Semi-Parabolic Implosion: Dimension 2

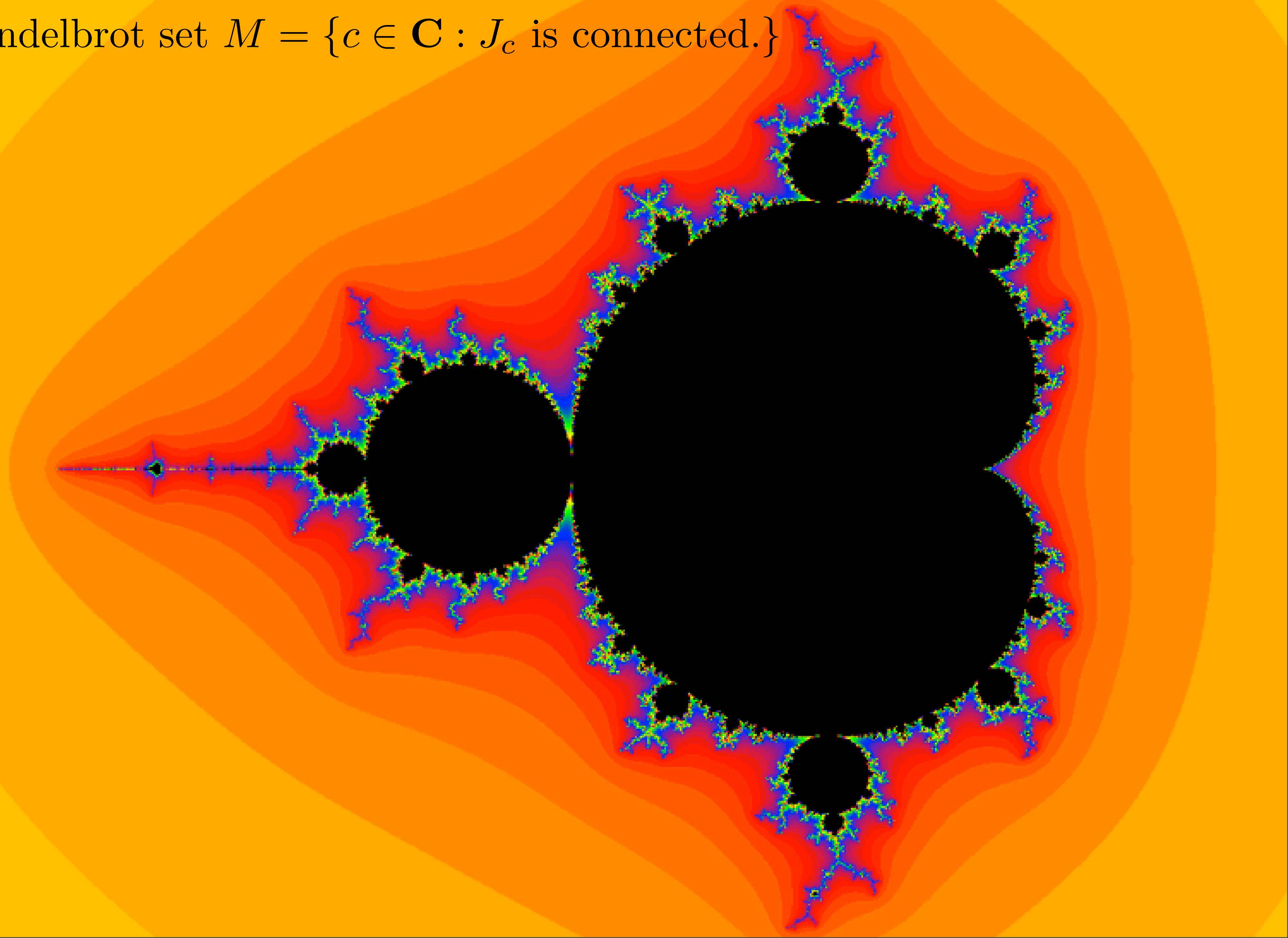
Work in progress with J. Smillie and T. Ueda

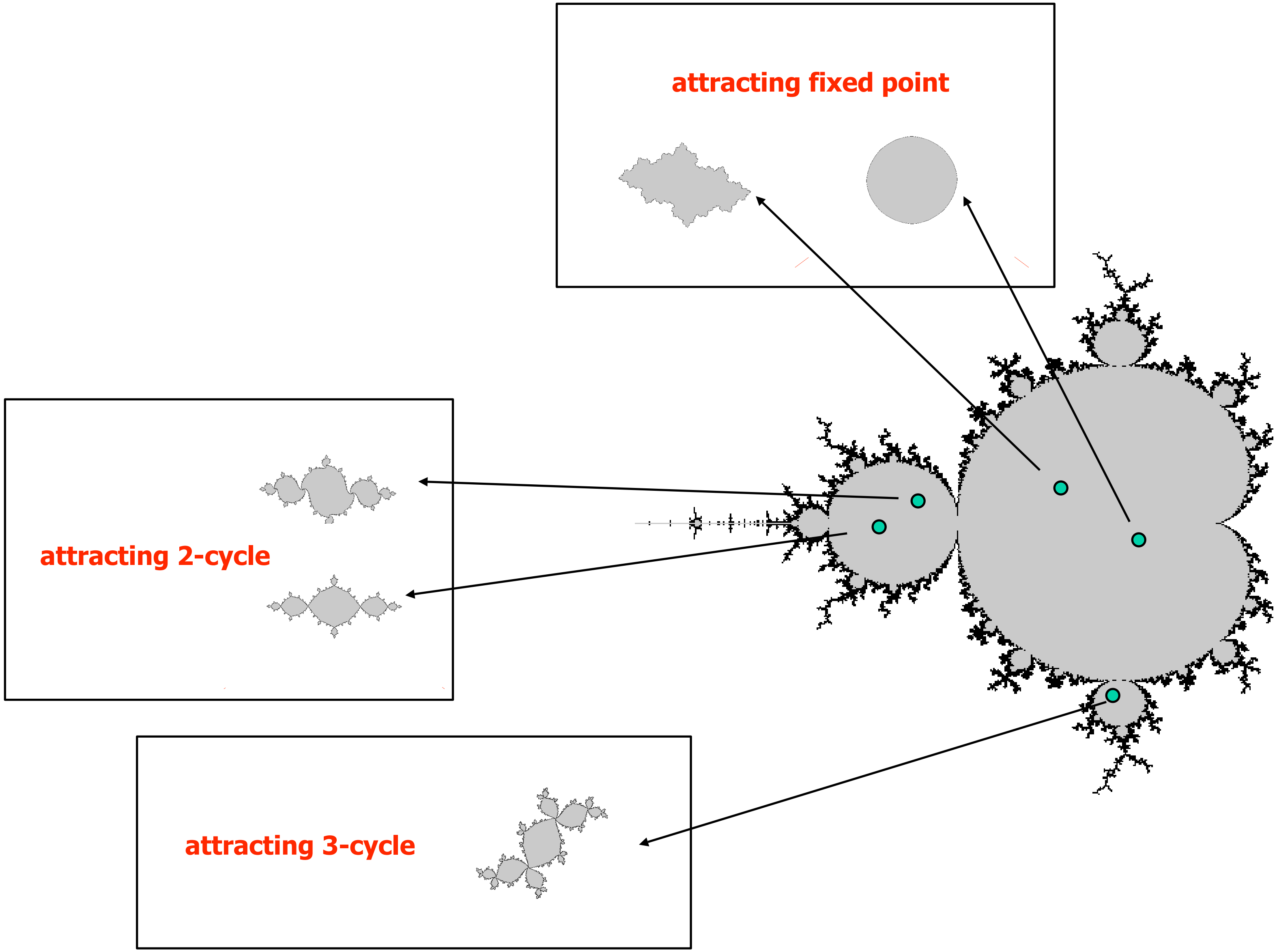
Plan of this talk

- Review 1-D implosion
- Eye candy
- Ueda's semi-attracting world (+ upgrades)
- Eggbeater dynamics
- Semi-parabolic Implosion

Quadratic family $\{p_c = z^2 + c\}$, parametrized by $c \in \mathbf{C}$.

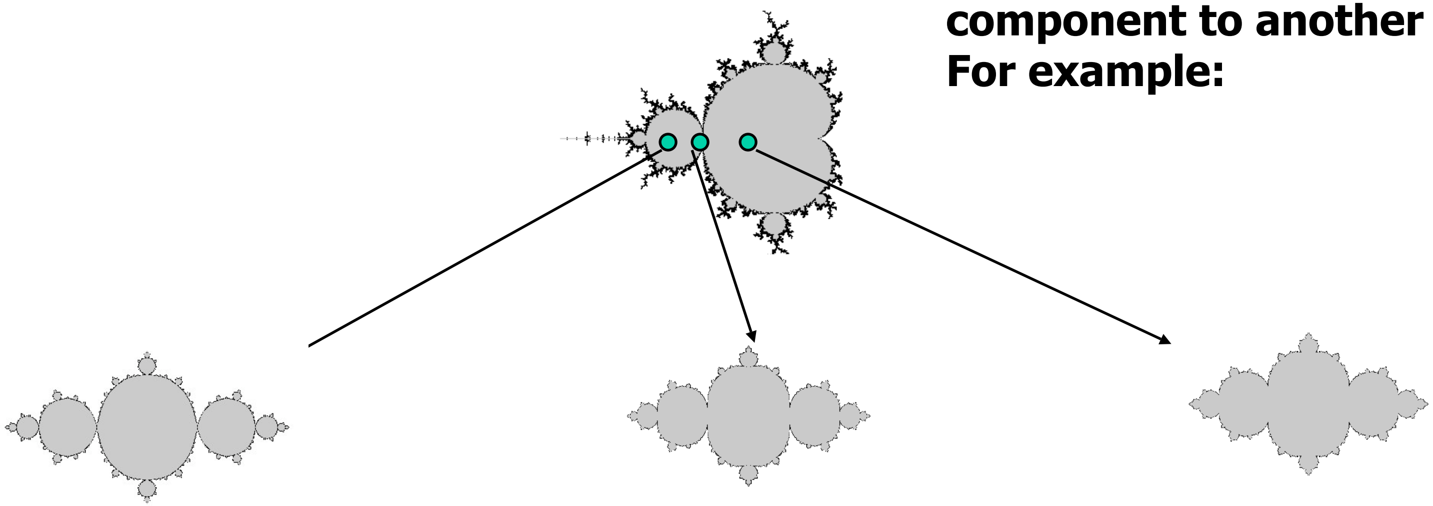
Mandelbrot set $M = \{c \in \mathbf{C} : J_c \text{ is connected.}\}$




attracting fixed point
attracting 2-cycle
attracting 3-cycle

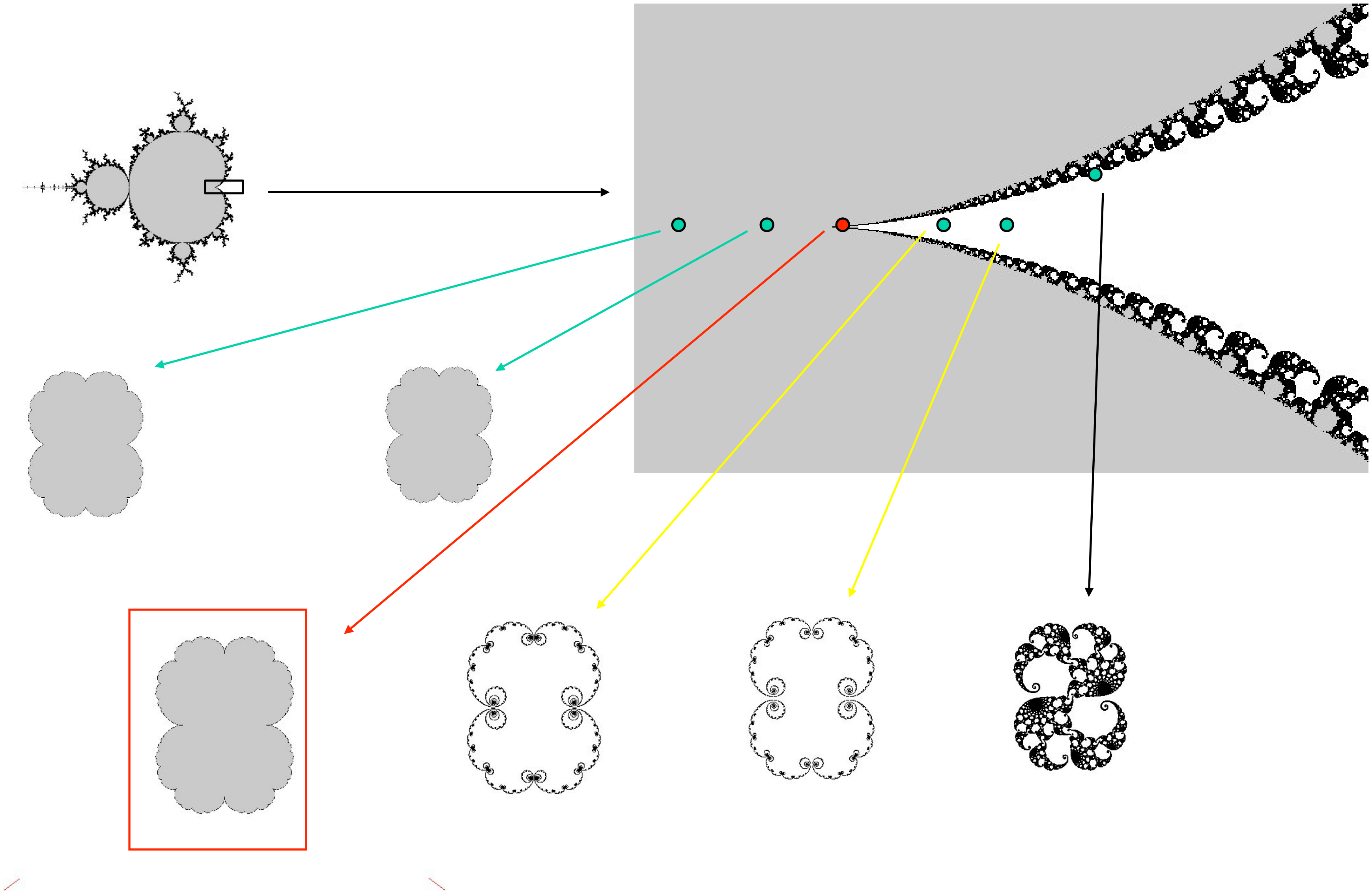
The Julia sets in each hyperbolic component are structurally stable, which implies that P-->J(P) is continuous in each hyperbolic component (and even defines a "holomorphic motion").

The passage from one hyperbolic component to another can be continuous. For example:



bifurcation: (attracting fixed point --> repelling fixed point + attracting 2-cycle)

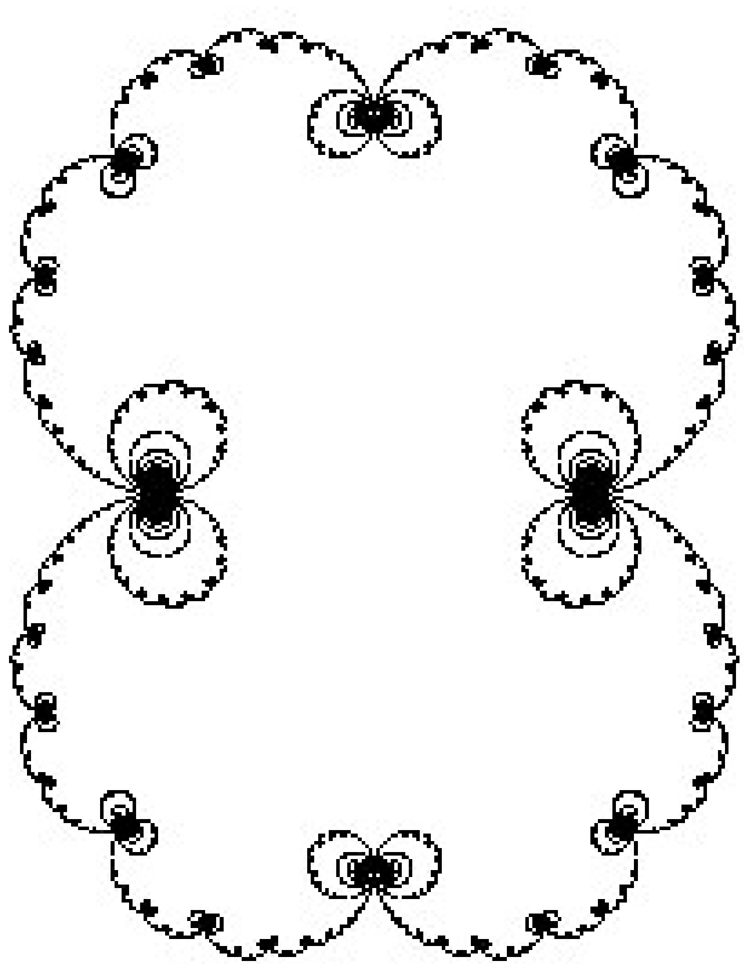
Bifurcation of the Cauliflower



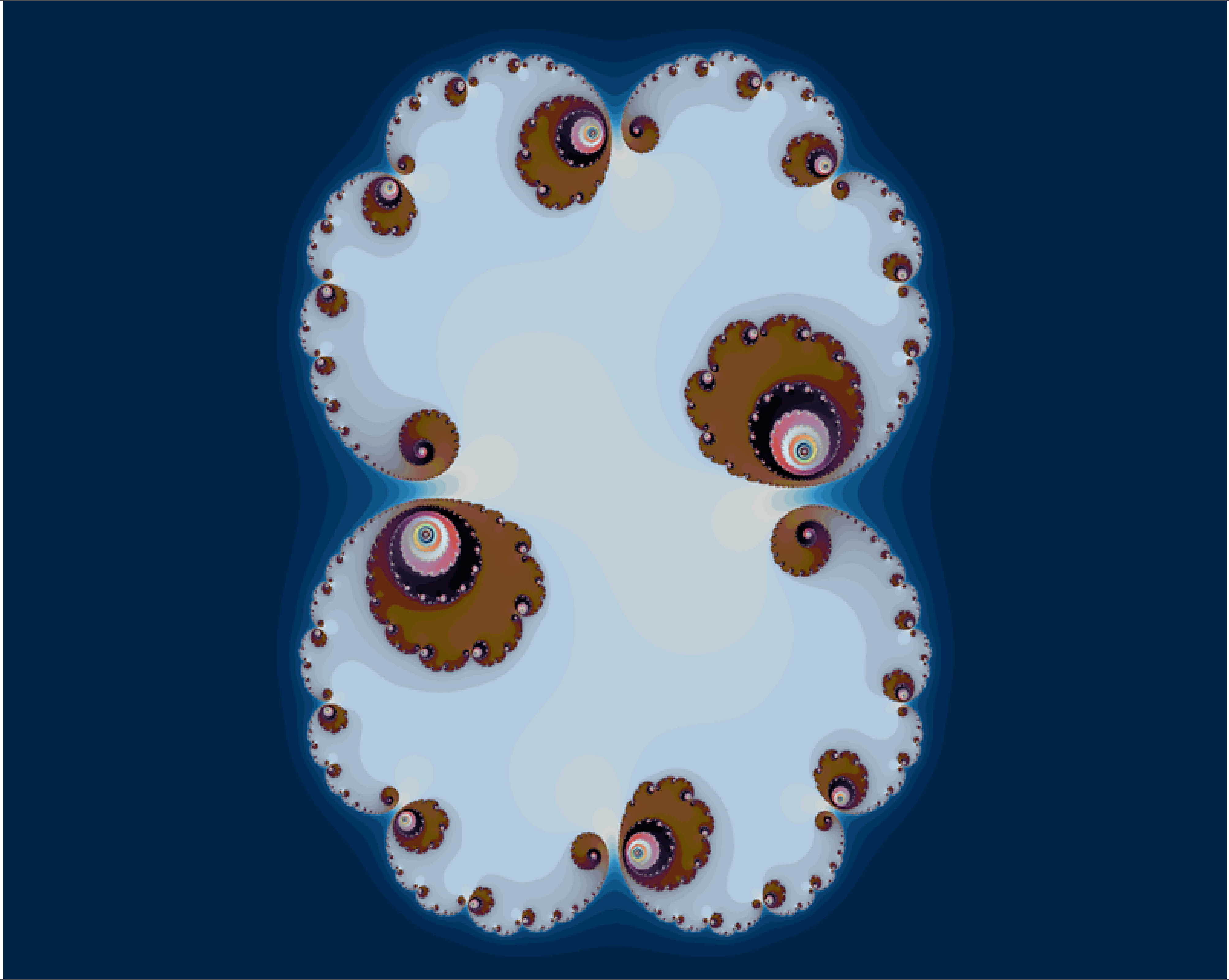
Parabolic Implosion: $J_{\frac{1}{2}} \neq \lim_{\epsilon \to 0^+} J_{\frac{1}{2}+\epsilon}$

The "inner curls" of $J_{\frac{1}{2}+\epsilon}$ suddenly disappear.

c=0.251

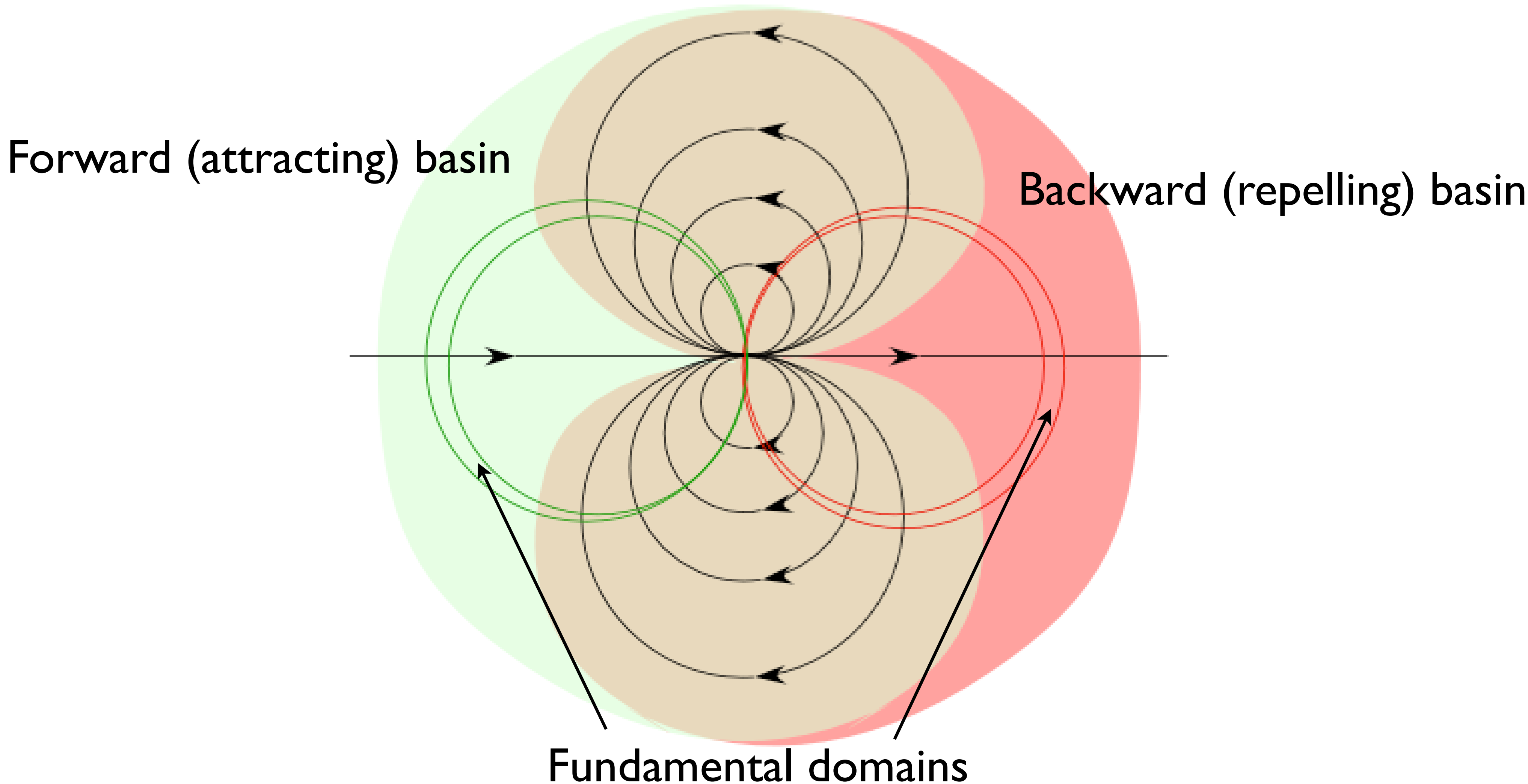


Lavaurs, Douady, Zinsmeister, ...



(Local) Parabolic Dynamics

$f: \ z \mapsto z + z^2 + \cdots$ $\qquad$ $f^{-1}: \ z \mapsto z - z^2 + \cdots$



Forward and backward Fatou coordinates on the attracting and repelling basins:

$$\Phi^+ : \mathcal{B}^+ \to \mathbf{C} \qquad \Phi^- : \mathcal{B}^- \to \mathbf{C}$$

$$\Phi^+ \circ f = \Phi^+ + 1 \qquad \Phi^- \circ f^{-1} = \Phi^- - 1$$

We map the crescent in the forward basin to C by the Fatou coordinate

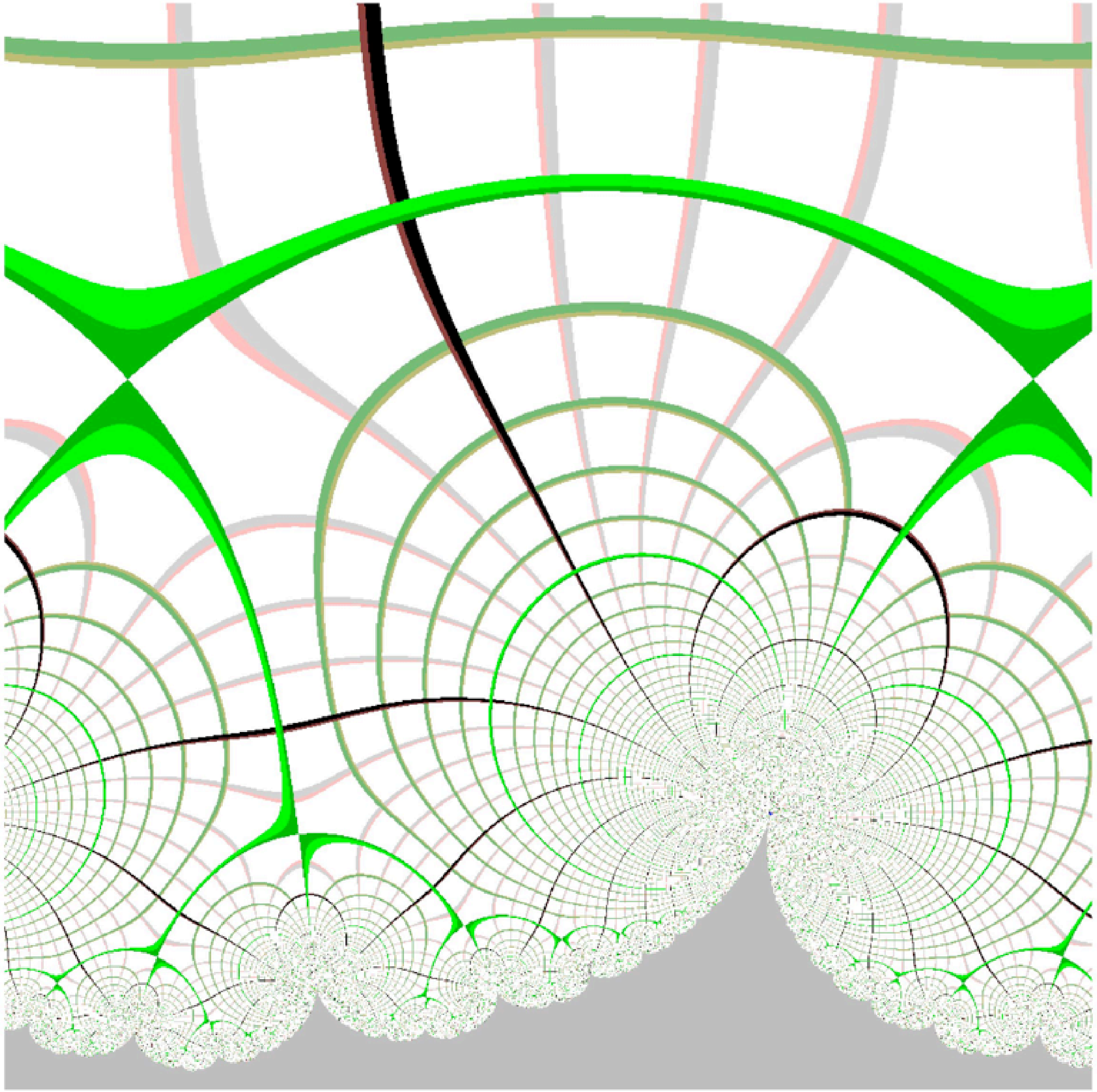
$$\Phi^- : \mathcal{B}^- \to \mathbf{C}$$

then we "graph" the Fatou coordinate

$$\Phi^+ : \mathcal{B}^+ \to \mathbf{C}$$

inside the image basin by showing the level sets of the real and imaginary parts. Our pictures are not the "normal" ones, since they take place only inside the crescents.

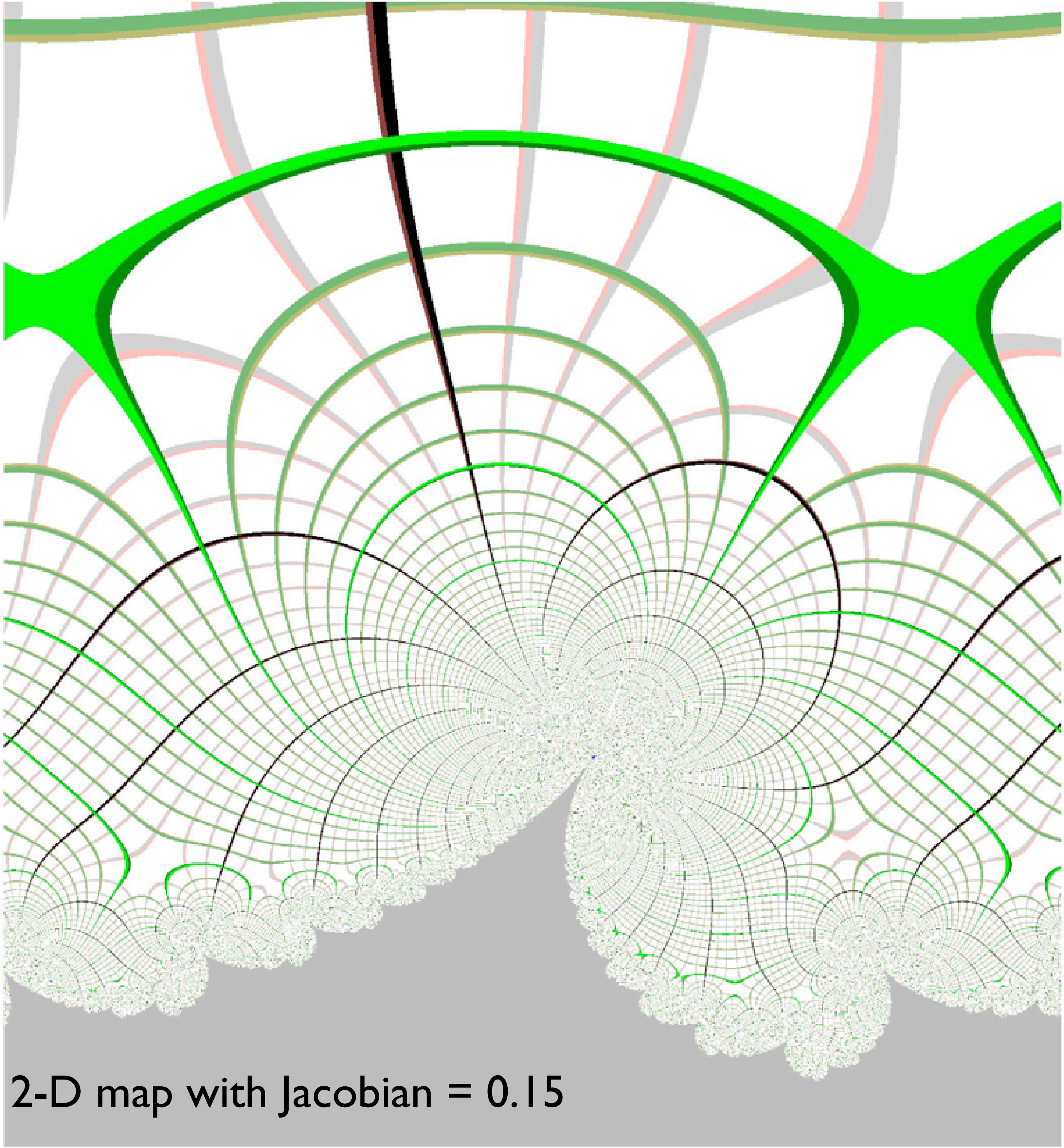
View of the 1-D quadratic map with c = .25 Note critical points (not part of the most local picture). View is truncated above (level curves are straight) and below (leave basin). There is a lot of gray, and then we encounter the "lower half" of this picture. Note the periodicity.



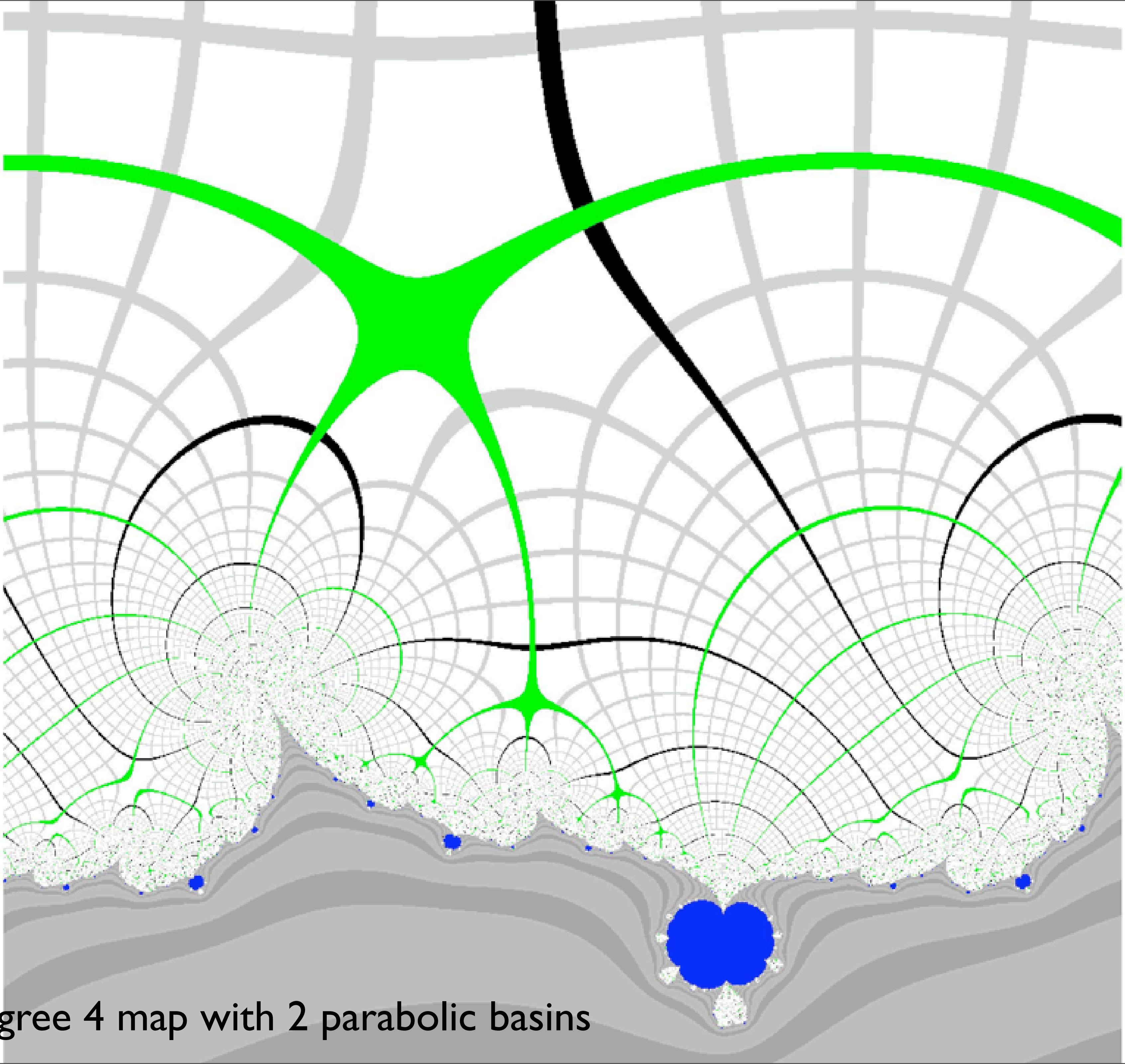
Move to 2-D mappings:

$$f(x,y) = ((1+a)x - ay + x^2 + bx^3 + cx^4, x)$$

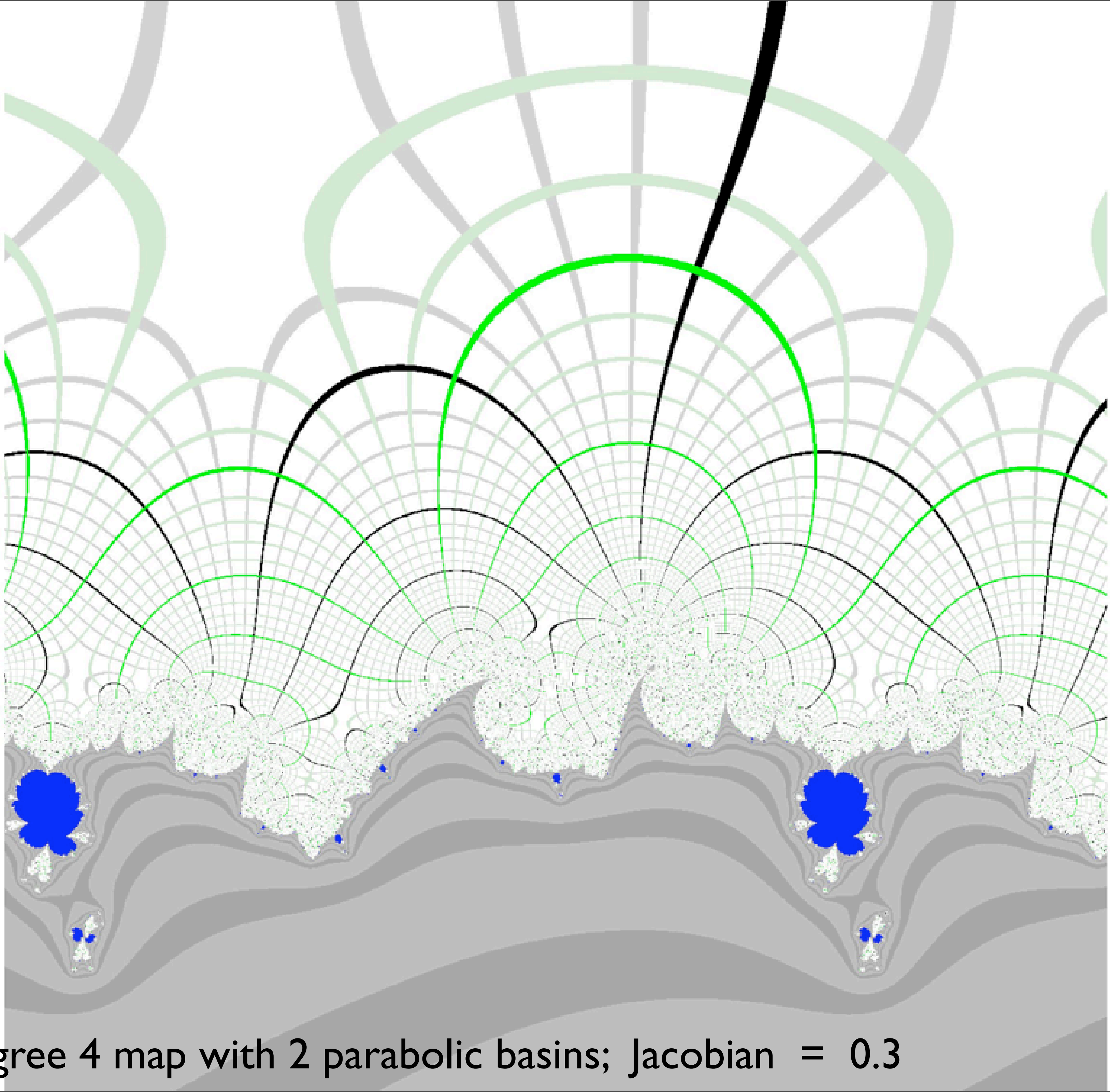
is a biholomorphic map. We have b = c = 0 in most pictures. The Jacobian is constant (= a). The origin (0,0) is fixed, and the eigenvalues at the origin are 1 and a. We work only with the case $0<|a|<1$. Values such as a = 0.3 are "very large".



2-D map with Jacobian = 0.15



1-D degree 4 map with 2 parabolic basins



2-D degree 4 map with 2 parabolic basins; Jacobian = 0.3

Forward and backward Fatou coordinates on the attracting and repelling basins:

$$\Phi^+ : \mathcal{B}^+ \to \mathbf{C} \qquad \Phi^- : \mathcal{B}^- \to \mathbf{C}$$

$$\Phi^+ \circ f = \Phi^+ + 1 \qquad \Phi^- \circ f^{-1} = \Phi^- - 1$$

A (partially defined) dynamical system on the overlap of forward/backward basins is given by the "transition function" or "Lavaurs map" (essentially visible in the previous pictures) between the two Fatou coordinates:

$$g_\alpha := (\Phi^+)^{-1} \circ T_\alpha \circ \Phi^-, \qquad T_\alpha(w) = w + \alpha$$

The translation parameter α is arbitrary since the Fatou coordinates are only defined modulo additive constants. The maps commute, and the pair (f, g_α) defines a new dynamical system. We define the dynamically invariant set:

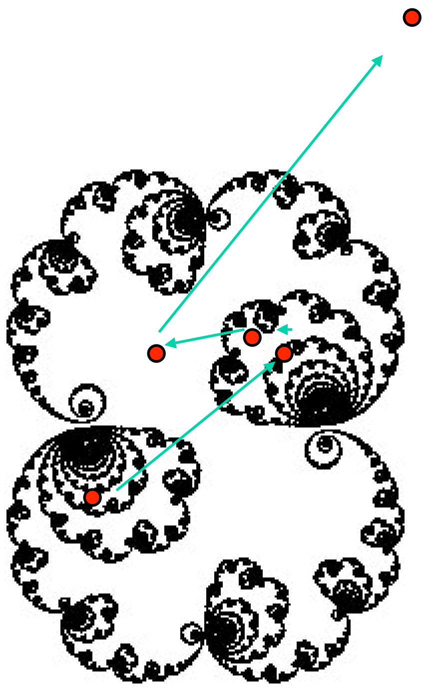
$$K_\infty(f, g_\alpha) := \{z : g_\alpha^n f^m(z) \in \mathcal{B},\ \forall n, m \geq 0\}$$

Julia-Lavaurs set:

Here we apply a map g_α to a point of the filled Julia set of

$$p(z) = z^2 + \frac{1}{4}$$

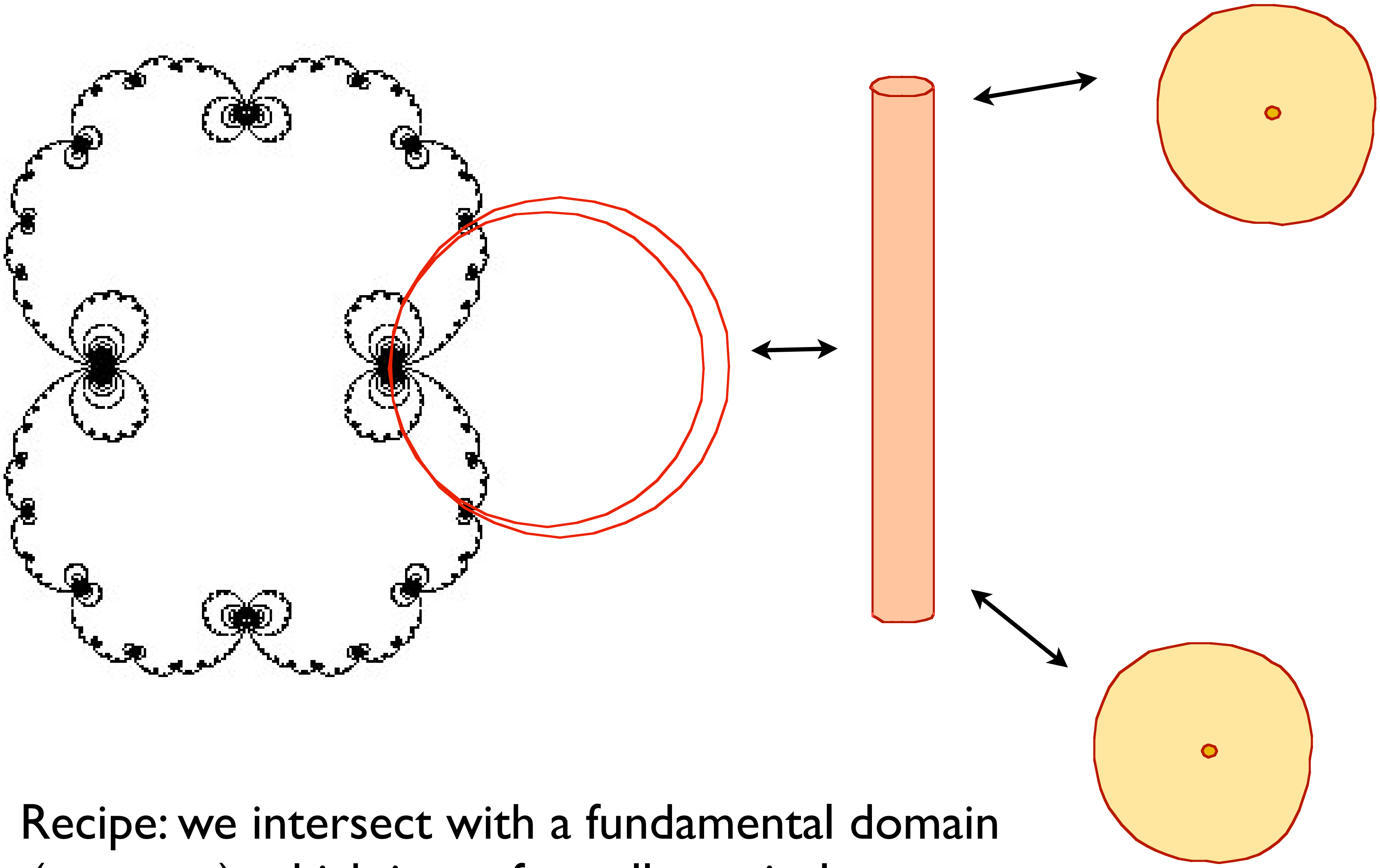
The red point is not in the filled Julia-Lavaurs set because it escapes. Lavaurs-Julia sets give a "geometric estimate" on the amount of discontinuity that takes place in parabolic implosion:



Theorem. *For $\epsilon_j \to 0$ with $Im(\epsilon_j) \approx c(Re(\epsilon_j)^2)$, there is a subsequence and α such that*

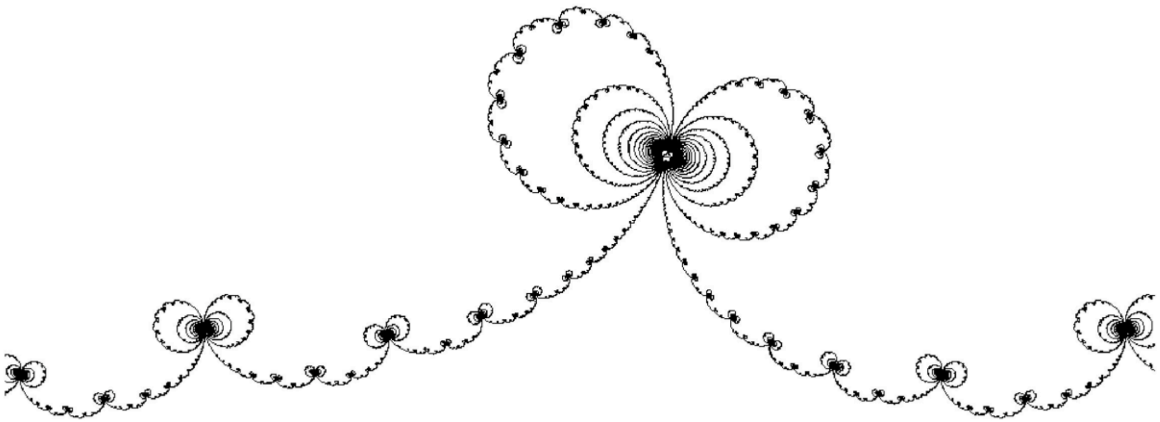
$$\liminf_{\epsilon_j \to 0} J_{\frac{1}{4}+\epsilon_j} \supset J(p_{\frac{1}{4}}, g_\alpha).$$

Another view of the Julia-Lavaurs set

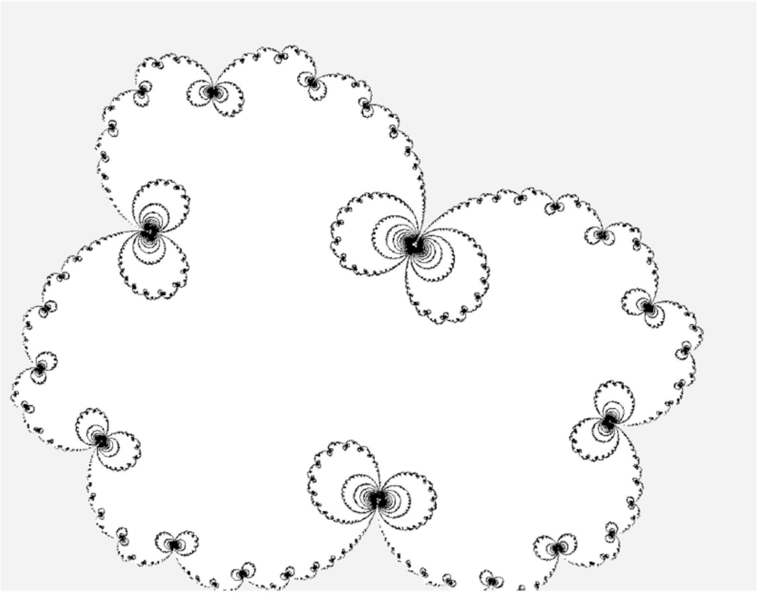


Recipe: we intersect with a fundamental domain (crescent), which is conformally equivalent to a cylinder. Each "end" of the cylinder is equivalent to a disk. Now draw the dynamically invariant Lavaurs-Julia set in the cylinder or disk.

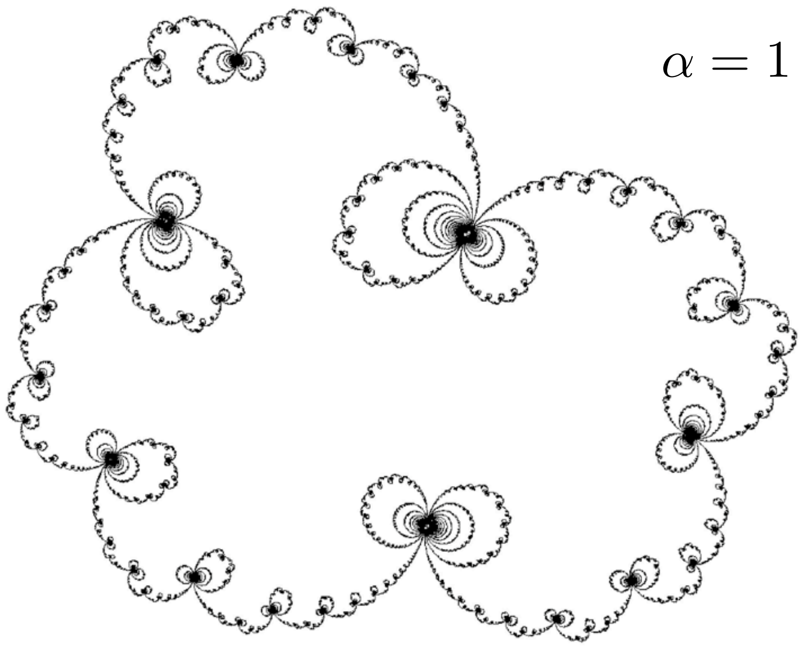
Previous Julia-Lavaurs set redrawn inside the cylinder



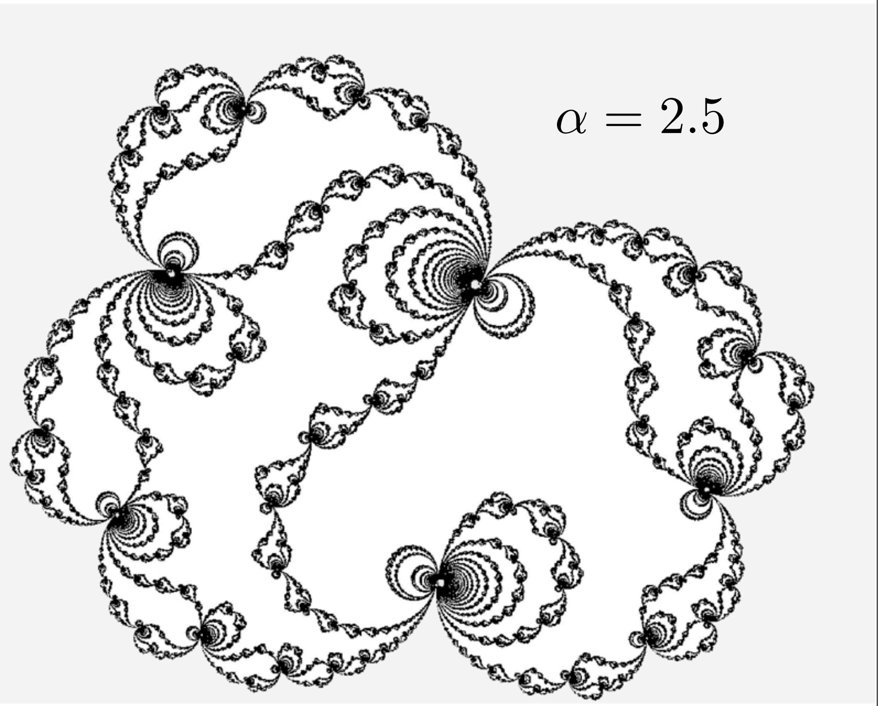
And inside the upper disk



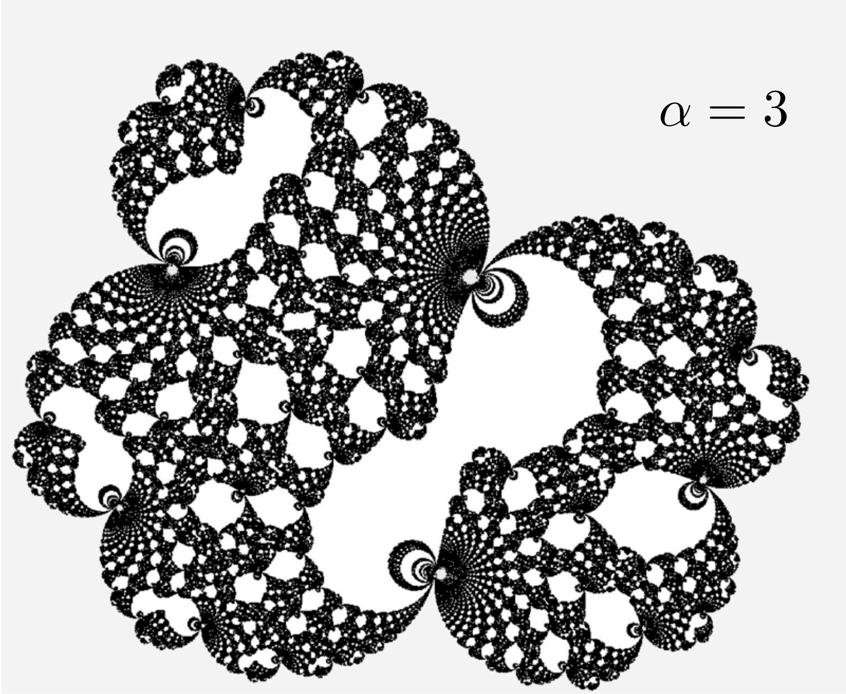
Effect of varying the parameter for the 1-D map c = .25



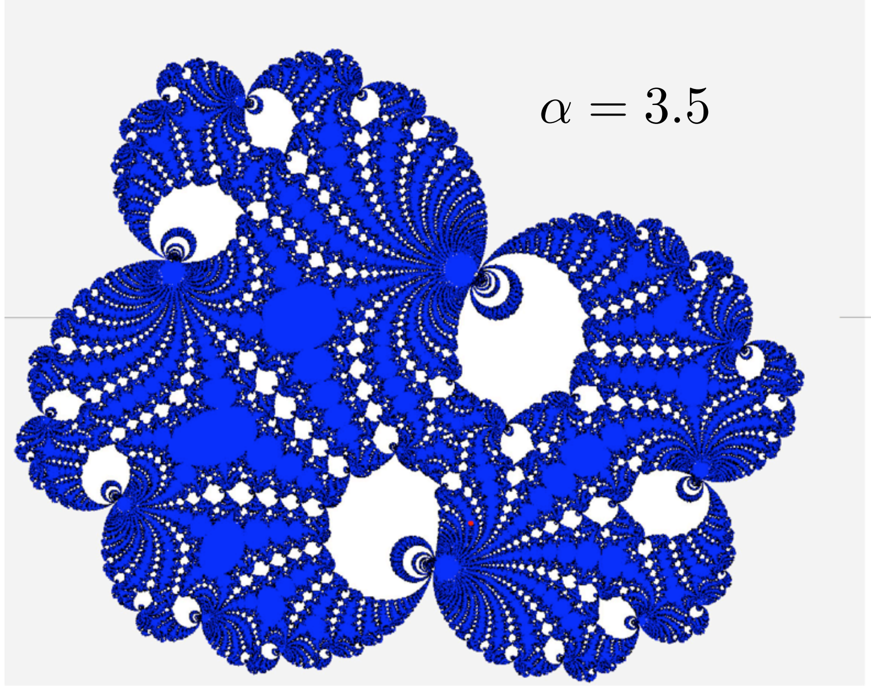
$\alpha = 1$



$\alpha = 2.5$

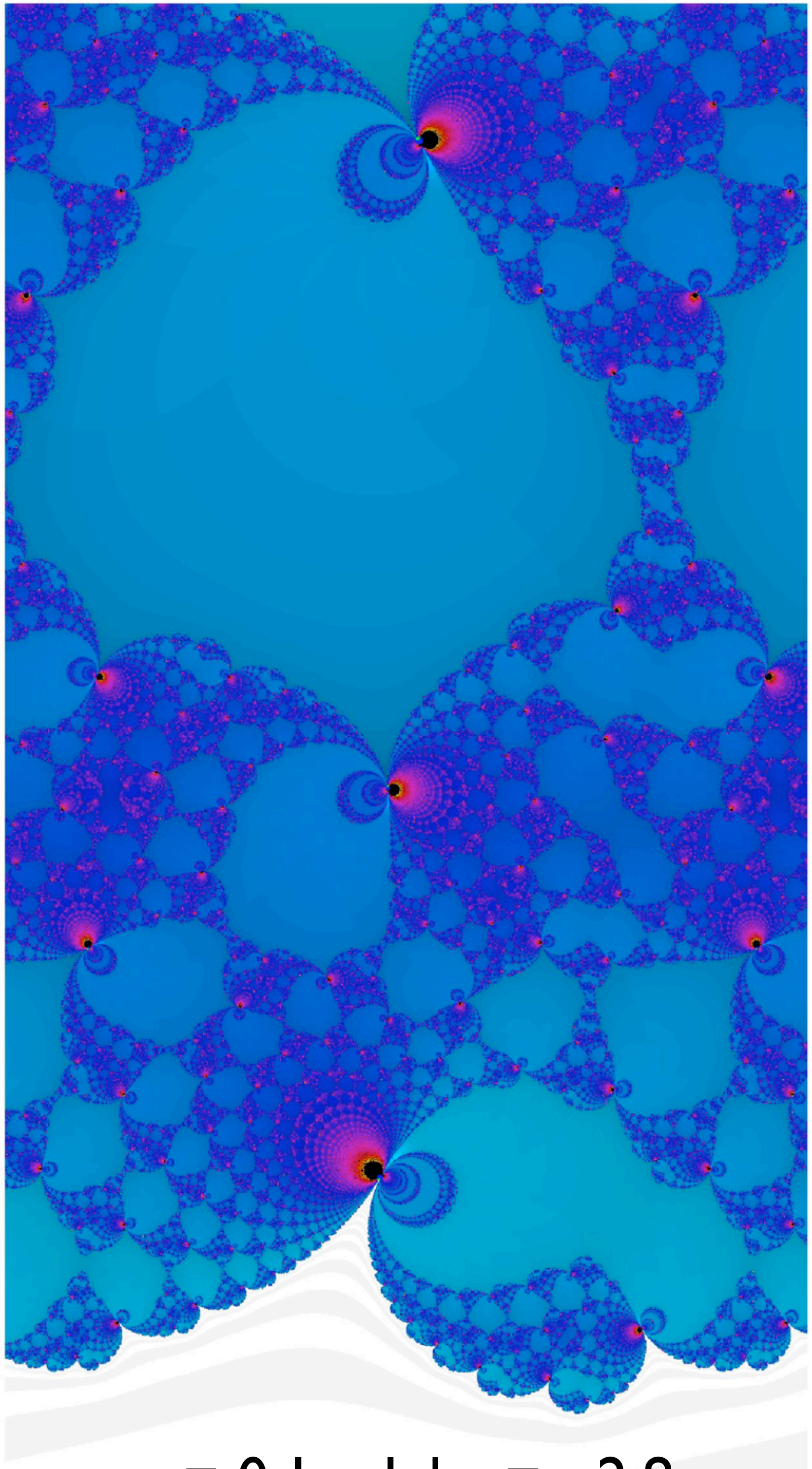


$\alpha = 3$

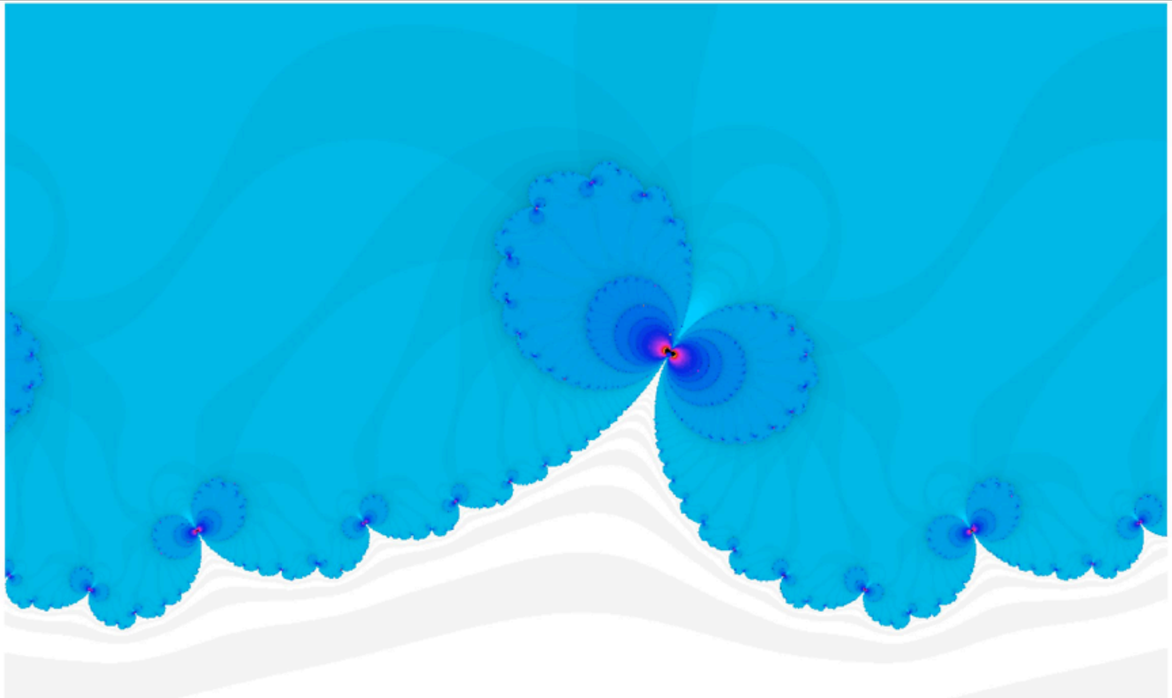


$\alpha = 3.5$

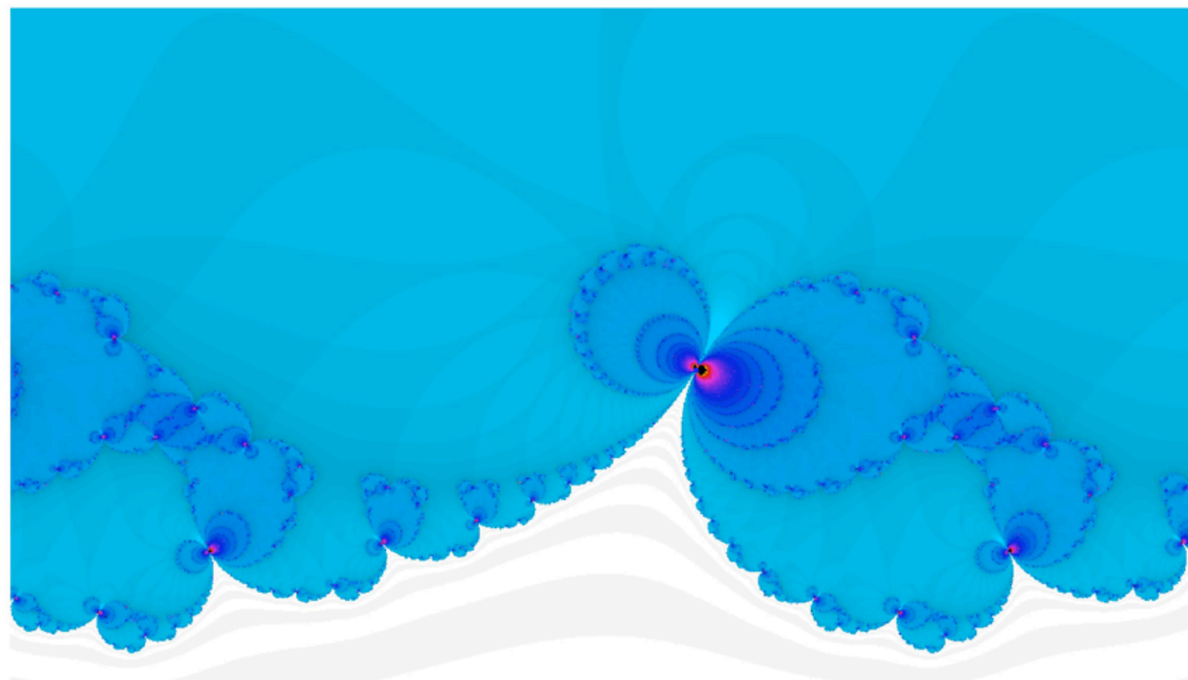
The dynamical sets of the transition maps measure parabolic implosion (as in the 1-D case):



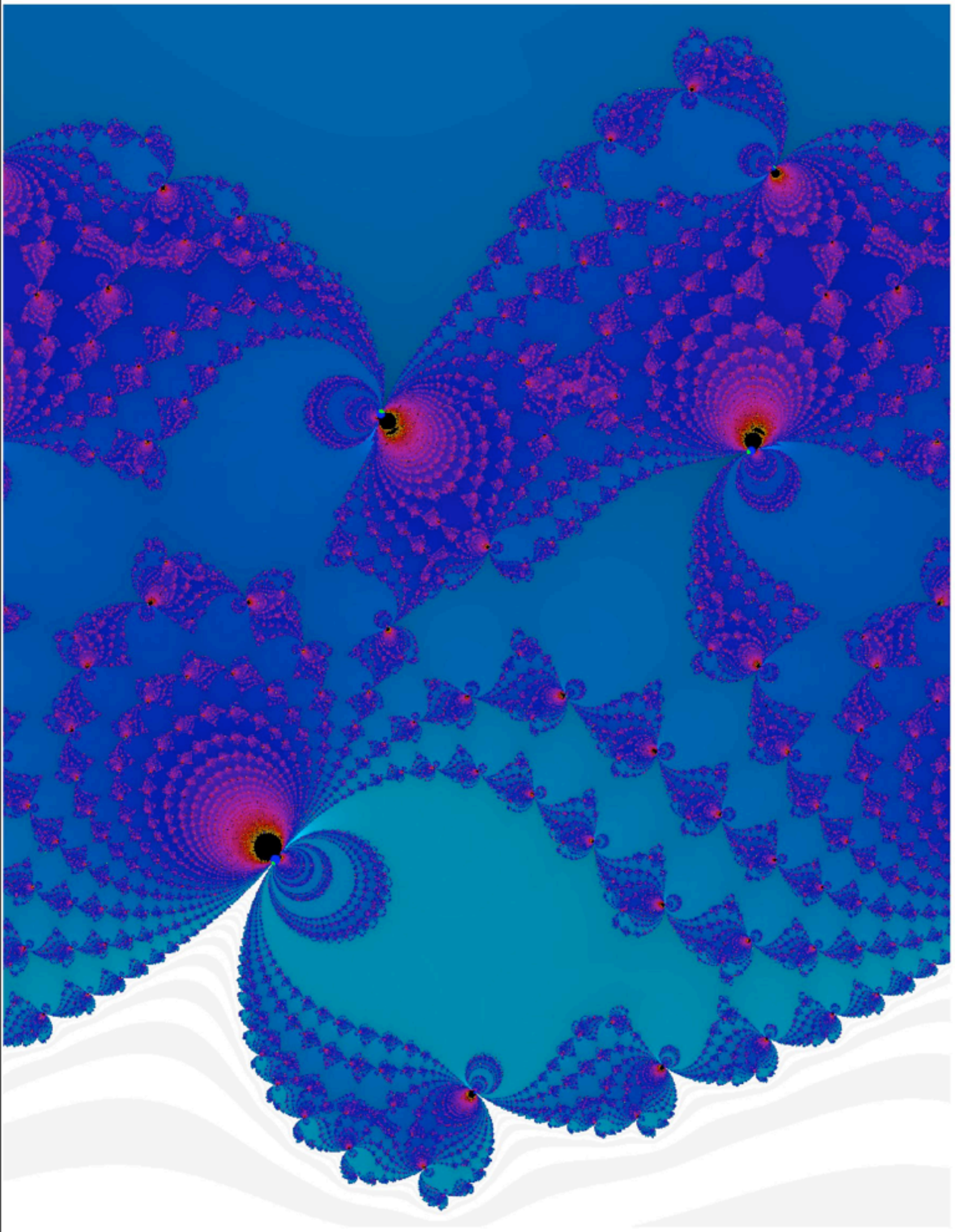
a = 0.1, alpha = -2.8



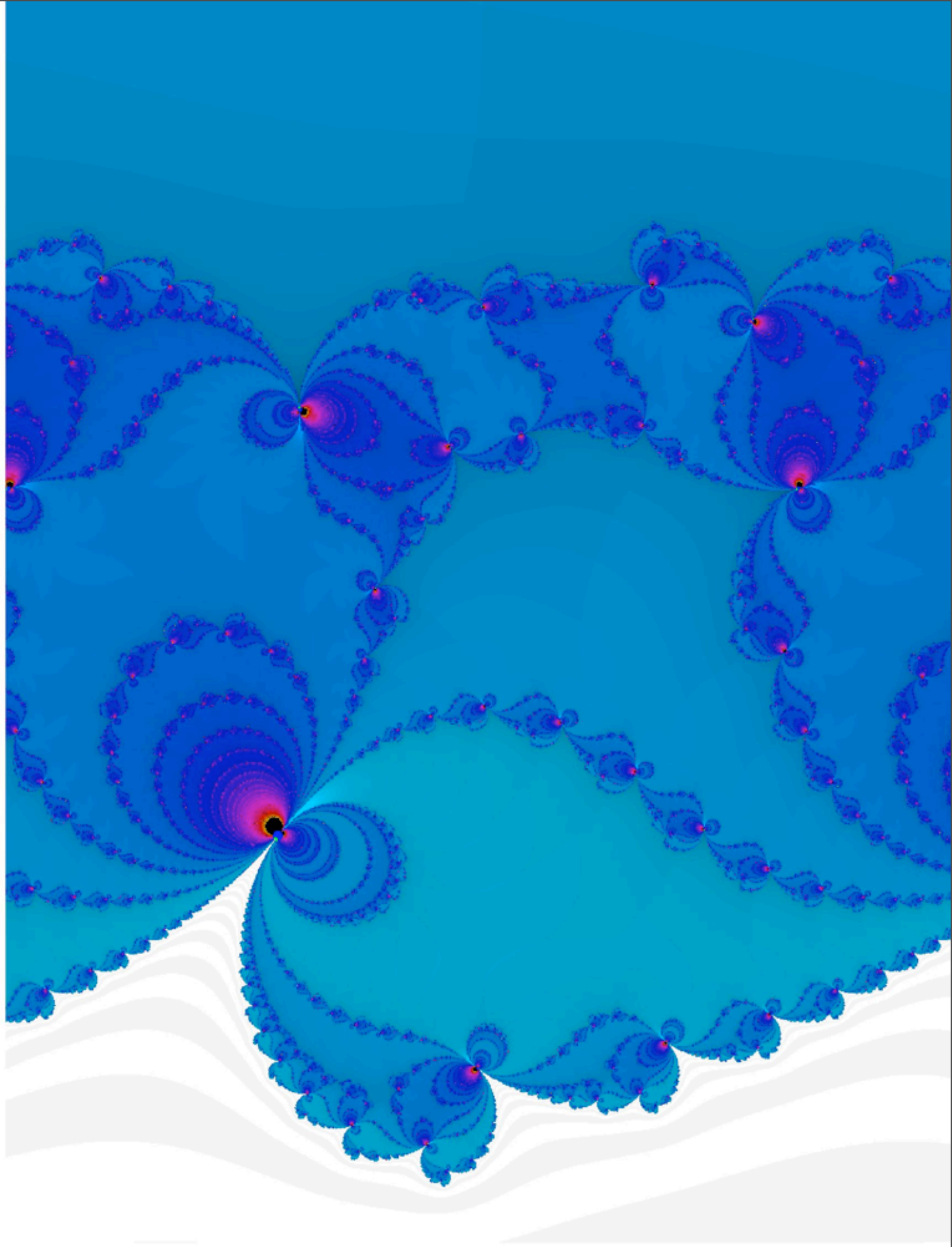
a = 0.1, alpha = 0



a = 0.1, alpha = 1.5

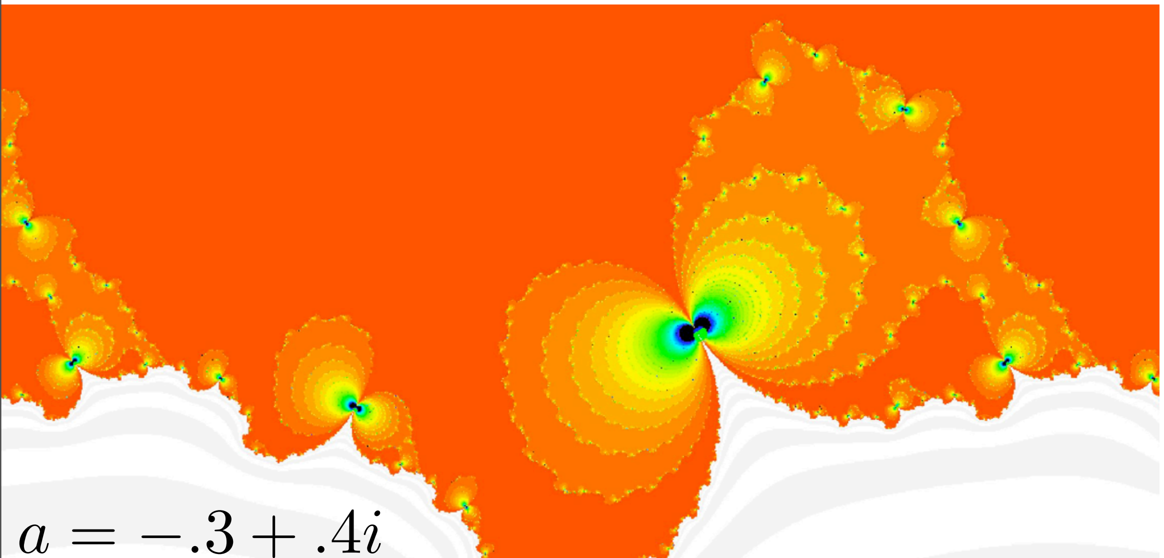


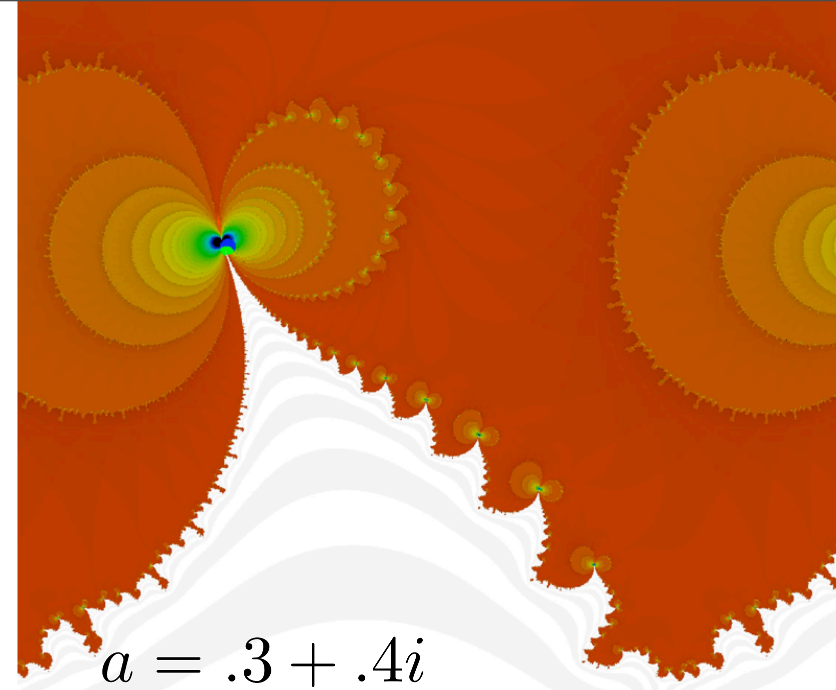
a = 0.2, alpha = -3.5

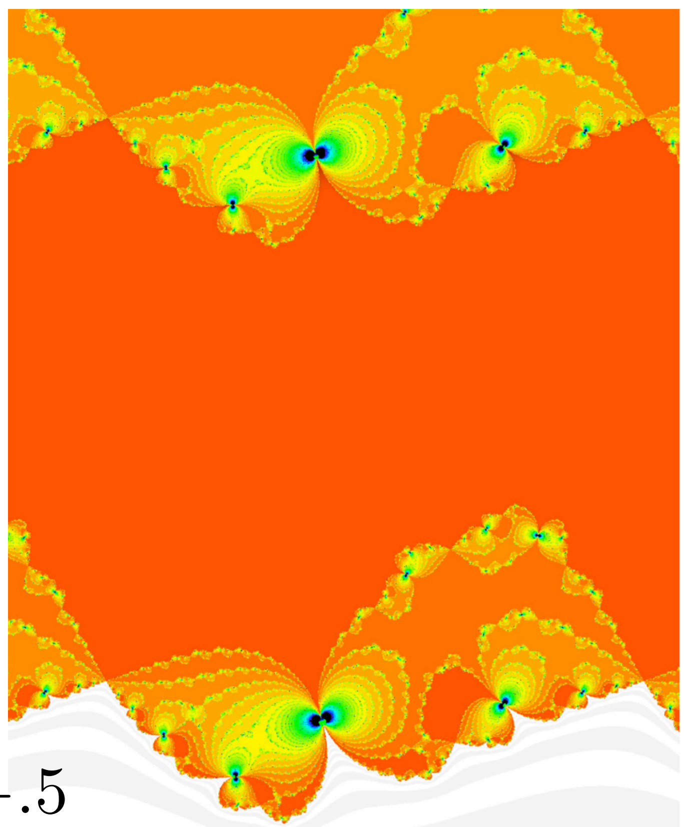


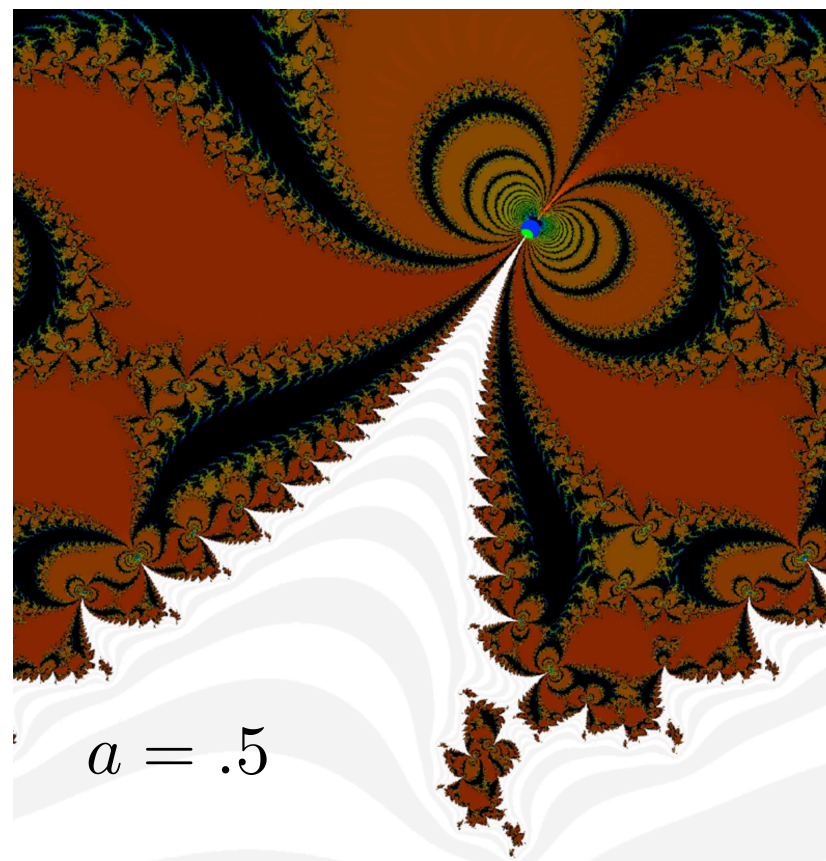
a = 0.2, alpha = -2.8

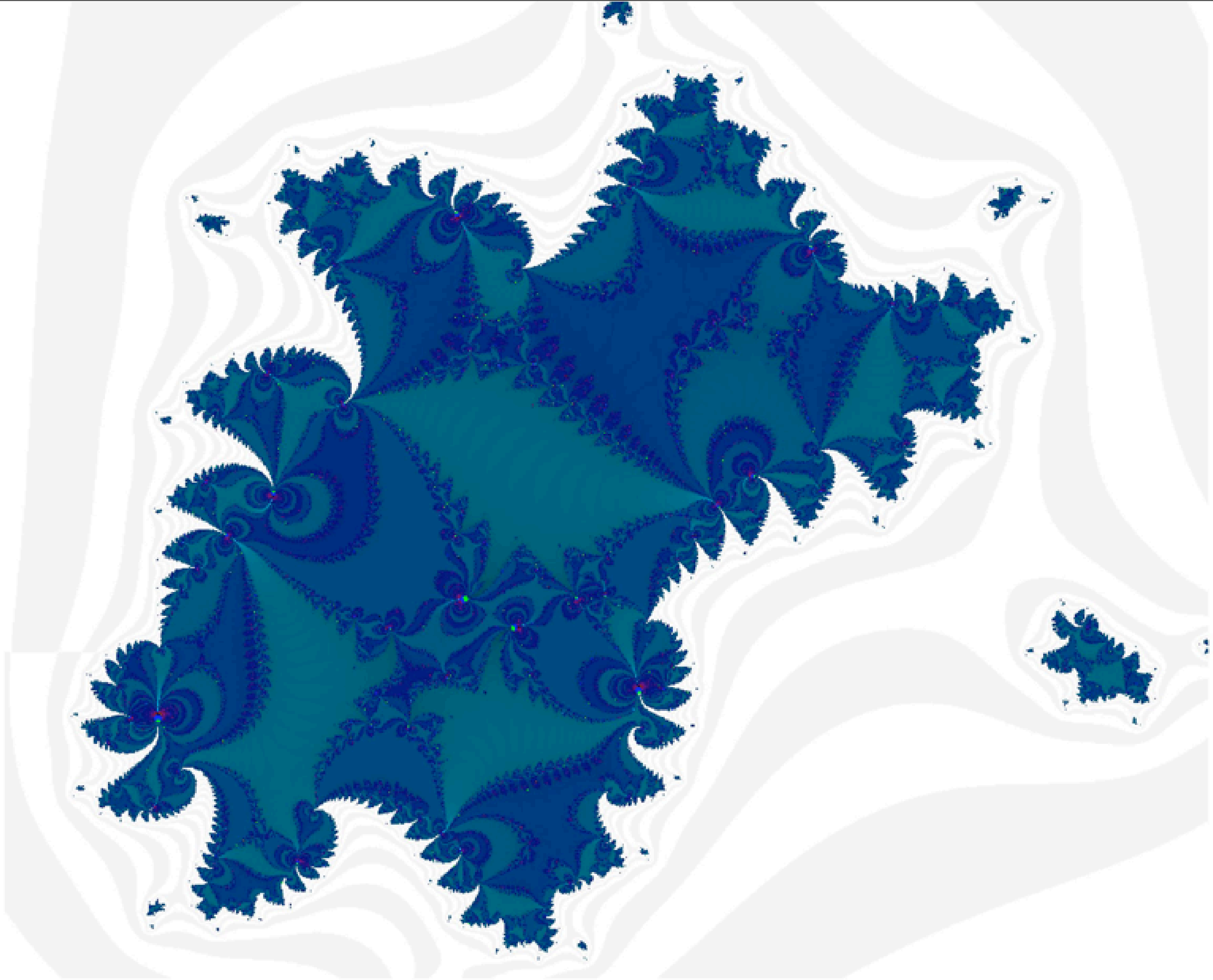
Pictures are in the cylinder; alpha = 0;
parameter a = 0 means map is 1-dimensional.











a = 0.6, alpha = 2.0

Dynamical set drawn inside the "disk at infinity" in the cylinder.