# *Applications of Parabolic Renormalization*

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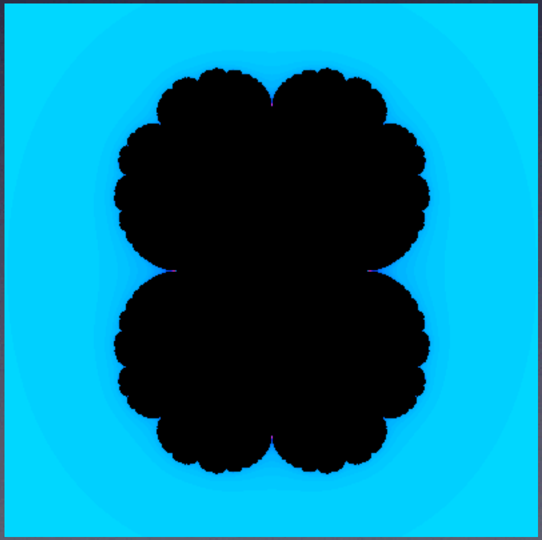


When a parabolic point is perturbed....

$$f_0(z) = z + z^2$$

Discontinuous change of Julia sets, etc.

# Inou's talk

$$f_0(z) = z + a_2 z^2 + \dots \qquad a_2 \neq 0$$

$E_{f_0} = \Phi_{attr} \circ \Phi_{rep}^{-1}$ horn map = Ecalle-Voronin invariant

Want to consider iteration

$$\mathbb{C}/\mathbb{Z} \ni z, \;\; E_{f_0}(z), \; E_{f_0}^{\,2}(z), \; E_{f_0}^{\,3}(z), \dots$$

$\mathcal{R}_0 f_0 = \mathrm{Exp}^\sharp \circ E_{f_0} \circ (\mathrm{Exp}^\sharp)^{-1}$ Parabolic renormalization

$$\mathrm{Exp}^\sharp(z) = e^{2\pi i z} : \mathbb{C}/\mathbb{Z} \xrightarrow{\simeq} \mathbb{C}^*$$

Want to iterate the process

$$f_0 \rightsquigarrow \mathcal{R}_0 f_0 \rightsquigarrow \mathcal{R}_0^2 f_0 \rightsquigarrow \mathcal{R}_0^3 f_0 \rightsquigarrow \dots$$

Main Theorem 1: $\exists \mathcal{F}_1 \quad \mathcal{R}_0(\mathcal{F}_1) \subset \mathcal{F}_1$

# Continued fraction and Renormalization (for rotation-like dynamics)

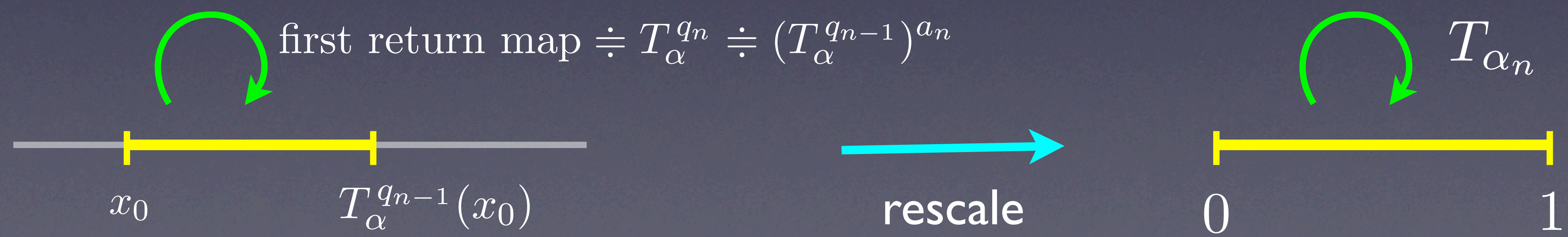
$$\alpha = \frac{1}{a_1 + \alpha_1} = \frac{1}{a_1 + \cfrac{1}{a_2 + \alpha_2}} = \cdots = \frac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

$$\alpha_0 = \alpha \in (0,1) \setminus \mathbb{Q}, \qquad a_n = \left[\frac{1}{\alpha_{n-1}}\right] \in \mathbb{N}, \quad \alpha_n = \frac{1}{\alpha_{n-1}} - a_n \in (0,1)$$

$$\alpha \quad \longleftrightarrow \quad T_\alpha : x \mapsto x + \alpha \text{ on } \mathbb{R}/\mathbb{Z}$$

$$\text{convergents } \frac{p_n}{q_n} \to \alpha \quad \longleftrightarrow \quad T_\alpha^{q_n} \to id \quad \text{“closest returns”}$$

$$\text{first return map} \doteqdot T_\alpha^{q_n} \doteqdot (T_\alpha^{q_{n-1}})^{a_n}$$



# Renormalization for rotation-like dynamics

## The same construction for non-linear maps

$$f_0 = f$$

$$f_{n+1} = \text{(first return map of } f_n \text{ to a fundamental domain)}$$
$$\text{up to rescaling}$$

$$\alpha_n = \text{rotation number of } f_n \qquad\qquad f_{n+1} \doteqdot f_n^{a_n}$$

# Yoccoz renormalization for Siegel-Bruno Theorem

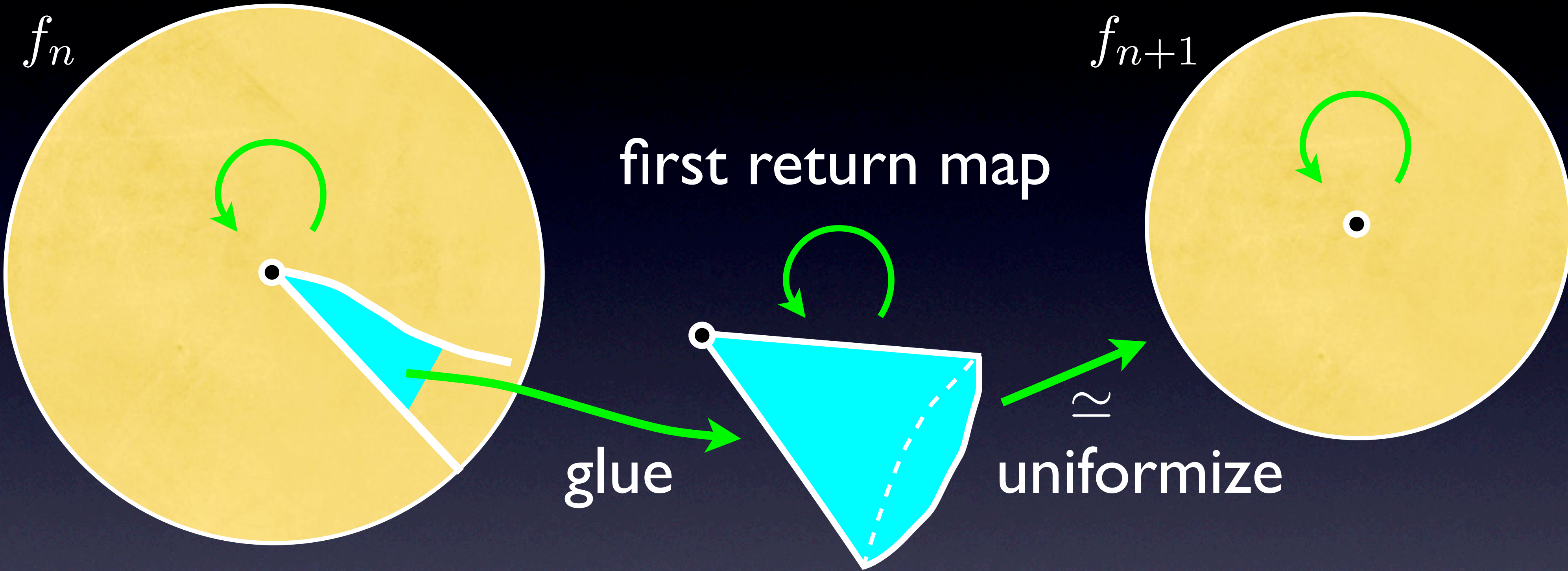
$$f(z) = e^{2\pi i \alpha} z + \ldots, \sum \log \frac{q_{n+1}}{q_n} < \infty \implies f \text{ is conjugate to } z \mapsto e^{2\pi i \alpha} z$$

$$f_n(z) = e^{2\pi i \alpha_n} z + \ldots \;\rightsquigarrow\; f_{n+1}(z) = e^{2\pi i \alpha_{n+1}} z + \ldots$$
$$= \text{first return map of } f_n$$
$$\text{to a fundamental domain}$$
$$\text{up to } \textit{uniformization}$$

$$\alpha_{n+1} \equiv -\frac{1}{\alpha_n} \quad (mod\ \mathbb{Z})$$

# Yoccoz renormalization for Siegel-Bruno Theorem

$$f_n(z) = e^{2\pi i \alpha_n} z + \dots \rightsquigarrow f_{n+1}(z) = e^{2\pi i \alpha_{n+1}} z + \dots$$




Optimality of Bruno condition (Yoccoz)

- inverse construction for germs(*unrenormalization*)
- can add extra fixed /periodic points
- another argument for quadratic polynomials

# Near-parabolic renormalization (cylinder renormalization)

take cresent-shaped fundamental region

quotient = cylinder $\mathbb{C}/\mathbb{Z} \quad \simeq \mathbb{C}^*$

Advantages:

quotient cylinder is canonical

can include critical point

when $\alpha$ is small the fundamental region does not shrink

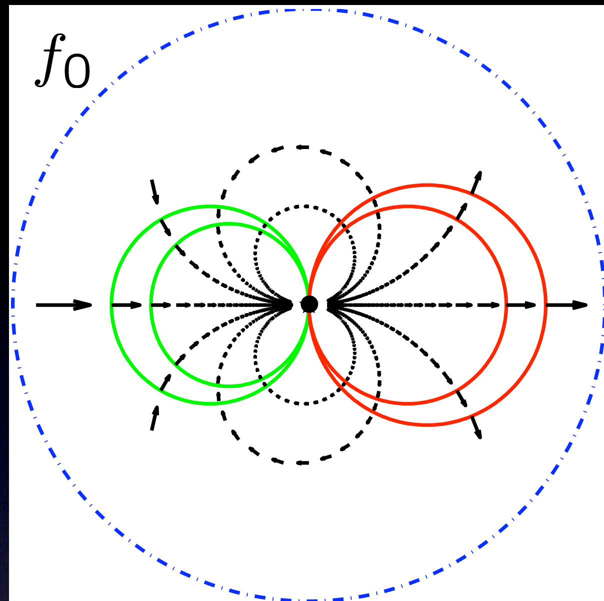
limit can be described by horn map

Disadvantage:

only applies to $\alpha$ with large continued fraction coefficients $\quad a_n \geq N$

# Horn map and Parabolic Renormalization

$$f_0(z) = z + a_2 z^2 + \ldots$$
$$a_2 \neq 0$$




Horn map

$$E_{f_0} = \Phi_{attr} \circ \Phi_{rep}^{-1}$$

Parabolic Renormalization

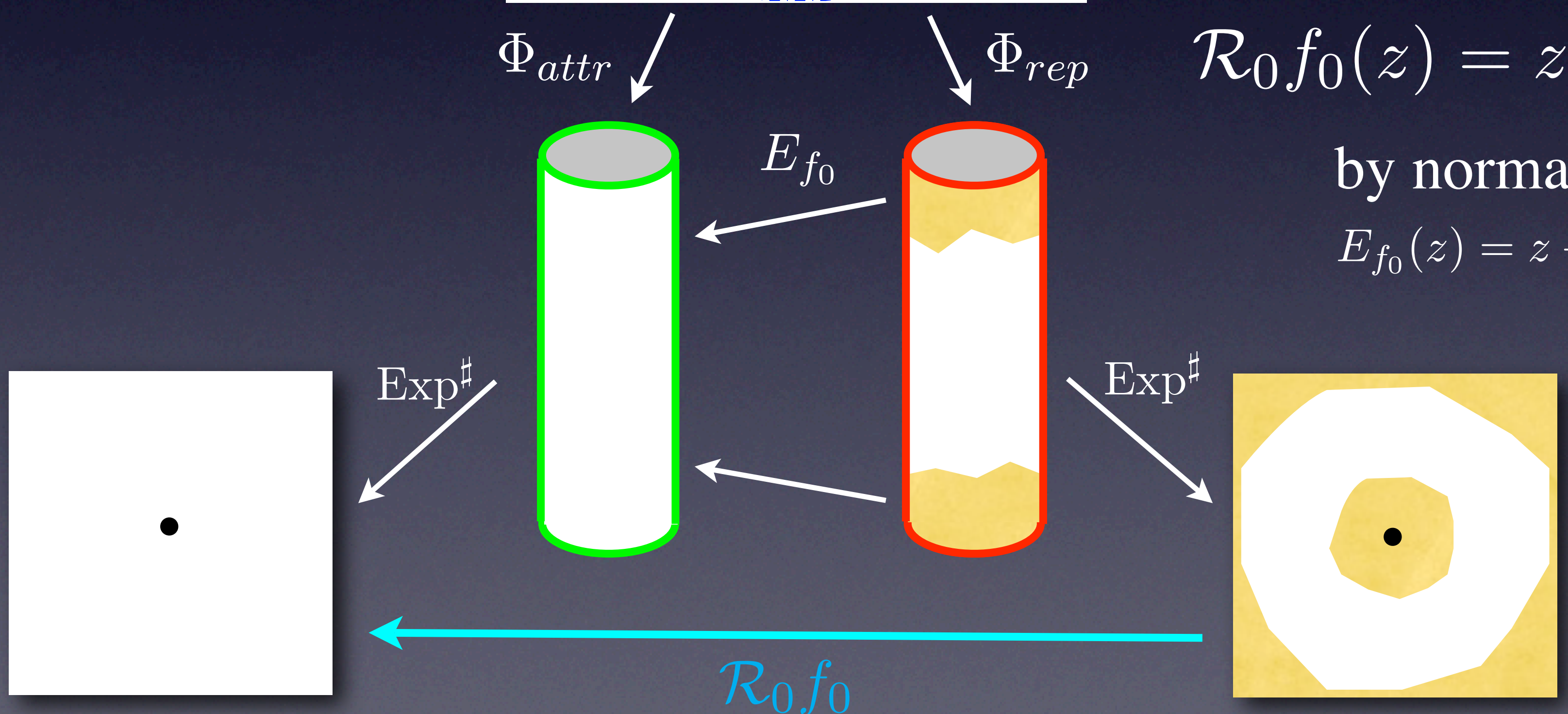
$$\mathcal{R}_0 f_0 = \mathrm{Exp}^\sharp \circ E_{f_0} \circ (\mathrm{Exp}^\sharp)^{-1}$$

$$\mathrm{Exp}^\sharp(z) = e^{2\pi i z} : \mathbb{C}/\mathbb{Z} \xrightarrow{\simeq} \mathbb{C}^*$$

$$\mathcal{R}_0 f_0(z) = z + \ldots$$

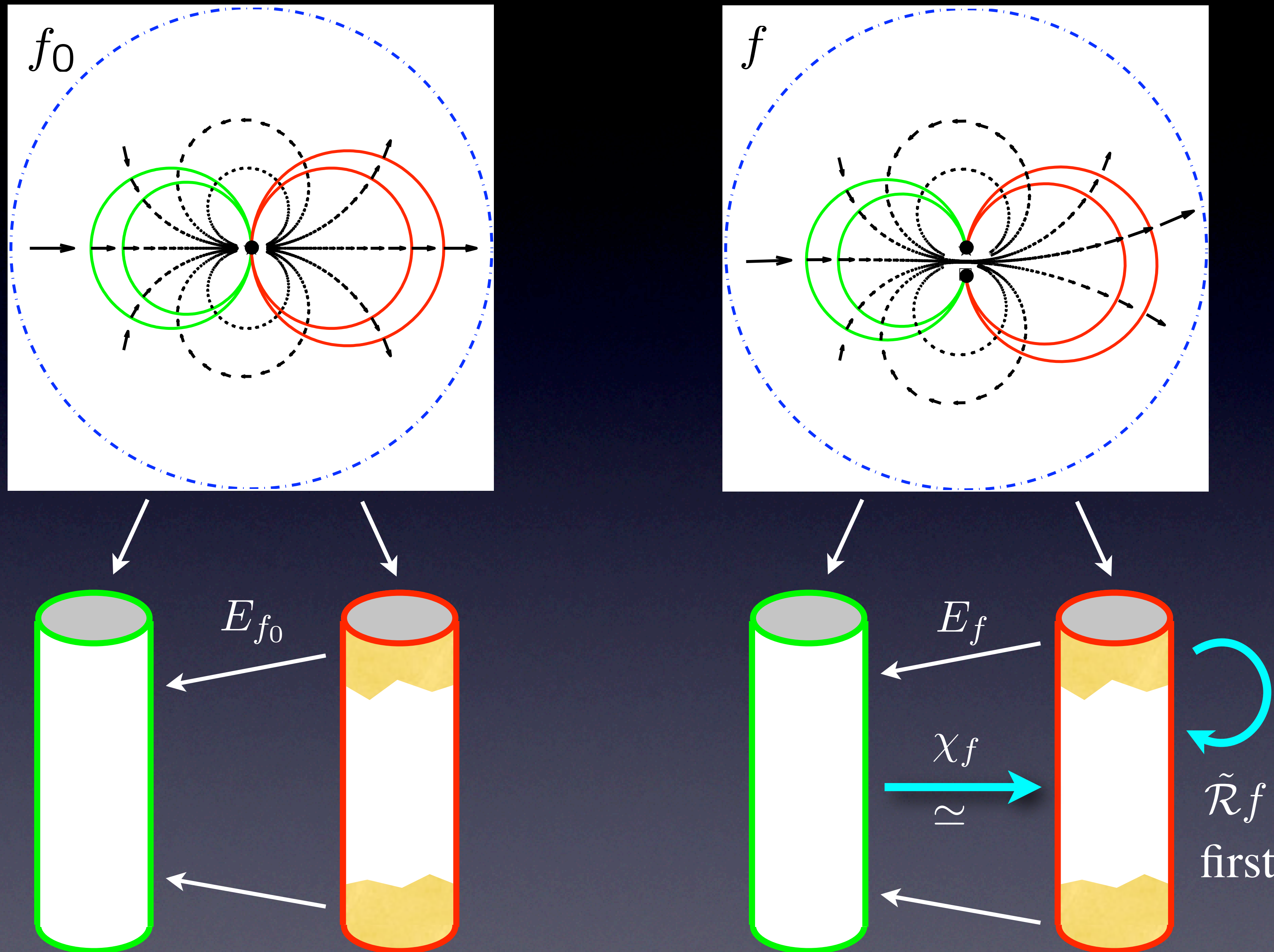
by normalization

$$E_{f_0}(z) = z + o(1) \quad (\mathrm{Im}\, z \to +\infty)$$



# Perturbation

$f'(0) = e^{2\pi i \alpha}$, $\alpha$ small $|\arg \alpha| < \frac{\pi}{4}$




$\tilde{\mathcal{R}} f = \chi_f \circ E_f$

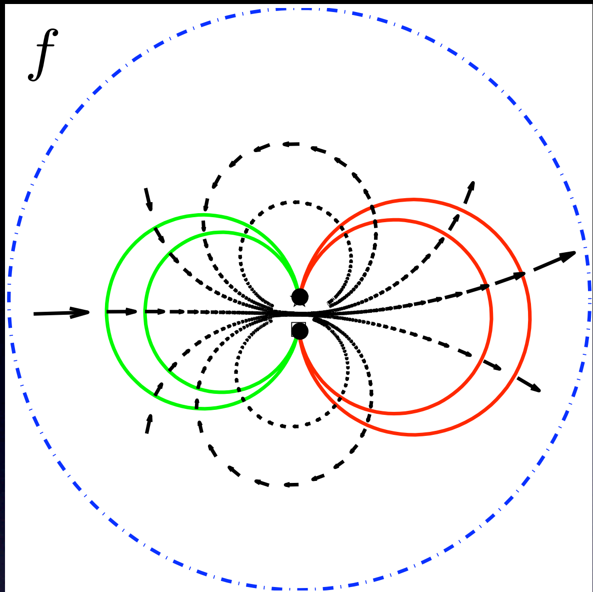
first return map

$E_f$ depends continuously on $f$

(after a suitable normalization)

$$\chi_f(z) = z - \frac{1}{\alpha}$$

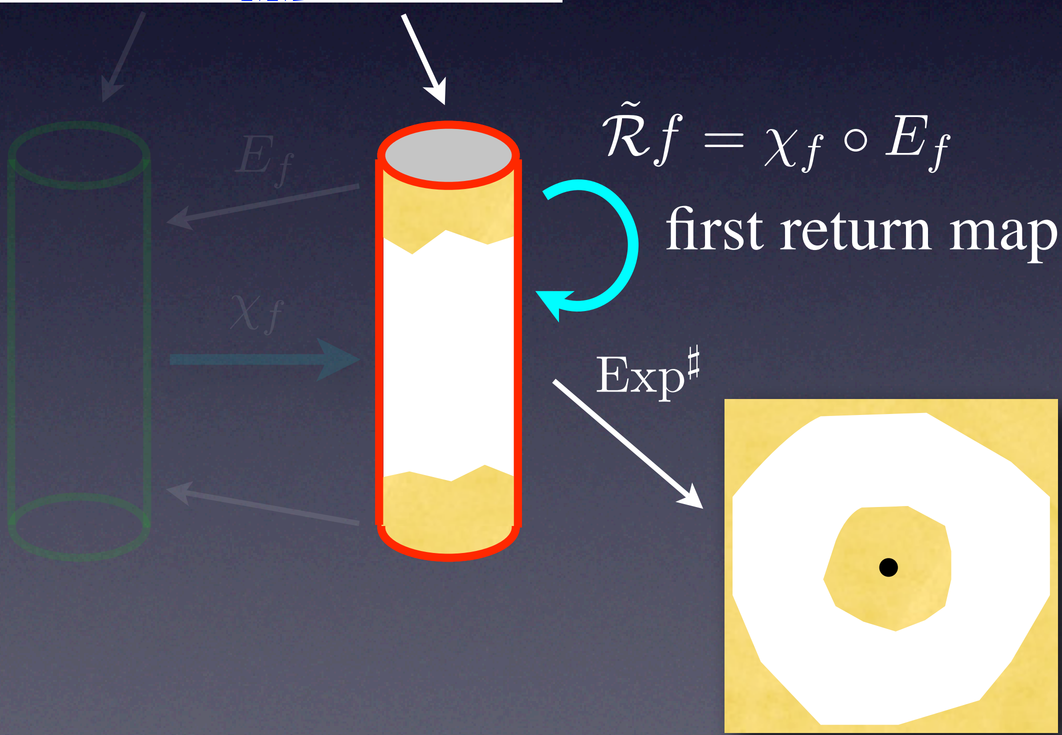
# Near-parabolic Renormalization (cylinder renorm.)




$$\mathcal{R}f = \mathrm{Exp}^\sharp \circ \tilde{\mathcal{R}}f \circ (\mathrm{Exp}^\sharp)^{-1}$$
$$= \mathrm{Exp}^\sharp \circ \chi_f \circ E_f \circ (\mathrm{Exp}^\sharp)^{-1}$$
$$= e^{2\pi i \beta} z + O(z^2)$$

where $\beta = -\frac{1}{\alpha} \pmod{\mathbb{Z}}$

or $\alpha = \frac{1}{m - \beta}$ $(m \in \mathbb{N})$




$$\mathcal{R}f \doteqdot e^{2\pi i \beta} \mathcal{R}_0 f_0$$

write $f = e^{2\pi i \alpha} f_0$

$\alpha \mapsto -\frac{1}{\alpha} \pmod{\mathbb{Z}}$

$f_0 \mapsto \mathcal{R}_0 f_0$

Suppose the near-parabolic renormalization can be iterated:

$$f_0 \xrightarrow{\mathcal{R}} f_1 \xrightarrow{\mathcal{R}} f_2 \xrightarrow{\mathcal{R}} f_3 \xrightarrow{\mathcal{R}} \dots$$

high iterates of $f_0$ corresponds to low iterates of $f_1$

highly recurrent behavior of $f_0$ can be analyzed through $f_i$'s

fine structure of orbits or invariant sets are magnified

$\mathcal{R}$ = a dynamical system in the space of certain type of dynamical systems

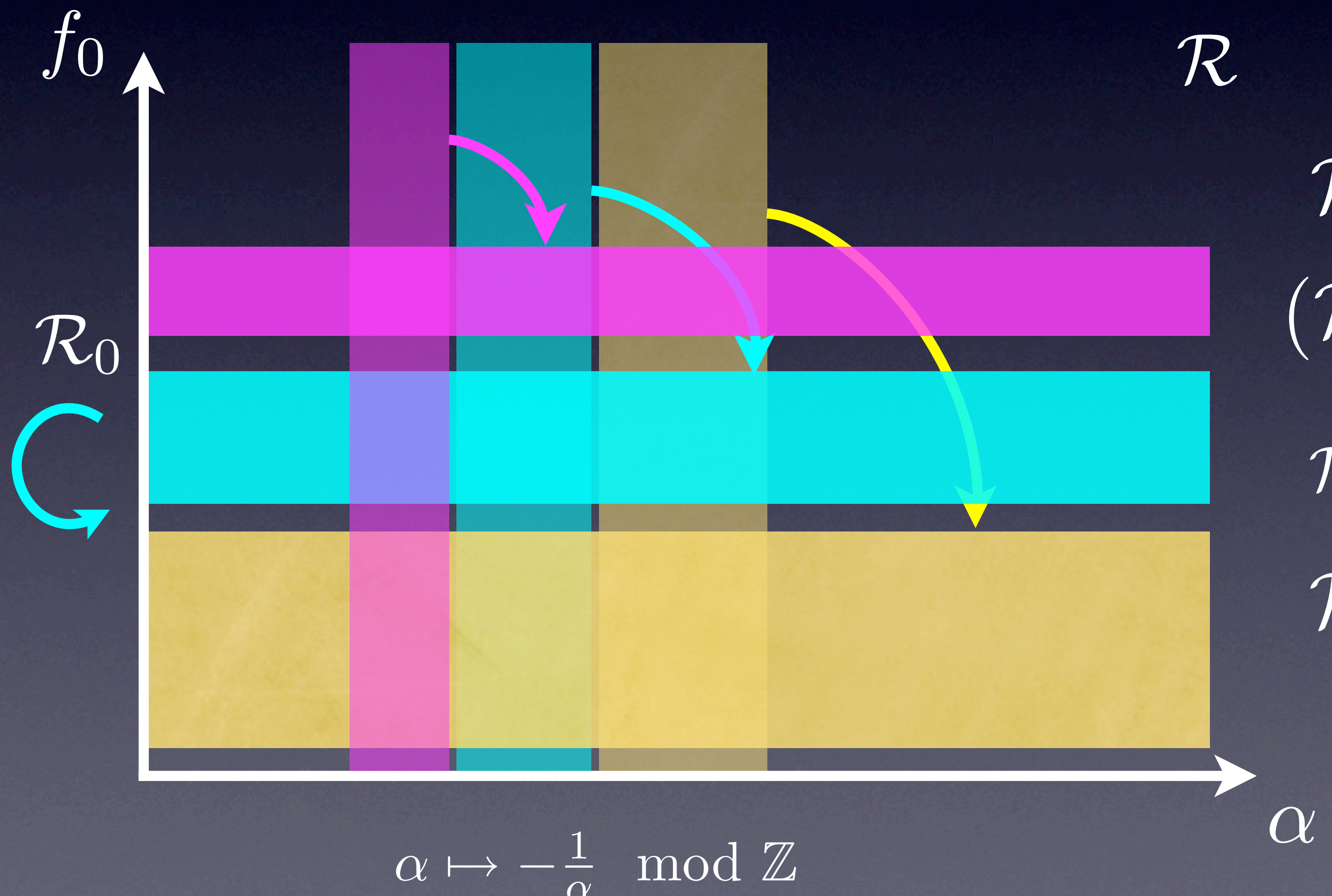
But, ... Can you really iterate infinitely many times?

# Renormalization: The Picture

$f(z) = e^{2\pi i\alpha} z + O(z^2) = e^{2\pi i\alpha} f_0(z)$ where $f_0(z) = z + O(z^2)$ 1-parabolic

$f \ \leftrightarrow (\alpha, f_0)$

Write $\mathcal{R}f(z) = e^{-2\pi i \frac{1}{\alpha}} \mathcal{R}_\alpha f_0(z)$ then $\mathcal{R} : (\alpha, f_0) \mapsto (-\frac{1}{\alpha}, \mathcal{R}_\alpha f_0)$




$\mathcal{R}$ hyperbolic?

($\mathcal{R}_\alpha$ contracting?)

$\mathcal{R}_\alpha f_0 \to \mathcal{R}_0 f_0 \ (\alpha \to 0)$

$\mathcal{R}_0$ contracting?

YES for $\alpha$ small

# Main Theorems

**Theorem 1** *Let $P(z) = z(1+z)^2$. There exist bounded simply connected open sets $V$ and $V'$ with $0 \in V \subset \overline{V} \subset V' \subset \mathbb{C}$ such that the class*

$$\mathcal{F}_1 = \left\{ f = P \circ \varphi^{-1} : \varphi(V) \to \mathbb{C} \;\middle|\; \begin{array}{c} \varphi : V \to \mathbb{C} \text{ is univalent} \\ \varphi(0) = 0,\ \varphi'(0) = 1 \end{array} \right\}$$

*satisfies the following:* univalent = holomorphic and injective

(0) *every $f \in \mathcal{F}_1$ is non-degenerate;*

(i) $\mathcal{F}_0 \setminus \{$*quadratic polynomial*$\}$ *can be naturally embedded into $\mathcal{F}_1$ (in particular, $\mathcal{R}_0^n(z+z^2) \in \mathcal{F}_1$ $n = 1, 2, \dots$);*

(ii) *The renormalization $\mathcal{R}_0$ is well defined on $\mathcal{F}_1$ so that $\mathcal{R}_0(\mathcal{F}_1) \subset \mathcal{F}_1$ ;*

(iii) *If we write $\mathcal{R}_0 f = P \circ \psi^{-1}$, then $\psi$ can be extended univalently to $V'$;*

(iv) *$f \mapsto \mathcal{R}_0 f$ is "holomorphic."*

**Theorem 2** *The above statements hold for $\mathcal{R}_\alpha$ for $\alpha$ small. Hence there exists an $N$ such that the above holds for*

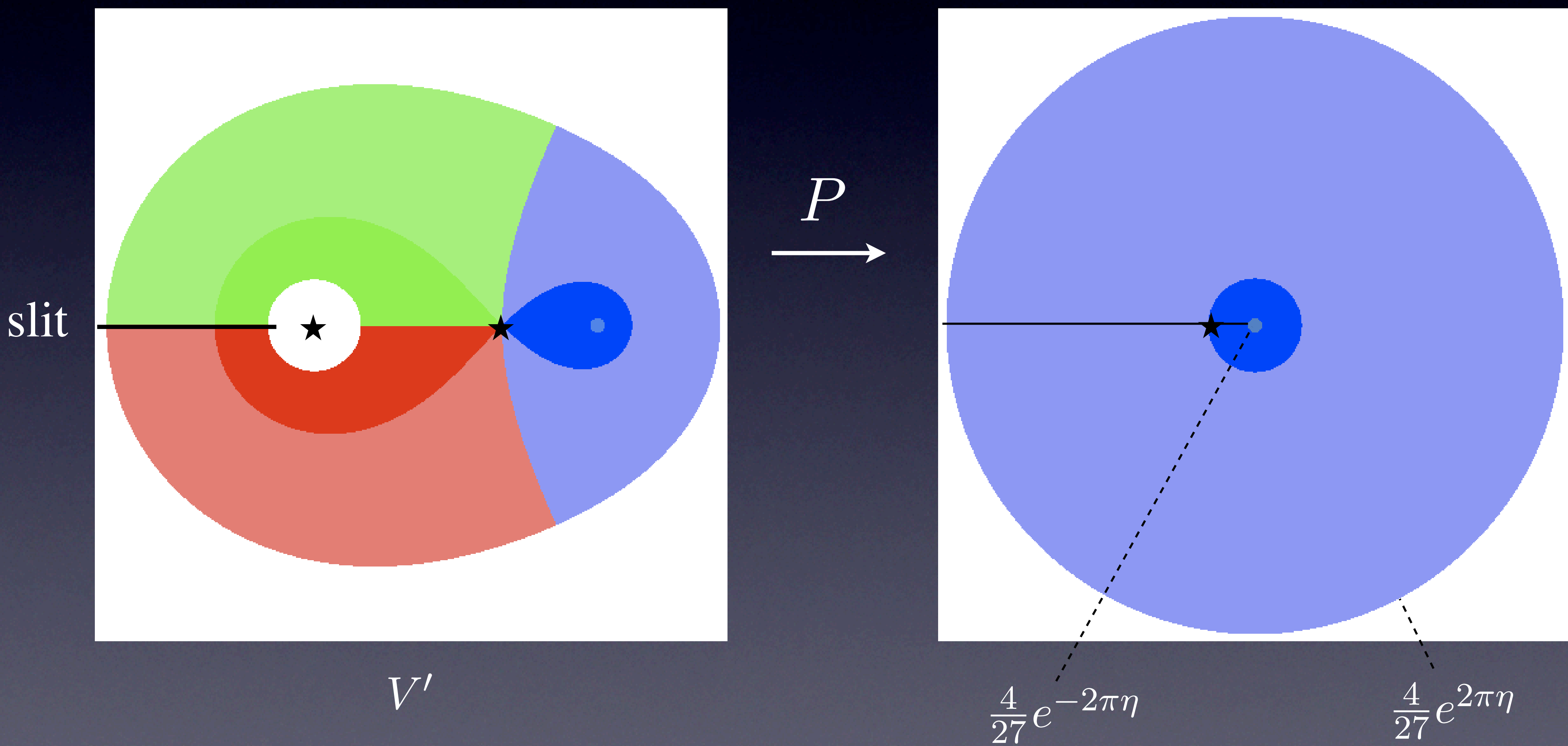
$$\alpha = \frac{1}{m+\beta} \quad \text{with } m \in \mathbb{N},\ \beta \in \mathbb{C} \text{ and } |\beta| \le 1.$$

# $P(z) = z(1+z)^2$ and $V$, $V'$

$P(0) = 0, \quad P'(0) = 1$

critical points: $-\dfrac{1}{3}$ and $-1$ critical values: $P(-\frac{1}{3}) = -\frac{4}{27}$ and $P(-1) = 0$

$\eta = 2$




$V$ slightly smaller domain than $V'$

# Applications

**Theorem 2'.** *There exists a large $N$ such that the following holds: If $f(z) = e^{2\pi i\alpha}h(z)$ with $h \in \mathcal{F}_1$ (or $h(z) = z + z^2$) and*

$$\alpha = \pm\cfrac{1}{a_1 \pm \cfrac{1}{a_2 \pm \cfrac{1}{\ddots}}} \qquad \textit{where } a_i \in \mathbb{N} \textit{ and } a_i \geq N,$$

*then the sequence of near-parabolic renormalizations*

$$f = f_0 \xrightarrow{\mathcal{R}} f_1 \xrightarrow{\mathcal{R}} f_2 \xrightarrow{\mathcal{R}} f_3 \xrightarrow{\mathcal{R}} \dots$$

*is defined so that $f_n = e^{2\pi i\alpha_n}h_n(z)$, $h_n \in \mathcal{F}_1$ (possibly with complex conjugation).*

**Corollary.** *Under the assumption of Theorem 2', the critical orbit stays in the domain of $f$ and can be iterated infinitely many times. Moreover if $f$ is (a part of) a rational map, then the critical orbit is not dense.*

# Julia set with positive area

**Theorem** (Buff-Chéritat). *There exists an irrational number $\alpha$ such that the Julia set of the quadratic polynomial $P_\alpha(z) = e^{2\pi i\alpha}z + z^2$ has positive Lebesgue measure.*

## Adrien Douady's plan

$$P_{\alpha(0)} \xrightarrow{\text{perturb}} P_{\alpha(1)} \xrightarrow{\text{perturb}} P_{\alpha(2)} \xrightarrow{\text{perturb}} P_{\alpha(3)} \xrightarrow{\text{perturb}} \dots \longrightarrow P_\alpha$$

Each $\alpha(n)$ is of bounded type, hence $P_{\alpha(n)}$ has a Siegel disk.

$\alpha(0) = [0, N, N, N, N, \dots]$

$\alpha(n) \rightsquigarrow \alpha(n+1)$: modify only $a_k$ (larger), for some large $k$ .

## Want:

Do not lose too much area in each step $K(P_{\alpha(n)}) \rightsquigarrow K(P_{\alpha(n+1)})$.

$\alpha = \lim_{n\to\infty} \alpha(n)$ non Bruno, $|J(P_\alpha)| = |K(P_\alpha)| \geq \limsup_{n\to\infty} |K(P_{\alpha(n)})| > 0$

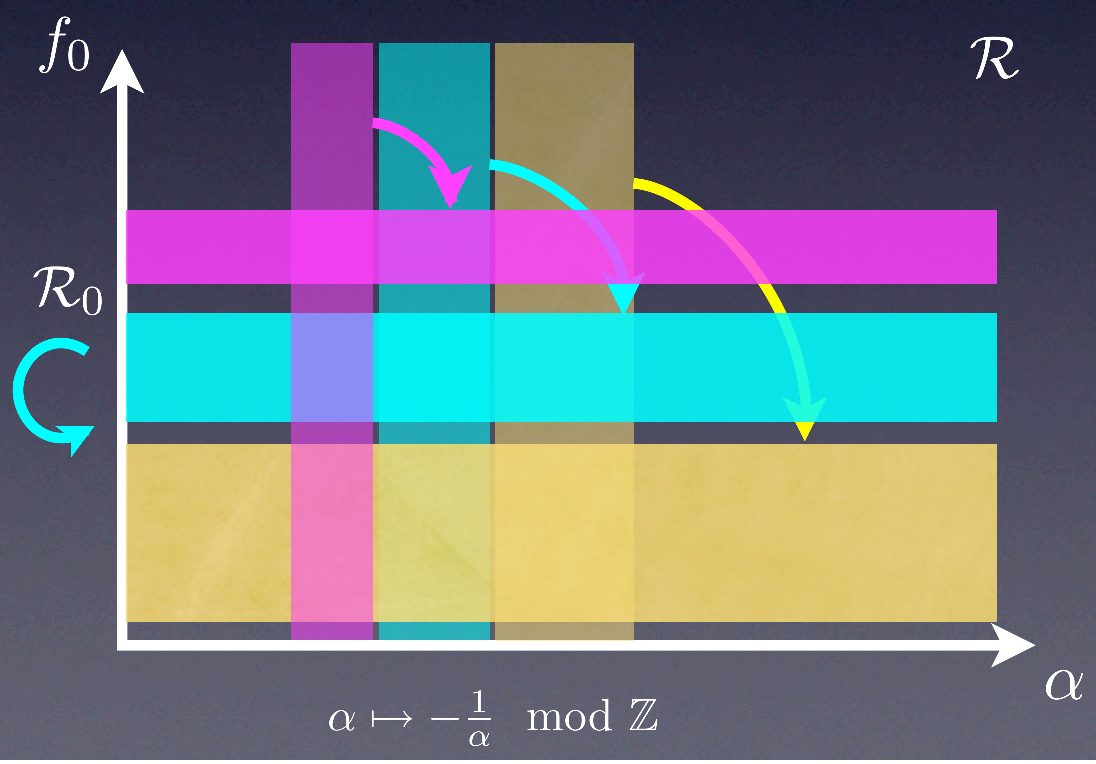
Need to control the boundary of Siegel disks, which is the closure of the critical orbit. "Semi-continuity" by our Theorem 2' which yields a uniform control on the sequence of renormalizations, even when $a_k$ are taken to be very big.

Also uses McMullen's renormalization result on Siegel disks of bounded type rotation number.

# Contraction and Hyperbolicity

**Theorem 3** *Modifying the definition slightly (requiring that $\varphi$ has a quasi-confomal extension to $\mathbb{C}$), $\mathcal{F}_1$ is in one to one correspondence with the Teichmüller space $Teich(W)$ of $W = \mathbb{C} \smallsetminus \overline{V} (\simeq \mathbb{D}^*)$. The induced map $\mathcal{R}_0^{Teich}$ is a uniform contraction with respect to the Teichmüller distance. (The Lipshitz constant $\leq \exp(-2\pi\, mod(V' \smallsetminus \overline{V}))$.)*

**Theorem 4** *The above statements hold for the fiber map $\mathcal{R}_\alpha$ for $\alpha$ small. Hence the total renormalization $\mathcal{R}$ is hyperbolic in this region.*




$$f = e^{2\pi i\alpha} f_0$$

$$\mathcal{R} : (\alpha, f_0) \mapsto (-\frac{1}{\alpha}, \mathcal{R}_\alpha f_0)$$

$$\mathcal{R}_\alpha f_0 \to \mathcal{R}_0 f_0 \ (\alpha \to 0)$$

# Proof of Theorems 3 and 4

$\mathcal{F}_1 \ni f = P \circ \varphi^{-1} \rightsquigarrow [\tilde{\varphi}|_W] \in Teich(W) = \{\psi : W \to \mathbb{C} \text{ qc }\}/\sim$

where $\tilde{\varphi}$ is a quasiconformal extension of $\varphi$ to $\mathbb{C}$.

Teichmüller space is like the unit disk with Poincaré metric.
holomorphic self map does not expand the distance,
and is often a contraction.

Royden-Gardiner Theorem (Teichmüller distance = Kobayashi distance)
cotangent space = {integrable holomorphic quadratic differentials}
modulus-area inequality for holom. quad. differentials
isoperimetric inequality for holom. quad. differentials
modified Carleman's inequality

# Application of contraction

**Theorem.** *Suppose $f$ and $f'$ satisfy the assumption of Theorem 2', with the same rotation number $\alpha$. Then they have small periodic cycles $\zeta_n$ and $\zeta'_n$ around 0 with period $q_n$. Let $\lambda(\zeta_n)$, $\lambda(\zeta'_n)$ be their multipliers. The differences*

$$\left|\lambda(\zeta_n) - \lambda(\zeta'_n)\right| \quad \text{and} \quad \left|\frac{1}{1-\lambda(\zeta_n)} - \frac{1}{1-\lambda(\zeta'_n)}\right|$$

*tends to 0 exponentially fast as $n \to \infty$ with a uniform rate.*

**Corollary.** *Not arbitrary germ with irrationally indifferent fixed point can be realized in such a map (for example, quadratic polynomials, or a perturbation of rational map with parabolic fixed point with only one critical point in its basin).*

*cf.* Yoccoz unrenormalization construction for the inverse of Siegel-Bruno Theorem.

Grazie!

A draft is available at
http://www.math.kyoto-u.ac.jp/~mitsu/pararenorm/