1}^{d} |\langle y\Lambda^\nu, y\rangle| \neq 0, \quad \forall y = (y_1, \dots, y_n), \quad \|y\| = 1. \quad (25)$$

We can write

$$\sum_{\nu=1}^{d} |\sum_{j=1}^{n_1} \xi_j^\nu (y_{2j-1}^2 + y_{2j}^2) + \sum_{j=n_1+1}^{n} y_j^2 \lambda_j^\nu| \neq 0, \quad \forall y, \quad \|y\| = 1. \quad (26)$$

We define $t = (t_1, \dots, t_n)$, $t \in \mathbb{R}_+^n$, $|t| = 1$ by $t_j = (y_{2j-1}^2 + y_{2j}^2)/2$ if $j \leq n_1$ and $t_j = y_j^2$ if $j > 2n$. Noting that $\xi^n u_j (y_{2j-1}^2 + y_{2j}^2) = 2t_j \xi_j^\nu = t_j(\xi_j^\nu + i\eta_j^\nu + \xi_j^\nu - i\eta_j^\nu)$ we see that (26) is written in $\sum_{\nu=1}^{d} |\sum_{j=1}^{n} t_j \lambda_j^\nu| \neq 0$ for every $t \in \mathbb{R}_+^n$ and $|t| = 1$. This is equivalent to the cone condition by definition. Hence we have proved the equivalence of (a) and (b).

The condition (b) is equivalent to the existence of real numbers c_ν $(\nu = 1, \dots, d)$ such that $\sum_{\nu=1}^{d} c_\nu \mathcal{B}_\nu$ is a Poincaré vector field. By what we have proved in the above $(d = 1)$ this is equivalent to say that $\sum_{\nu=1}^{d} c_\nu \mathcal{B}_\nu$ is transversal to the sphere $\|x\|_g = r$. Hence we have proved the theorem.

Example. Let ρ be a $\mathbb{R}^2$ action in $\mathbb{R}^n$, $n \geq 4$ with $m = 3$. We choose a basis Λ_2 of $\mathbb{R}^3$ such that

$$\Lambda_2 = \left\{ {}^t(1, 1, \nu), {}^t(0, 1, \mu) \right\}, \qquad \nu, \mu \in \mathbb{R}. \tag{27}$$

(cf. D. Dickinson, T. Gramchev and M. Yoshino [DGY2002] for similar and more general reductions of commuting vector fields on the torus).

We will characterize the set of $(\nu, \mu) \in \mathbb{R}^2$ satisfying the cone condition, and determine the simultaneous resonances. By (14), $\Gamma[\Lambda_2]$ is generated by the set of vectors $\{(1, 0), (1, 1), (\nu, \mu)\}$. Hence the cone condition holds if and only if these vectors generate a proper cone, namely (ν, μ) is not in the set $\{(\nu, \mu) \in \mathbb{R}^2; \nu \leq \mu \leq 0\}$. We note that the interesting case is $\mu < \nu \leq 0$, where every generator in (13) is in a Siegel domain.

We will show that if a cone condition is violated, i.e., $\nu < \mu < 0$, then there exist (ν, μ) with the density of continuum such that the linearized overdetermined system of two homological equations has a divergent solution.

Next, we determine (ν,μ) so that a simultaneous resonance exists. If $\eta=(\eta_1,\eta_2,\eta_3)\in\mathbb{Z}_+^3(2)$ is a simultaneous resonance, we have the following set of equations:

$$
\begin{array}{ll}
(1) & \eta_1+\eta_2+\nu\eta_3=1,\ \eta_2+\mu\eta_3=0,\\
(2) & \eta_1+\eta_2+\nu\eta_3=1,\ \eta_2+\mu\eta_3=1,\\
(3) & \eta_1+\eta_2+\nu\eta_3=\nu,\ \eta_2+\mu\eta_3=\mu.
\end{array}
$$

By elementary computations, in order that one of these equations has a solution η the (ν,μ) satisfies the following:

a) Case $\nu\le\mu\le 0$. The resonance exists iff $(\nu,\mu)\in\mathbb{Q}_-\times\mathbb{Q}_-$, where $\mathbb{Q}_-$ is the set of nonpositive rational numbers. The resonance is given by $(1+(\mu-\nu)k,-\mu k,k)$ and $((\mu-\nu)k,1-k\mu,k)$ where $k\ge 1/(1-\nu)$, $k\in\mathbb{Z}_+$, and $((\nu-\mu)(1-k),\mu(1-k),k)$, where $k\ge(2-\nu)(1-\nu)$, $k\in\mathbb{Z}_+$.

b) Case $\nu>\mu$ and $\mu\le 0$. The resonance is given by $(0,-\mu/(\nu-\mu),1/(\nu-\mu))$, where $-\mu/(\nu-\mu)\in\mathbb{Z}_+$, $1/(\nu-\mu)\in\mathbb{Z}_+$ and $2\nu-\mu\le 1$.

c) Case $\mu>0$, $\nu\le\mu$. The resonance is given by $(0,0,1/\nu)$, when $\nu=\mu$, $\nu\le 1/2$, $\nu^{-1}\in\mathbb{Z}_+$, $(0,\nu,0)$, when $\nu=\mu\ge 2$, $\nu\in\mathbb{Z}_+$, $((\mu-\nu)/\mu,0,1/\mu)$, if otherwise, where $(\mu-\nu)/\mu\in\mathbb{Z}_+$, $1/\mu\in\mathbb{Z}_+$ and $\nu+\mu\le 1$.

d) Case $\nu>\mu\ge 0$. The resonance is given by $(\nu-\mu,\mu,0)$, where $\nu-\mu\in\mathbb{Z}_+$, $\mu\in\mathbb{Z}_+$ and $\nu\ge 2$.

Let now ν be a negative rational number, $\nu=-k_1/k_2$, $k_1,k_2\in\mathbb{Z}_+$, $k_2\ne 0$. Let μ be a rational number and satisfy $\mu<\nu$. Assume that the nonlinear part of X^2 is zero. If the nonlinear part of X^1 consists of the resonant terms of X^2, then we have $[X^1,X^2]=0$. We can easily see that the linearizability of X^1 holds provided $\mu\ne\nu-1/k_2=-(k_1+1)/k_2$.

5. Nonsolvability of overdetermined systems of LHE)

We now study the action ρ_{lin} which is in a Siegel domain and admits a Jordan block. We assume that the action is formally (simultaneously) linearizable and does not satisfy the cone condition and that the family of linear parts is Diophantine. We shall show that the unique formal solution of a linearized homological equation diverges.

Let $\mathbb{C}_2^n\{x\}$ be the set of n vector functions of convergent power series of x without constant and linear terms. We examine the system of the linearized homology equation

$$L_A v = {}^t(L_1 v, \ldots, L_d v) = f, \quad f := {}^t(f_1, \ldots, f_d) \in (\mathbb{C}_2^n\{x\})^d, \tag{28}$$

where L_j is a Lie bracket,

$$L_j v = [A_j x, v] = \langle A_j x, \partial_x \rangle v - A_j v, \quad j = 1, \ldots, d,$$

under the compatibility conditions

$$L_j f_k = L_k f_j, \qquad j, k = 1, \ldots, d. \tag{29}$$

First we consider a 2-$\mathbb{C}$ action studied in the example in the previous section. We assume that there exists a vector field in the two-dimensional Lie algebra which is not semisimple. We can choose a base X_1, X_2 with linear parts $A_j \in GL(4; \mathbb{C})$ satisfying $\mathrm{spec}\,(A_1) = \{1, 1, \nu, \nu\}$ and $\mathrm{spec}\,(A_2) = \{0, 1, \mu, \mu\}$, respectively, where $\nu \le \mu \le 0$, $(\nu, \mu) \notin \mathbb{Q} \times \mathbb{Q}$, and

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \nu & \varepsilon \\ 0 & 0 & 0 & \nu \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mu & \varepsilon_0 \varepsilon \\ 0 & 0 & 0 & \mu \end{pmatrix}, \tag{30}$$

where $\varepsilon \neq 0$ and $\varepsilon_0 \in \mathbb{C}$. We can make $|\varepsilon| > 0$ arbitrarily small by an appropriate linear change of variables.

Let $\sigma \geq 1$. We say that a formal power series $f(x) = \sum_\alpha f_\alpha x^\alpha$ is in a Gevrey space $G_2^\sigma(\mathbb{C}^4)$ if $f_\alpha = 0$ for $|\alpha| \leq 1$ and, there exist $C > 0$ and $R > 0$ such that

$$|f_\alpha| \leq CR^{|\alpha|}|\alpha|!^{\sigma-1}, \quad \forall \alpha \in \mathbb{Z}_+^4.$$

We consider the following equation

$$L_A v := {}^t(L_1 v, L_2 v) = f, \quad f = {}^t(f_1, f_2) \in (\mathbb{C}_2^4\{x\})^2,\ x \in \mathbb{C}^4, \tag{31}$$

where ${}^t(f_1, f_2)$ satisfies the compatibility condition $L_1 f_2 = L_2 f_1$. Then we have

Theorem Assume that $\varepsilon_0 \neq 0$ is a real number. Then, if $(\nu, \mu) \in \mathbb{Q} \times \mathbb{Q}$ and $\nu < \mu \leq 0$, then there exists $f = {}^t(f_1, f_2) \in (\mathbb{C}_2^4\{x\})^2$ such that $L_1 f_2 = L_2 f_1$ and the equation (3.6) has a formal power series solution $v \notin \bigcup_{1 \leq \sigma < 5/2} G_2^\sigma(\mathbb{C}^4)$.

For more details, see the paper [YoGr2006]

7. Intersections with spheres.

Before addressing the transversality issue in the presence of Jordan blocks we consider examples of non-diagonalizable linear complex flows.

First, let

$$L_0 = (w_1 + \varepsilon w_2)\partial_{w_1} + w_2\partial_{w_2}, \qquad w \in \mathbb{C}^2.$$

It corresponds to the linear complex action defined by the matrix

$$\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}, \qquad \varepsilon \in \mathbb{C}.$$

Straightforward calculations imply that L_0 is transversal to S^3 iff $|\varepsilon| \ll 1$.

Secondly, let

$$N_0 = \varepsilon w_2 \partial_{w_1}$$

. It corresponds to the linear complex action defined by the matrix

$$\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}, \qquad \varepsilon \in \mathbb{C}.$$

Straightforward calculations imply that N_0 is not transversal to S^3, the set of tangency is (topologically) S^1 while each transversal intersection produces a cycle.

Next, we consider a linear holomorphic flow Az which is in the Poincaré domain. Assume that it admits exactly m Jordan blocks, $1 \leq m \leq n$ with eigenvalues

$$\lambda_k = \alpha_k + i\beta_k, \quad k = 1, \ldots, m.$$

Proposition. Let the nilpotent part of A satisfy a smallness condition. Then $F[A]$ intersects $S^{2n-1}(r)$ transversally. Then if if

$$\alpha_k\beta_j - \alpha_j\beta_k \neq 0 \quad \text{for all } j,k = 1,\dots,m,\ k \neq j, \tag{32}$$

then X admits exactly m periodic orbits.

Let now $z \notin \mathbb{I}^{2n_j}_{eig}$. Then the curve $\ell[z]$ defined by $F(s,t;z) =$ could be parameterized by the implicit function theorem by $s = \theta(t) = \theta(t,z)$ and

$$\ell[z] : Z(t) = \Gamma(\theta(t),t;z), \qquad t \in \mathbb{R}, \tag{33}$$

is not periodic and satisfies the following properties

$$\lim_{t\to+\infty} \operatorname{dist}(Z(t), O[\ell[\overline{k}(z)]]) = 0 \tag{34}$$

$$\lim_{t\to-\infty} \operatorname{dist}(Z(t), O[\ell[\underline{k}(z)]]) = 0 \tag{35}$$

where $\overline{k}(z)$ (respectively $\underline{k}(z)$) stands for the largest (respectively) smallest integer $k \in \{1,\dots,m\}$ such that $z^k \neq 0$. Here $O[\ell_k] = \{Z^k_{per}(t) : t \in \mathbb{R}\}$ stands for the orbit of the periodic curve ℓ_k.

Next, if the condition is not satisfied, then X has infinitely many periodic orbits.

Finally, $F[A] \bigcap S^{2n-1}(r)$ defines a Hopf type foliation, i.e., every orbit of X is periodic, provided the vectors (α_j,β_j), $j = 1,\dots,m$, lie on a half–line containing the origin, i.e.,

$$\frac{\xi_1}{\alpha_1} = \ldots = \frac{\xi_m}{\alpha_m} =: \tau; \tag{36}$$

for every $k \in \{1,\dots,m\}$.

8. Concluding remarks

Research in progress: on divergence solutions in the presence of higher order resonances, convergent simultaneous normal forms in Siegel domains.

The issue of holomorphic solutions for analogues of the harmonic oscillator (might be viewed in the realm of infinite dimensional systems.