Invariant varieties for a dicritic diffeomorphism in $(\mathbb{C}^2, 0)$

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Given $F \in \mathrm{Diff}(\mathbb{C}^n, 0)$,
Main problem: to find formal, topological, C^r or analytic invariants that determine F.
i.e if F and G have the same invariants then there exists φ formal, continuous, C^r or analytic local diffeomorphism at 0, such that the diagram

$$\begin{array}{ccc} (\mathbb{C}^n, 0) & \xrightarrow{F} & (\mathbb{C}^n, 0) \\ \varphi \downarrow & & \downarrow \varphi \\ (\mathbb{C}^n, 0) & \xrightarrow{G} & (\mathbb{C}^n, 0) \end{array}$$

commutes.

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- (Siegel, Brjuno) $DF(0)$ satisfies a Brjuno-condition $\Rightarrow$ analytic conjugation.

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- (J. Rey) C_*^{k+1}-conjugated with $g(x) = x + x^{k+1}$, i.e., conjugation by a continuous diffeomorphism that is C^{k+1} at every point except the origin.

Let $F(z) = z + P_k(z) + P_{k+1}(z) + \cdots$, $z = (z_1, \ldots, z_n)$ where $P_k \not\equiv 0$. A *characteristic direction* for F is a point $[v] \in \mathbb{P}^{n-1}$ such that $P_k(v) = \lambda v$, for some $\lambda \in \mathbb{C}$; it is *nondegenerate* if $\lambda \neq 0$.

F is called *dicritic* if $P_k(z) = p(z)z$ where $p(z)$ is a homogeneous polynomial of degree $k-1$.
A *parabolic curve* for F is an injective holomorphic map $\varphi : \Omega \to \mathbb{C}^n$, where Ω is a simply connected domain in $\mathbb{C}$ with $0 \in \partial\Omega$ such that

- φ is continuous at the origin, and $\varphi(0) = 0$.
- $F(\varphi(\Omega)) \subset \varphi(\Omega)$ and $F^{\circ k}(p)$ converges to 0 when $k \to +\infty$, for $p \in \varphi(\Omega)$.

The natural generalization of Leau-Fatou theorem is given by Hakim in the generic case

Theorem (Hakim)

Let F be a tangent to the identity germ of diffeomorphism of $(\mathbb{C}^n, 0)$. For any nondegenerate characteristic direction $[v]$ there exist $\mathrm{ord}(F) - 1$ disjoint parabolic curves tangent to $[v]$ at the origin.

When $n = 2$, Abate proved that the nondegeneracy condition can be dismissed.

Theorem (Abate)

Let F be a tangent to the identity germ of diffeomorphism of $\mathbb{C}^2$ such that 0 is an isolated fixed point. Then there exist $\mathrm{ord}(F) - 1$ disjoint parabolic curves for F at the origin.

Let $\pi : (M, D) \to (\mathbb{C}^2, 0)$ be the blow up of $\mathbb{C}^2$ at the origin, where $D = \pi^{-1}(0) = \mathbb{P}^{n-1}$.

In the dicritic case we can say that almost all orbits have a flower dynamic.

Theorem (B.M.)

Let $F(z) = z + p(z)z + \cdots$ be a dicritic diffeomorphism, and $\tilde{F} : (M, D) \to (M, D)$ such that $\pi \circ \tilde{F} = F \circ \pi$ and $\tilde{F}|_D = id|_D$, then there exist open sets U^+, U^-, such that

- *$U^+ \cup U^-$ is a neighborhood of $\{[z] \in D | p(z) \neq 0\}$.*
- *if $P \in U^+$, then $\tilde{F}^{\circ n}(P)$ converges to a point in D when $n \to \infty$.*
- *if $P \in U^-$, then $\tilde{F}^{\circ n}(P)$ converges to a point in D when $n \to -\infty$.*

Generalization of Szekeres normal form in dimension 2.
A formal change $\varphi(x, v)$ of coordinates is called *semiformal* in U if its components are in $\mathcal{O}(U)[[x]]$.

Theorem (B.M.)

Let $F(x, y) = \begin{pmatrix} x+p(x,y)x+\cdots \\ y+p(x,y)y+\cdots \end{pmatrix}$ *be a dicritic diffeomorphism, then there exists a unique rational function* $q(v)$ *such that* $\tilde{F}$ *is semiformally conjugated with*

$$\tilde{F}_{p,q} = \begin{pmatrix} x + x^{k+1}p(1, v) + x^{2k+1}q(v) \\ v \end{pmatrix}$$

in $\{p([v_1, v_2]) \neq 0\}$*. In addition,* $q(v) = \frac{s(v)}{p(1,v)^{2k+1}}$ *where* $s(v)$ *is a polynomial of degree* $2k + 2 + 2k \deg_v(p(1, v))$*.*

Observe that if $F, G \in \mathrm{Diff}(\mathbb{C}^2, 0)$ are formally conjugated then $\tilde{F}$ and $\tilde{G}$ are semiformally conjugated, but the reciprocal is not true.
In fact, let

$$G_m(x,y) = \begin{pmatrix} x + x^{k+1} + y^m \\ y + x^k y + x^m \end{pmatrix},$$

If $m_1, m_2 > 2k+1$ then $\tilde{G}_{m_1}$ and $\tilde{G}_{m_2}$ are semiformally conjugated, but G_{m_1} and G_{m_2} are not formally conjugated.
In addition, they are not topologically conjugated because their Milnor numbers are different.

Fixing the coordinates $(x, v) \in M$, then for any point $a \in D \setminus \{(x : y) | p(x, y) = 0\}$ and any simply connected neighborhood $U \subset D \setminus \{(x : y) | p(x, y) = 0\}$ of a there exists an analytic change of coordinates $\varphi \in \mathrm{Diff}(M, U)$ such that $\varphi \circ \tilde{F} \circ \varphi^{-1} = G$ where

$$\begin{aligned} g_1(x, v) &= x + x^{k+1}p(v) + r(v)x^{2k+1} + O(x^{2k+2}) \\ g_2(x, v) &= v + O(x^{2k+1}). \end{aligned}$$

By a linear change of coordinates we suppose $a = 0$.

Making a ramificated change of coordinates $z = \frac{1}{x^k}$, between $\mathbb{C}^*$ and $\mathbb{C}_k^* = \cup U_j$, and fixing a chart $U_j \sim \mathbb{C}$, then the diffeomorphism G takes the form

$$G(z,v) = \begin{pmatrix} g_1(z,v) \\ g_2(z,v) \end{pmatrix} = \begin{pmatrix} z - p(v) - \frac{s(v)}{z} + O(z^{-1-\frac{1}{k}}) \\ v + O(z^{-2}) \end{pmatrix}$$

thus, G is semiformally conjugated with

$$G_{p,s} = \begin{pmatrix} z - p(v) - \frac{s(v)}{z} \\ v \end{pmatrix}$$

For every $R > 0$ and $r > 0$ let

$$U^{j,+}_{\theta,R,r} = \left\{ (z,v) \,\middle|\, \left|\arg\left(\frac{z}{p(v)} - R\right)\right| < \theta, |v| < r \right\}$$

For every $r \geq 0$ such that $p(v) \neq 0$ for all $|v| \leq 2r$ and for every $0 < \theta < \pi$, there exists $R \gg 0$ such that $G^{\circ j}(z,v) \in U^{j,+}_{\theta,R,2r}$ for all $j \in \mathbb{N}$ and $(z,v) \in U^{j,+}_{\theta,R,r}$
The same way we define

$$U^{j,-}_{\theta,R,r} = \left\{ (z,v) \,\middle|\, \left|\arg\left(\frac{z}{p(v)} + R\right)\right| < \pi - \theta, |v| < r \right\}$$

Using the stability in sectorial domains, it follows that there exists

$$h_{j,+} : U^{j,+}_{\theta,R,2r} \to h_{j,+}(U^{j,+}_{\theta,R,2r})$$

a unique holomorphic diffeomorphism such that

$$h_{j,+} \circ G|_{U^{j,+}_{\theta,R,r}} = G_{p,s} \circ h_{j,+}|_{U^{j,+}_{\theta,R,r}}$$

In addition, the asymptotic development $\hat{h}_{j,+}$ of h at (∞, v_0) conjugates the asymptotic developments $\hat{f}$ and $\hat{g}$ at (∞, v_0), for $|v_0| < r$.

The same way there exists

$$h_{j,-} : U^{j,-}_{\theta,R,2r} \to h_{j,-}(U^{j,-}_{\theta,R,2r})$$

a unique holomorphic diffeomorphism such that

$$h_{j,-} \circ G^{-1}|_{U^{j,-}_{\theta,R,r}} = G^{-1}_{p,s} \circ h_{j,-}|_{U^{j,-}_{\theta,R,r}}$$

therefore

$$h_{j,+} \circ h^{-1}_{j,-} \circ G_{p,s} = G_{p,s} \circ h_{j,+} \circ h^{-1}_{j,-}$$

in $h_{j,-}(U^{j,-}_{\theta,R,r} \cap U^{j,+}_{\theta,R,r})$

By the classical theory of analytic classification of tangent to the identity diffeomorphism in dimension 1, it is know that the $2k$ sectorial maps

$$h_{j,+} \circ h_{j,-}^{-1}, \quad h_{j,+} \circ h_{j+1,-}^{-1}, \qquad j = 1, \ldots k$$

determine the analytic class of the diffeomorphism, locally at a neighborhood of $0 \in D$.

In addition there exist $2k$ C^∞-diffeomorphisms $\psi_{j,+}$, $\psi_{j,-}$, flat at infinity, that commute with $G_{p,s}$ and

$$h_{j,+} \circ h_{j,-}^{-1} = \psi_{j,+} \circ \psi_{j,-}^{-1}$$

so

$$\Psi = \psi_{j,+}^{-1} \circ h_{j,+} = \psi_{j,-}^{-1} \circ h_{j,-}$$

is a C^∞-diffeomorphism that conjugates G and $G_{p,s}$ at a neighborhood of $(0,0) \in D$.

Theorem

There exist $R \gg 0$ *and* $r > 0$ *such that* G *and* $G_{p,s}$ *are* C^{∞} *conjugated in* $U = \{|z| > R\} \times \{|y| < r\}$, *i.e. there exists a* C^{∞}*-diffeomorphism* $\Psi : U \to \varphi(U)$ *such that* $G \circ \Psi = \Psi \circ G_{p,s}$.

Now, since $S := (v = 0)$ is an invariant curve for $G_{p,s}$ then $\Psi^{-1}(S)$ is an invariant curve for G.

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- These invariants depend analytically of v.
- Thus, using analytic extension, we can construct a set of analytic function defined in the universal cover of $D \setminus \{p = 0\}$.

This set determines the analytic class of the diffeomorphism?