Some Extensions and Analysis of Flux and Stress Theory

Reuven Segev

Department of Mechanical Engineering
Ben-Gurion University

Structures of the Mechanics of Complex Bodies
October 2007
Centro di Ricerca Matematica, Ennio De Giorgi
Scuola Normale Superiore

Cauchy's Flux Theorem in Light of Geometric Integration Theory

Objective: *Presentation of the theory of Cauchy fluxes in the framework of geometric integration theory as formulated by H. Whitney and extended recently by J. Harrison.*

Traditional Approach:

In terms of scalar extensive property in space, one assumes:

- *Balance*: $T(\partial\mathscr{B}) + S(\mathscr{B}) = 0$
- *Regularity*: $S(\mathscr{B}) = \int_{\mathscr{B}} \beta_{\mathscr{B}}\,\mathrm{d}V$ and $T(\partial\mathscr{B}) = \int_{\partial\mathscr{B}} \tau_{\mathscr{B}}\,\mathrm{d}A$
- *Locality (pointwise)*: $\beta_{\mathscr{B}}(x) = \beta(x)$ and $\tau_{\mathscr{B}}(x) = \tau(x, \boldsymbol{n})$
- *Continuity*: $\tau(\cdot, \boldsymbol{n})$ is continuous.

Cauchy's Theorem

asserts that $\tau(p, \boldsymbol{n})$ depends linearly on $\boldsymbol{n}$. There is a vector field $\boldsymbol{h}$ such that

$$\tau = \boldsymbol{h} \cdot \boldsymbol{n}.$$

Considering smooth regions such that Gauss-Green Theorem may be applied, the balance may be written in the form of a differential equation as

$$\operatorname{div} \boldsymbol{h} + \beta = 0.$$

Contributions in Continuum Mechanics - I

Noll (1957): $t(\boldsymbol{n})$ implied by local dependence on open sets of the boundary.

Gurtin & Williams (1967): Interaction $I(A, B)$ on a universe of bodies

$$
\begin{aligned}
\textit{bi-additive:} \quad & I(A \curlyvee B, C) = I(A, C) + I(B, C), \\
\textit{bounded:} \quad & |I(A, B)| \leq l \cdot \text{area}(\partial A \cap \partial B) + k \cdot \text{volume}(A), \\
\textit{Pairwise balanced:} \quad & I(A, B) = -I(B, A), \\
\textit{Continuity:} \quad & t(\cdot, \boldsymbol{n}) \text{ is continuous (omitted in later works).}
\end{aligned}
$$

Continued later by Noll (1973,1986), Gurtin, Williams & Ziemer (1986), Noll & Virga (1988), etc.

Contributions in Continuum Mechanics - II

Gurtin & Martins (1975): Relaxing the continuity of $t(p, \boldsymbol{n})$ in p, proved linearity in $\boldsymbol{n}$ almost everywhere.

Šilhavý (1985,1991): Admissible bodies are sets of finite perimeter in E^n, and the assumptions and results are assumed to hold for "almost every subbody", in a way which allows singularities. The resulting flux vector t has an L^p weak divergence.

Degiovanni & Marzocchi & Musesti (1999) generalize Šilhavý by considering fluxes which are only locally integrable. The field $b = -\operatorname{div} \boldsymbol{\tau}$ is meaningful only in the weak sense.

Šilhavý (recent work): Admissible bodies are general open sets, fluxes are divergence measure fields, problem with the normal trace—the generalization of $\boldsymbol{\tau}$.

Geometric measure theory [de Giorgi, Federer, Fleming] *is used for specifying the class of bodies, generalized definitions of* $\boldsymbol{n}$*, generalized Gauss Theorem.*

Previous work:

Segev 1986, 1991 Stress theory for manifolds without a metric using a weak formulation. Stresses may be as irregular as measures. Works for continuum mechanics of any order.

Segev 2000, Segev & Rodnay 1999: Classical Cauchy approach on general manifolds using differential forms

Reference:

G. Rodnay & R. Segev, 2003, Cauchy's Flux Theorem in Light of Geometric Integration Theory, Journal of Elasticity, 71 (Truesdell Memorial Volume), 183–203.

The Proposed Formulation

Uses *Geometric Integration Theory* by Whitney (1947, 1957), Wolf (1948), and later Harrison (1993,1998), rather than *Geometric Measure Theory* (e.g., [de Giorgi, Federer, Fleming]).

- Building blocks: r-dimensional oriented cells in E^n.
- Formal vector space of r-cells: polyhedral r-chains.
- Complete w.r.t a norm: Banach space of r-chains.
- Elements of the dual space: r-cochains.

Relevance to Continuum Mechanics

- The total flux operator on regions is modelled mathematically by a cochain.
- Cauchy's flux theorem is implied by a representation theorem for cochains by forms.

Features of the Proposed Formulation

- It offers a common point of view for the analysis of the following aspects: *class of domains, integration, Stokes' Theorem, and fluxes.*
- *Irregular domains and flux fields.* Smoother fluxes allow less regular domains and vise versa in an optimal way. Examples:
  - Domains as irregular as Dirac measure and its derivatives—differentiable flux fields.
  - L^1 regions—bounded and measurable flux fields
- Codimension not limited to 1. Allows membranes, strings, etc. Not only the boundary is irregular, but so is the domain itself.
- Compatible with the formulation on general manifolds.

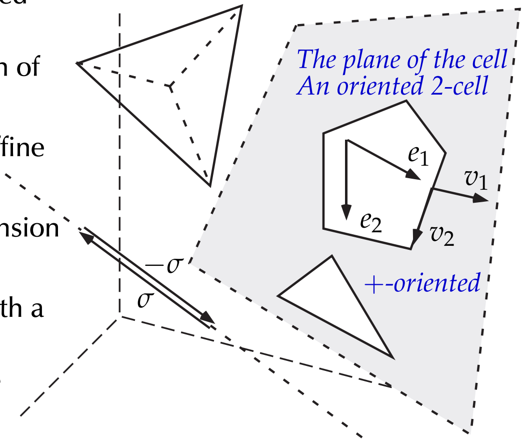
The Structure of the Presentation

- Cells and polyhedral chains
- Algebraic cochains
- Norms and the complete spaces of chains (flat, sharp, natural)
- The representation of cochains by forms:
  - Multivectors and forms
  - Integration
  - Representation
  - Coboundaries and balance equations

Cells and Polyhedral Chains

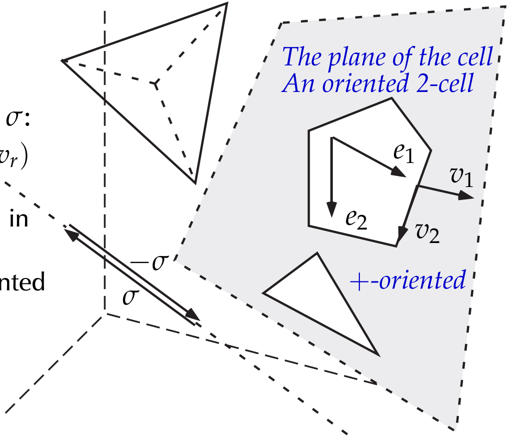
Oriented Cells

- A *cell*, σ, is a non empty bounded subset of E^n expressed as an intersection of a finite collection of half spaces.
- The *plane of σ* is the smallest affine subspace containing σ.
- The *dimension* of σ is the dimension of its plane, an r-cell.
- An *oriented* r-cell is an r-cell with a choice of one of the two orientations of the vector space associated with its plane.



Oriented Cells (continued)

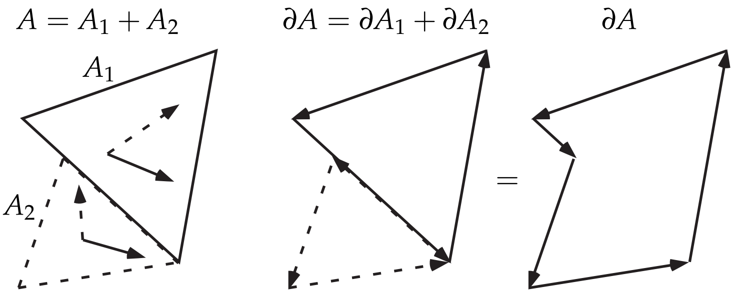
- The orientation of $\sigma' \in \partial\sigma$ is determined by the orientation of σ:
  - Choose independent $(v_2, \ldots, v_r)$ in σ'.
  - Order them such that given v_1 in σ which points out at σ', $(v_1, \ldots, v_r)$ are positively oriented relative to σ.



Polyhedral Chains

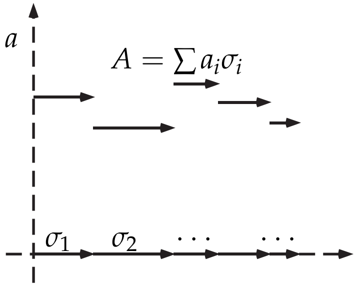
- A *polyhedral r-chain* in E^n is an element of the vector space spanned by formal linear combinations of r-cells, together with:
  - The polyhedral chain 1σ is identified with the cell σ.
  - We associate multiplication of a cell by -1 with the operation of inversion of orientation, i.e., $-1\sigma = -\sigma$.
  - If σ is cut into $\sigma_1, \ldots, \sigma_m$, then σ and $\sigma_1 + \ldots + \sigma_m$ are identified.
- The space of polyhedral r-chains in E^n is now an *infinite-dimensional vector space* denoted by $\mathscr{A}_r(E^n)$.
- The *boundary of a polyhedral r-chain* $A = \sum a_i \sigma_i$ is $\partial A = \sum a_i \partial \sigma_i$. Note that ∂ is a linear operator $\mathscr{A}_r(E^n) \longrightarrow \mathscr{A}_{r-1}(E^n)$.

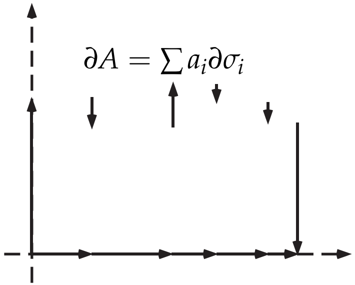
Polyhedral Chains: Illustration



$$\partial \colon \mathscr{A}_r \to \mathscr{A}_{r-1}$$

A Polyhedral Chain as a Function

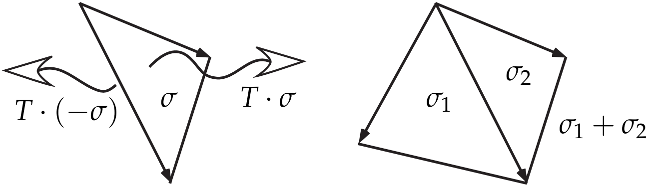




Total Fluxes as Cochains

A *cochain:* Linear $T\colon \mathscr{A}_r \to \mathbb{R}$.
Algebraic implications:

- additivity,
- interaction antisymmetry.



$$T\cdot(-\sigma) = -T\cdot\sigma, \qquad T\cdot(\sigma_1+\sigma_2) = T\cdot\sigma_1 + T\cdot\sigma_2$$

Norms and the Complete Spaces of Chains

The Norm Induced by Boundedness

Boundedness: $|T_{\partial B}| \leqslant N_2\,|\partial B|$, $|T_{\partial B}| \leqslant N_1\,|B|$. Setting $A = \partial B, \ldots$

As a cochain: $|T \cdot A| \leqslant N_2\,|A|$, $|T \cdot \partial D| \leqslant N_1\,|D|$, $A \in \mathscr{A}_r$, $D \in \mathscr{A}_{r+1}$.

Thus, for any $D \in \mathscr{A}_{r+1}$, and $A \in \mathscr{A}_r$:

$$
\begin{aligned}
|T \cdot A| &= |T \cdot A - T \cdot \partial D + T \cdot \partial D| \\
&\leqslant |T \cdot A - T \cdot \partial D| + |T \cdot \partial D| \\
&\leqslant N_2\,|A - \partial D| + N_1\,|D| \\
&\leqslant C_T\,(|A - \partial D| + |D|)\,,
\end{aligned}
$$

Basic Idea

Regard the flux as a *continuous linear functional* on the space of chains w.r.t. a norm

$$|T \cdot A| \leqslant C_T \|A\|,$$

where the *flat norm* (smallest) is given as

$$\|A\| = |A|^{\flat} = \inf_{D}\{|A - \partial D| + |D|\}.$$

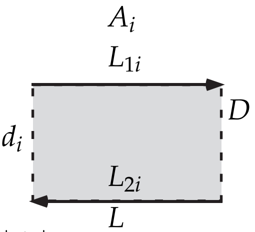
Flat Chains

- The *mass* of a polyhedral r-chain $A = \sum a_i \sigma_i$ is $|A| = \sum |a_i||\sigma_i|$.
- The *flat norm*, $|A|^\flat$, of a polyhedral r-chain:

$$|A|^\flat = \inf\{|A - \partial D| + |D|\},$$

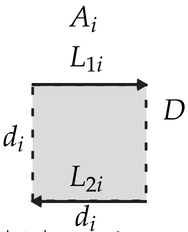
  using all polyhedral $(r+1)$-chains D.
-
  - Taking $D = 0$, $|A|^\flat \leqslant |A|$.
  - If $A = \partial B$, taking $D = B$ gives $|A|^\flat \leqslant |B|$. Hence, $|\partial B|^\flat \leqslant |B|$.
- Completing $\mathscr{A}_r(E^n)$ w.r.t the flat norm gives a Banach space denoted by $\mathscr{A}_r^\flat(E^n)$, whose elements are *flat* r-chains in E^n.
- Flat chains may be used to represent continuous and smooth submanifolds of E^n and even irregular surfaces.
- The *boundary of a flat $(r+1)$-chain* $A = \lim^\flat A_i$, is the a flat r-chain $\partial A = \lim \partial A_i$.

Flat Chains, an Example (Illustration - I):



$|A_i| = 2L,$

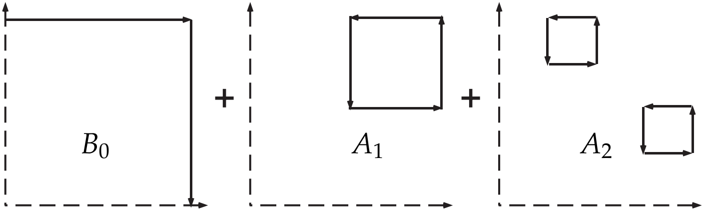
$|A_i|^\flat \leqslant (L+2)d_i \to 0.$



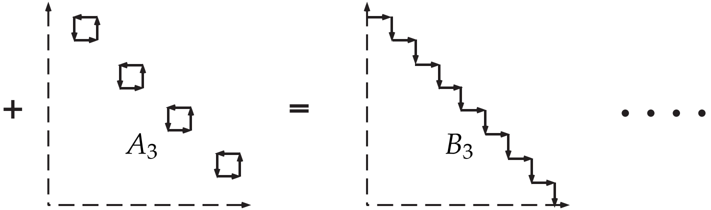
$|A_i| = 2d_i,$

$|A_i|^\flat \leqslant 2d_i \to 0.$

Example: The Staircase



The dashed lines are for reference only.

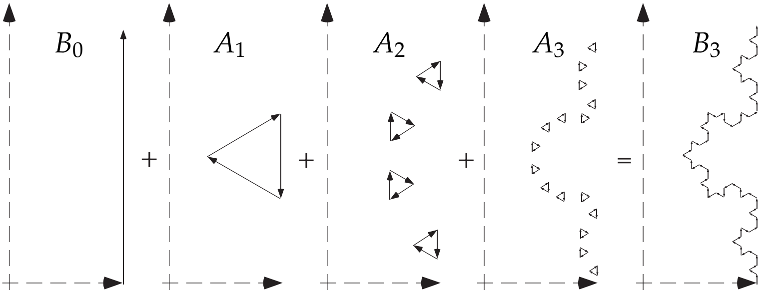


$$|A_i|^\flat \leqslant 2^{i-1}2^{-2i} = 2^{-i}/2 \quad \Longrightarrow \quad (B_i) \text{ a convergent series.}$$

Note, $\left|B_i - B_j\right| = \left|\sum_{k=j+1}^{i} A_k\right| \leq \sum_{k=j+1}^{i} |A_k| \leq \sum_1^\infty |A_k| \leqslant \sum_1^\infty 2^{-i}/2, \quad \forall \quad i, j.$

Example: the Van Koch Snowflake

A_i contains 4^i triangles of side length 3^{-i}. Each time the length increases by $2 \cdot 3^{-i} \cdot 4^i = 2\left(\frac{4}{3}\right)^i$. Hence, $|B_i| \to \infty$.



$$|A_i|^\flat \leqslant 4^i \frac{\sqrt{3}}{2} 3^{-i} 3^{-i} = \frac{\sqrt{3}}{2}\left(\frac{2}{3}\right)^i$$

Flat Chains: Federer's Point of View

- Flat chains are *distributions* defined on the space of smooth differential forms.
- The *flat semi-norm of a smooth differential form* ϕ, supported in some compact set, is given by

$$\|\phi\| = \sup_x \{|\phi(x)|, |d\phi(x)|\}.$$

- The *flat semi-norm of a linear functional* T is the dual norm

$$\|T\| = \sup_\phi \frac{T(\phi)}{\|\phi\|}.$$

Sharp Chains

- Add regularity to the cochains by requiring that

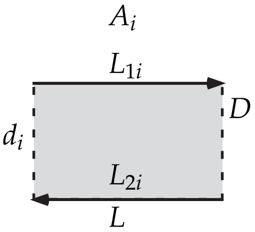
$$|T\cdot(\sigma-\mathrm{trans}_v\,\sigma)|\leqslant C_T\,|\sigma|\,|v|\,,$$

  where trans_v is a *translation operator*, which moves $p\in\sigma$ to $p+v$.
- This will be implied by continuity if we use the *sharp norm* $|A|^\sharp$ of a polyhedral r-chain $A=\sum a_i\sigma_i$:

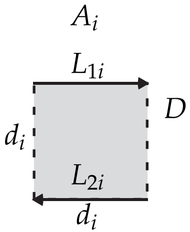
$$|A|^\sharp=\inf\left\{\frac{\sum|a_i||\sigma_i||v_i|}{r+1}+\left|\sum a_i\,\mathrm{trans}_{v_i}\,\sigma_i\right|^\flat\right\},$$

  using all vectors $v_i\in E^n$.
- Completing $\mathscr{A}_r(E^n)$ w.r.t the sharp norm, gives $\mathscr{A}_r^\sharp(E^n)$ whose elements are *sharp* chains.
- Setting all $v_i=0$, we conclude that $|A|^\sharp\leqslant|A|^\flat$. Hence, $\mathscr{A}_r^\flat(E^n)$ is a Banach subspace of $\mathscr{A}_r^\sharp(E^n)$.

Sharp Chains, an Example (Illustration - I):

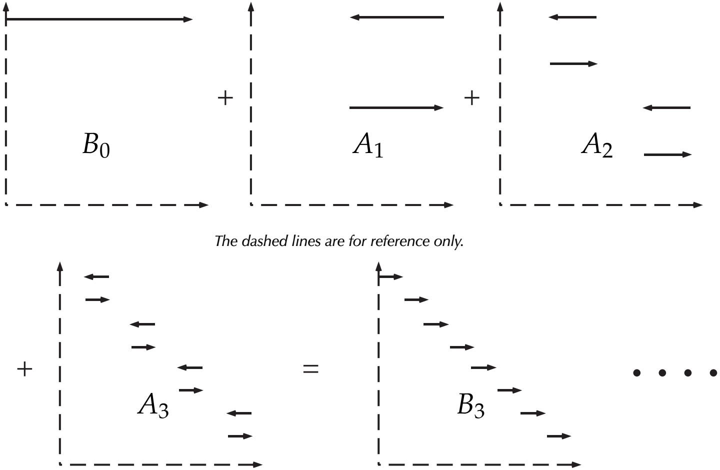


$$|A_i| = 2L,$$
$$|A_i|^\flat \leqslant (L+2)d_i \to 0.$$
$$|A_i|^\sharp \leqslant L d_i \to 0.$$



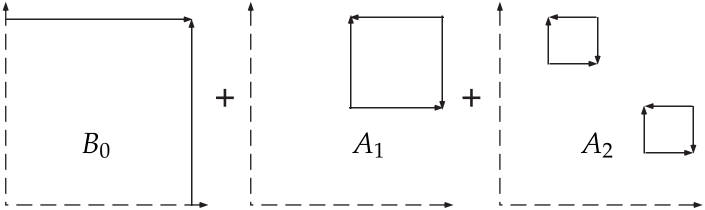
$$|A_i| = 2d_i,$$
$$|A_i|^\flat \leqslant 2d_i \to 0.$$
$$|A_i|^\sharp \leqslant d_i^2/2.$$

The Staircase Strainer

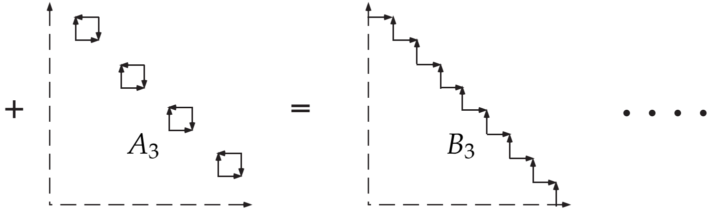


$$|A_i|^\sharp \leqslant 2^{i-1}(1/2^i)^2/2 = 2^{-i}/4$$

The Staircase Mixer:



The dashed lines are for reference only.



$$|A_i|^\sharp \leqslant 2^{-i}/2$$

Harrison's Theory: Dipoles

- A *simple r-dimensional 0-dipole*: r-simplex σ^0 with $\operatorname{diam}(\sigma^0) \leqslant 1$.
- A simple r-dimensional 1-dipole: $\sigma^1 = \sigma^0 - \operatorname{trans}_{v_1} \sigma^0$, such that $|v_1| \leqslant 1$ and $\operatorname{trans}_{v_1} \sigma^0$ disjoint from σ^0.
- A simple r-dimensional j-dipole: an r-chain

$$\sigma^j = \sigma^{j-1} - \operatorname{trans}_{v_j} \sigma^{j-1},$$

  such that $|v_j| \leqslant 1$ and $\operatorname{trans}_{v_j} \sigma^{j-1}$ disjoint from σ^{j-1}.
- A simple j-dipole is determined by σ^0 and $v_1, \ldots, v_j$.
- A *j-dipole* is a simplicial chain

$$D^j = \sum_i a_i \sigma_i^j$$

  of simple j-dipoles.

The Natural Norm I

- The *j-dipole mass* of a simple j-dipole is defined by

$$\left|\sigma^j\right|_j = \left|\sigma^0\right| |v_1| \cdots |v_j| .$$

- The *j-dipole mass* of the j-dipole $D^j = \sum_i a_i \sigma_i^j$ is defined as

$$\left|D^j\right|_j = \sum_i |a_i| \left|\sigma_i^j\right|_j .$$

- The *k-natural norm* on the space of polyhedral chains:

$$|A|_k^\natural = \inf\left\{\sum_{s=0}^{k} |D^s|_s + |C|_{k-1}\right\},$$

over decompositions $A = \sum_{s=0}^{k} D^s + \partial C$, for dipoles D^s.

- Completing $\mathscr{A}_r(E^n)$ w.r.t the k-natural norm, gives $\mathscr{A}_r^k$ whose elements are *k-natural r-chains*.

The Natural Norm II

- The 0-natural norm equivalent to the flat norm.
- As k-increases, the the spaces of natural chains increase.
- The *Riemann integral* over a natural r-chain $A = \lim A_i$, is defined by

$$\int_A \tau = \lim \int_{A_i} \tau.$$

  For the r-form τ with $k-1$ bounded derivatives and k-th derivative Lipschitz, the limit exists.
- The *boundary operator is a continuous linear operator* $\partial \colon \mathscr{A}_r^k \to \mathscr{A}_{r-1}^{k-1}$.

The Representation of Cochains by Forms

Basic Problem:

A representation theorem for cochains in terms of fields.

Why not a vector?

- Say the flux is represented by
  - $\boldsymbol{\tau}_0$ with respect to the *reference* coordinate system (Piola),
  - $\boldsymbol{\tau}$ relative to the *space* coordinate system (Cauchy).
- The relation between the two is given by

$$\boldsymbol{\tau}_0 = |F|\, F^{-1}(\boldsymbol{\tau}),$$

  F is the deformation gradient.
- We would expect a transformation of form

$$\boldsymbol{\tau}_0 = F^{-1}\boldsymbol{\tau},$$

  if the flux were a vector field.

Multivectors

- A *simple r-vector* in V is an expression of the form $v_1 \wedge \cdots \wedge v_r$, where $v_i \in V$.
- An r-*vector* in V is an element of the vector space V_r of formal linear combinations of simple r-vectors, together with:

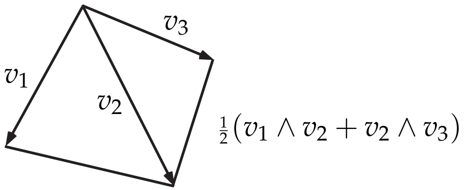
$$
\begin{aligned}
&(1)\ \ v_1 \wedge \cdots \wedge (v_i + v_i') \wedge \cdots \wedge v_r \\
&\qquad\qquad = v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_r + v_1 \wedge \cdots \wedge v_i' \wedge \cdots \wedge v_r; \\
&(2)\ \ v_1 \wedge \cdots \wedge (a v_i) \wedge \cdots \wedge v_r = a(v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_r); \\
&(3)\ \ v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_r \\
&\qquad\qquad = -v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_r.
\end{aligned}
$$

- The dimension of V_r is $\dim V_r = \frac{n!}{(n-r)!r!}$.
- Given a basis $\{e_i\}$ of V, the r-vectors $\{e_{\lambda_1 \dots \lambda_r} = e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_r}\}$, such that $1 \leq \lambda_1 < \cdots < \lambda_r \leq n$, *form a basis* of V_r.

Multivectors and Polyhedral Chains

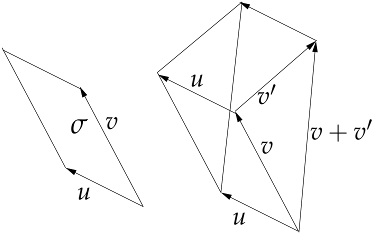
- Given an oriented r-*simplex* σ in E^n, with vertices $\{p_0 \ldots p_r\}$, the r-*vector of* σ, $\{\sigma\}$, is $\{\sigma\} = v_1 \wedge \cdots \wedge v_r / r!$, where the v_i are defined by $v_i = p_i - p_0$ and are ordered such that they belong to σ's orientation.
  $\{\sigma\}$ represents the *plane*, *orientation* and *area* of σ—the relevant aspects.
- The *r-vector of a polyhedral r-chain* $\sum a_i \sigma_i$, is

$$\{\sum a_i \sigma_i\} = \sum a_i \{\sigma_i\}.$$



Why an r-covector?

For the 3-dimensional example, we want to measure the flux through any cell σ, $\{\sigma\} = v \wedge u$.



- Denote by $\bar{\tau}(\sigma)$ the flux through that infinitesimal element.
- As τ depends only the plane, orientation and area, we expect

$$\bar{\tau}(\sigma) = \tilde{\tau}(\{\sigma\}).$$

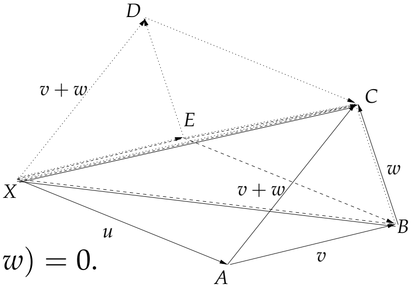
- Balance: $\tilde{\tau}$ is linear

$$\bar{\tau}(\sigma) = \tau \cdot \{\sigma\}.$$

Rough Proof

Consider the infinitesimal tetrahedron X, A, B, C generated by the three vectors u, v, w.

— Use right-handed orientation.



— Balance implies:

$$J(v,u)+J(v,w)+J(u,v+w)-J(u+v,w)=0.$$

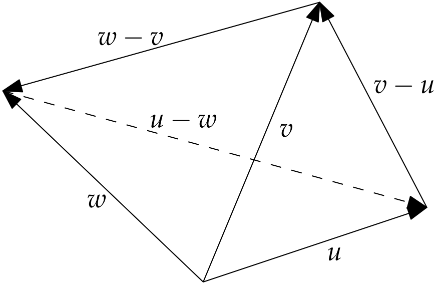
— Same for X, B, C, E and X, C, D, E

$$J(w,u)+J(u+v,w)+J(v,u)-J(v,w+u)=0$$
$$J(w,u)-J(v+w,u)-J(v,w)+J(v,w+u)=0.$$

— Add up to obtain: $J(u,v+w)=J(u,v)+J(u,w)$.

Or Using Multi-Vectors

- Consider the infinitesimal tetrahedron D generated by the three vectors u, v, w and let $A = \partial D$.
- $|A|^{\flat} \leqslant |A - \partial D| + |D| \to 0$, as the volume of the tetrahedron decreases.
- Thus, $\lim J(\{A\}) = 0$.



— Use right-handed orientation.

Thus: $J(u \wedge v) + J(v \wedge w) + J(w \wedge u) + J((w - v) \wedge (v - u)) = 0.$

Using: $(w - v) \wedge (v - u) = w \wedge v - w \wedge u + v \wedge u = -u \wedge v - v \wedge w - w \wedge u,$

we conclude: $\boxed{J(u \wedge v + v \wedge w + w \wedge u) = J(u \wedge v) + J(v \wedge w) + J(w \wedge u).}$

Multi-Covectors

- An r-*covector* is an element of V^r—the dual space of V_r.
- r-covectors can be expressed using *covectors*:

$$V^r = (V^*)_r = L_A^r(V, \mathbb{R}).$$

$(V^*)_r$ is the space of *multi-covectors*, i.e., constructed as V_r using elements of the dual space V^*:

$$\tau = f_{\lambda_1 \cdots \lambda_r} e^{\lambda_1} \wedge \cdots \wedge e^{\lambda_r}, \quad \lambda_i \leqslant \lambda_{i+1}.$$

- r-covectors may be identified with *alternating multilinear* mappings:

$$V^r = L_A^r(V, \mathbb{R}), \quad \text{by} \quad \tau(v_1 \wedge v_2 \wedge \cdots \wedge v_r) = \bar{\tau}(v_1, \ldots, v_r).$$

Riemann Integration of Forms Over Polyhedral Chains

- An r-*form* in $Q \subset E^n$ is an r-covector valued mapping in Q.
- An r-form is continuous if its components are continuous functions.
- The *Riemann integral* of a continuous r-form τ over an r-*simplex* σ is defined as

$$\int_\sigma \tau = \lim_{k\to\infty} \sum_{\sigma_i \in \mathcal{S}_k\sigma} \tau(p_i) \cdot \{\sigma_i\},$$

  where $\mathcal{S}_i\sigma$ is a sequence of *simplicial subdivisions* of σ with mesh $\to 0$, and each p_i is a point in σ_i.
- The Riemann integral of a continuous r-form over a *polyhedral r-chain* $A = \sum a_i\sigma_i$, is defined by $\int_A \tau = \sum a_i \int_{\sigma_i} \tau$.

Lebesgue Integral of Forms over Polyhedral Chains

- An r-form in E^n is *bounded and measurable* if all its components are bounded and measurable.
- The *Lebesgue integral* of an r-form τ over an r-cell σ is defined by

$$\int_\sigma \tau = \int_\sigma \tau(p) \cdot \frac{\{\sigma\}}{|\sigma|}\, dp,$$

  where the integral on the right is a Lebesgue integral of a real function.
- This is extended by linearity to domains that are polyhedral chains by

$$\int_A \tau = \sum a_i \int_{\sigma_i} \tau,$$

  if $A = \sum_i a_i \sigma_i$.

The Cauchy Mapping

- The *Cauchy mapping*, D_T, for the *cochain* T, gives $D_T(p,\alpha)$, at the *point* p *in the direction* α *defined by the cell* σ, defined as:

$$D_T(p,\alpha) = \lim_{i\to\infty} T \cdot \frac{\sigma_i}{|\sigma_i|}, \quad \alpha = \frac{\sigma_i}{|\sigma_i|}$$

  where all σ_i contain p, have r-direction α and $\lim_{i\to\infty} \operatorname{diam}(\sigma_i) = 0$.
- The Cauchy mapping for a given cochain T, of r-directions is analogous to the dependence of the flux density on the unit normal.

The Representation Theorem

Whitney:

- *The analog to Cauchy's flux theorem.* For each r-cochain T the Cauchy mapping D_T may be extended to an r-form that represents T by

$$T \cdot A = \int_A D_T,$$

  for every chain A, i.e., D_T is linear in α. (We use the same notation for the form and the Cauchy mapping.)
- There is an isomorphism between sharp r-cochains T and bounded Lipschitz r-forms D_T, called sharp r-forms.
- For flat r-forms D_T is not unique. There is an isomorphism between flat r-cochains and equivalence classes of bounded and measurable r-forms under equality almost everywhere, that are called flat r-forms.
- There is an isomorphism between k-natural cochains T and r-forms with the first k derivatives bounded and Lipschitz k-th derivative.

Coboundaries and Balance Equations

- The *coboundary* dT of an r-cochain T is the $(r+1)$-cochain defined by

$$dT \cdot A = T \cdot \partial A,$$

  i.e., it is *the dual of the boundary operator* for chains.
- The coboundaries of flat and sharp cochains are flat.
- Hence, there is a flat cochain S satisfying the global balance equation:

$$S \cdot A + T \cdot \partial A = 0, \quad \forall A, \quad \Longrightarrow \quad dT + S = 0.$$

Stokes' theorem for Polyhedral Chains

- The *exterior derivative* of a *differentiable* r-form τ is an $(r+1)$-form $d\tau$ defined by

$$d\tau(p)\cdot(v_1\wedge\cdots\wedge v_{r+1}) = \sum_{i=1}^{r+1}(-1)^{i-1}\nabla_{v_i}\tau(p)\cdot(v_1\wedge\cdots\wedge\widehat{v}_i\wedge\cdots\wedge v_{r+1}$$

  where $\widehat{v}_i$ denotes a vector that has been omitted, and ∇_{v_i} is a directional derivative operator.

- Stokes' theorem for polyhedral chains, based on the fundamental theorem of differential calculus, states that

$$\int_A d\tau = \int_{\partial A} \tau$$

  for every differentiable r-form τ and an $(r+1)$-polyhedral chain A.

The Local Balance Equation

- For *natural cochains* D_T is differentiable and D_{dT} is the exterior derivative of D_T, i.e.,

  $$D_{dT} = dD_T.$$

  Thus, using τ for D_T and b for the form $-D_{dT}$ we get the differential balance equation:

  $$d\tau + b = 0.$$

- If $\tau = D_T$ is an arbitrary *flat form*, we may consider any $d_0\tau$ in the equivalence class of D_{dT}. Hence, we obtain the local $d_0\tau + b = 0$. Thus, one may write the differential balance in the general situation of *flat cochains*.
- If T is a *sharp cochain*, the coordinates of $\tau = D_T$ are Lipschitz, hence, $d\tau$ is defined almost everywhere. Furthermore, it turns out that $d_0\tau = d\tau$ almost everywhere.