# Some Extensions and Analysis of Flux and Stress Theory

Reuven Segev

Department of Mechanical Engineering
Ben-Gurion University

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# Forces and Cauchy Stresses on Manifolds

# Cauchy Stress Theory on Manifolds

Reminder:

- The classical Cauchy theory for the existence of stress uses the *metric structure* of the Euclidean space.
- How would you generalize the notion of stress and Cauchy's postulate so the theory can be formulated on a general manifold?

# Added Benefit

- Such a stress object will *unify* the classical Cauchy stress and Piola-Kirchhoff stress.
- If you consider a material body as a manifold, all configurations of the body, in particular, the current configuration and any reference configuration, are *equivalent charts* in terms of the manifold structure of the body.
- The transformation from the Cauchy stress to the Piola-Kirchhoff stress will be just a transformation rule for two different representations of the *same stress object*.

## In Classical Continuum Mechanics

The force on a body $\mathscr{B}$ in the material manifold $\mathbb{R}^3$ is given by

$$F_{\mathscr{B}} = \int_{\mathscr{B}} b_{\mathscr{B}} \, \mathrm{d}V + \int_{\partial\mathscr{B}} t_{\mathscr{B}} \, \mathrm{d}A.$$

$b_{\mathscr{B}}$ is the body force on $\mathscr{B}$;

$t_{\mathscr{B}}$ is the surface force on $\mathscr{B}$.

The *force system* $\{(b_{\mathscr{B}}, t_{\mathscr{B}})\}$ is considered as a set function.

# Cauchy's Postulates for the dependence on $\mathscr{B}$.

- The *body force* $b_{\mathscr{B}}$ does not depend on the body, i.e., $b_{\mathscr{B}}(x) = b(x)$.
- The *surface force* at a point on the boundary of a control volume depends on the normal to the boundary at that point, i.e., $t_{\mathscr{B}}(x) = \Sigma_x(\boldsymbol{n}(x))$.
- $\Sigma_x$ is assumed to be *continuous*.
- There is a vector field $s$ on the material manifold, the *ambient force* or *self force* (usually taken as zero), such that

$$I_{\mathscr{B}} = \int_{\mathscr{B}} b_{\mathscr{B}} \mathrm{d}v + \int_{\partial\mathscr{B}} t_{\mathscr{B}} \mathrm{d}a = \int_{\mathscr{B}} s \mathrm{d}v.$$

*Cauchy's Theorem: $\Sigma_x$ is linear.*

*Obstacles to the Generalization to Manifolds*:

- You cannot integrate vector fields on manifolds.
- You do not have a unit normal if you do not have a Riemannian metric. ✔

*Basic modifications:*

- Use *integration of forms* on manifolds to integrate scalar fields.✔
- Write the force in terms of *power* expanded for various velocity fields so you integrate a scalar field.
- Use dependence on the *tangent space* instead of direction of the normal.✔
- Use *restriction* of forms for Cauchy's formula.✔

# Preliminaries for Continuum Mechanics on Manifolds

$\mathscr{U}$ is the *material manifold*, $\dim \mathscr{U} = m$;

$\mathscr{B}$ a *body* is an $m$-dimensional submanifold on $\mathscr{U}$.

$\mathscr{M}$ is the *physical space* manifold, $\dim \mathscr{M} = \mu$.

*A configuration* of a body $\mathscr{B}$ is an embedding

$$\kappa\colon \mathscr{B} \to \mathscr{M}.$$

*A velocity* is a mapping

$w\colon \mathscr{B} \to T\mathscr{M}$ such that, $\tau_{\mathscr{M}} \circ w = \kappa$ is a configuration.

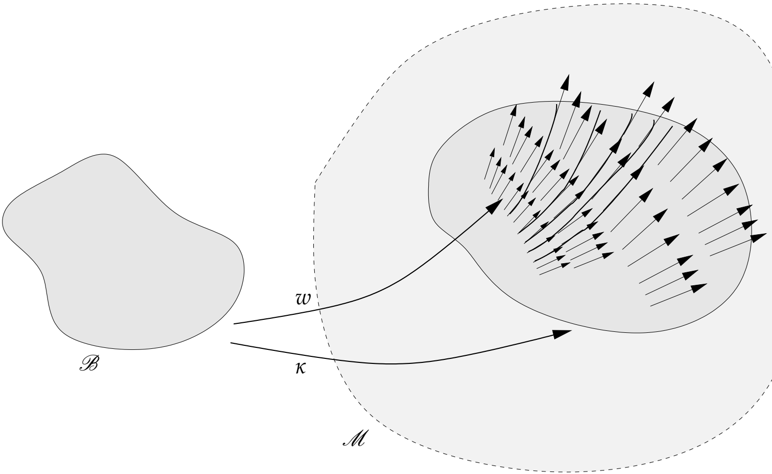
Alternatively, if

$$\kappa^*(\tau_{\mathscr{M}})\colon W = \kappa^*(T\mathscr{M}) \to \mathscr{U}$$

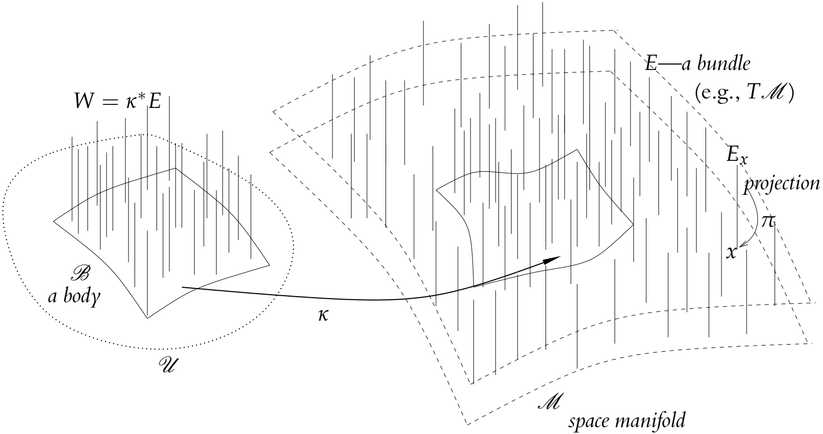
is the *pullback*, a velocity at $\kappa$ may be regarded as a section

$$w\colon \mathscr{U} \to W.$$

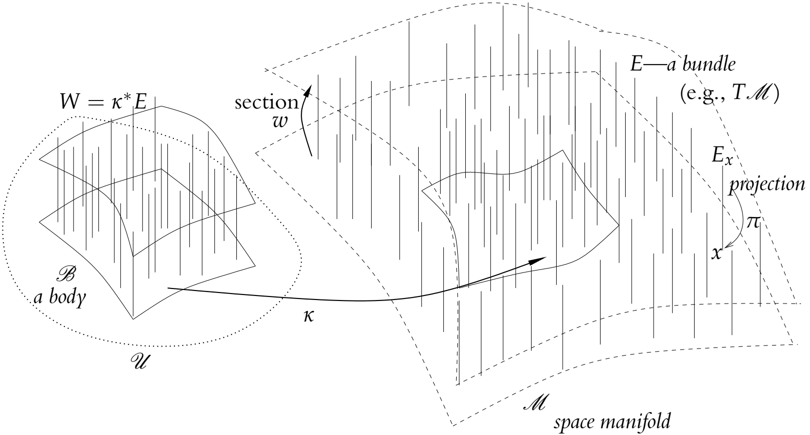
# Velocity Fields



# Bundles and Pullbacks



# Sections of Bundles



# Force Densities

$$F_{\mathscr{B}}(w) = \int_{\mathscr{B}} b_{\mathscr{B}}(w) + \int_{\partial\mathscr{B}} t_{\mathscr{B}}(w),$$

for linear

$$b_{\mathscr{B}}(x)\colon W_x \to \bigwedge^{m} T_x\mathscr{U}, \quad \text{and} \quad t_{\mathscr{B}}(y)\colon W_y \to \bigwedge^{m-1} T_y\partial\mathscr{B}.$$

- $b_{\mathscr{B}}$ is a section of

$$L(W, \bigwedge^{m}(T\mathscr{B})) = \bigwedge^{m}(T\mathscr{B}, W^*),$$

- $t_{\mathscr{B}}$ is a section of

$$L(W, \bigwedge^{m-1}(T\partial\mathscr{B})) = \bigwedge^{m-1}(T\partial\mathscr{B}, W^*)$$

— $W^*$-valued forms.

# Vector Valued Forms

- $\gamma_x \in L\big(W_x, \bigwedge^k (T_x P)\big)$, $P \subset \mathscr{U}$ a submanifold, $k \leq \dim(P)$.
- $\tilde{\gamma}_x \colon (T_x P)^n \to W_x^*$, alternating, multi-linear.

$$\tilde{\gamma}_x \in \bigwedge^k (T_x \mathscr{U}, W_x^*), \quad \text{a } \textit{(co-)vector valued form}.$$

- The requirement

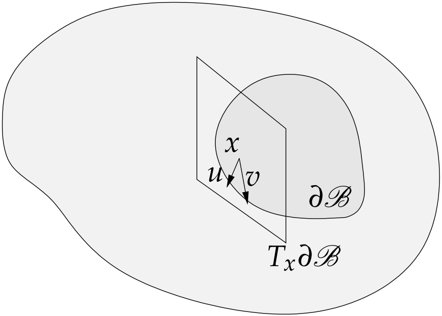
$$\tilde{\gamma}_x(v_1, \dots, v_k)(u) = \gamma_x(u)(v_1, \dots, v_k),$$

for any collection of $k$ vectors $v_1, \dots, v_k$, and $u \in W_x$, generates an isomorphism

$$L\big(W_x, \bigwedge^k (T_x P)\big) = \bigwedge^k (T_x \mathscr{U}, W_x^*).$$

# What Will Cauchy's Theorem and Formula Look Like?

For scalars, the flux form was an $(m-1)$-form $J$ on an $m$-dimensional manifold. By restriction, the Cauchy formula, $\tau_{\mathscr{B}} = \iota^*(J)$, induces an $(m-1)$-form on $T_x\partial\mathscr{B}$.

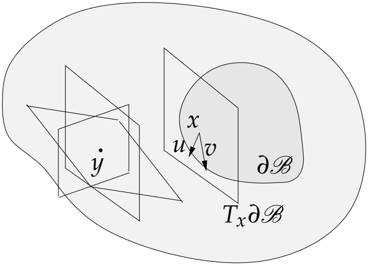



- For the case of force theory, $t_{\mathscr{B}}(w)$ is an $(m-1)$-form, the flux of power, where $t_{\mathscr{B}}(x) \colon W_x \longrightarrow \bigwedge^{m-1} T_x^*\partial\mathscr{B}$.
- The natural generalization: at each point $x$ there is a linear mapping $\sigma_x \colon W_x \longrightarrow \bigwedge^{m-1} T_x^*\mathscr{U}$, called *the stress* at $x$, such that $t_{\mathscr{B}}(w) = \iota^*\big(\sigma(w)\big)$. In other words,

$$t_{\mathscr{B}} = \iota^* \circ \sigma, \quad \textit{is the required Cauchy formula.}$$

# The Cauchy Postulates: Notes.

The dependence of $t_{\mathscr{B}}(x)$ on the subbody $\mathscr{B}$ through the tangent space to $\mathscr{B}$ is assumed to be continuous in the tangent space and point $x$. This aspect, that we neglected before, should be meaningful.




- The collection of hyperplanes, $G_{m-1}(T\mathscr{U})$-–*the Grassmann bundle*, i.e., $\left(G_{m-1}(T\mathscr{U})\right)_x$ is the manifold of $(m-1)$-dimensional *subspaces* of $T_x\mathscr{U}$.
- The mapping that assigns the surface forces to hyperplanes will be referred to as the *Cauchy section*. At each point it is a mapping

$$\Sigma_x \colon G_{m-1}(T_x\mathscr{U}) \to L\Big(W_x, \bigwedge^{m-1}\big(G_{m-1}(T_x\mathscr{U})\big)^*\Big).$$

# The Cauchy Postulates: The Cauchy Section

More precisely, consider the diagram

$$\begin{array}{ccccc} \pi_G^*(W) & \xrightarrow{\pi_G^*(\pi)} & G_{m-1}(T\mathscr{U}) & \longleftarrow & \bigwedge^{m-1}\big(G_{m-1}(T\mathscr{U})\big)^* \\ \uparrow & & \downarrow \pi_G & & \\ W & \xrightarrow{\pi} & \mathscr{U} & & \end{array}$$

Then, the Cauchy section is a *section*

$$\Sigma \colon G_{m-1}(T\mathscr{U}) \to L\Big(\pi_G^*(W), \bigwedge^{m-1}\big(G_{m-1}(T\mathscr{U})\big)^*\Big).$$

- It is assumed that $\Sigma$ is smooth.

# The Cauchy Postulates: Boundedness

We need the analog of the boundedness assumption

$$\left|\int_{\mathscr{B}} \beta + \int_{\partial\mathscr{B}} \tau_{\mathscr{B}}\right| \leq \int_{\mathscr{B}} s,$$

where eventually we get $\tau_{\mathscr{B}} = \iota^*(J)$ and $\int_{\partial\mathscr{B}} \tau_{\mathscr{B}} = \int_{\mathscr{B}} dJ$.

- We write the scalar boundedness assumption for the power, so $\beta = b(w)$ and $\tau_{\mathscr{B}} = t_{\mathscr{B}}(w)$.
- We anticipate that $t_{\mathscr{B}} = \iota^* \circ \sigma$. Hence, the bounded expression is

$$\left|\int_{\mathscr{B}} b(w) + \int_{\partial\mathscr{B}} t_{\mathscr{B}}(w)\right| = \left|\int_{\mathscr{B}} b(w) + \int_{\partial\mathscr{B}} \iota^*\big(\sigma(w)\big)\right| = \left|\int_{\mathscr{B}} b(w) + \int_{\mathscr{B}} d\big(\sigma(w)\big)\right|.$$

Thus, the expression should be bounded by the values of *both $w$ and its derivative*—the first *jet* $j^1(w)$.

# Consequences of the (Generalized) Cauchy Theorem

Since $t_{\mathscr{B}}(w) = \iota^*\big(\sigma(w)\big)$, the total power is given as

$$F_{\mathscr{B}}(w) = \int_{\mathscr{B}} b(w) + \int_{\partial\mathscr{B}} t_{\mathscr{B}}(w) = \int_{\mathscr{B}} b(w) + \int_{\mathscr{B}} d\big(\sigma(w)\big).$$

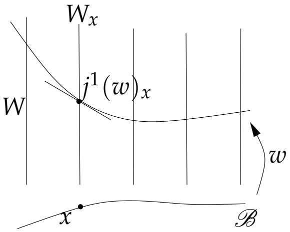
- The density of $F_{\mathscr{B}}(w)$ depends linearly on the values of $w$ and its derivative.
- For manifolds, there is no way to separate the value of the derivative of a section from the value of the section. Hence $j^1(w)$—the first jet of $w$ is a single invariant quantity that contains both the value and the value of the derivative.

Thus, the expression should be bounded by the values of *both $w$ and its derivative*—the first *jet* $j^1(w)$.

# Variational Stresses

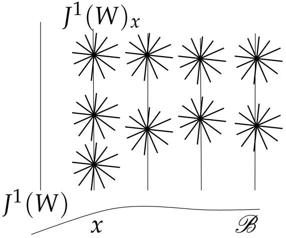
# Jets

*A jet* of a section at $x$ is an invariant quantity containing the values of both the section and its derivative.




$J^1(W)_x$ —the collection of all possible values of jets at $x$—the *jet space*.

$J^1(W)$—the collection of jet spaces, the *jet bundle*.



# Variational Stresses

We obtained

$$F_{\mathscr{B}}(w) = \int_{\mathscr{B}} \Big( b(w) + d\big(\sigma(w)\big) \Big).$$

- The value of the power density at a point is linear in the jet of $w$.
- Hence, there is a section $S$, such that

$$S_x \colon J^1(W)_x \longrightarrow \bigwedge^m T_x^* \mathscr{U} \quad \text{such that} \quad S_x\big(j^1(w)_x\big) = b(w) + d\big(\sigma(w)\big).$$

- We will refer to such a section $S$ of $L\big(J^1(W), \bigwedge^m (T^* \mathscr{U})\big)$ as a *variational stress density*. It produces power from the jets (gradients) of the velocity fields.
- Thus,

$$F_{\mathscr{B}}(w) = \int_{\mathscr{B}} \Big( b(w) + d\big(\sigma(w)\big) \Big) = \int_{\mathscr{B}} S\big(j^1(w)\big).$$

*Conclusion:*

*A Cauchy stress $\sigma$ and a body force $b$ induce a variational stress density $S$.*

# Variational Stress Densities:

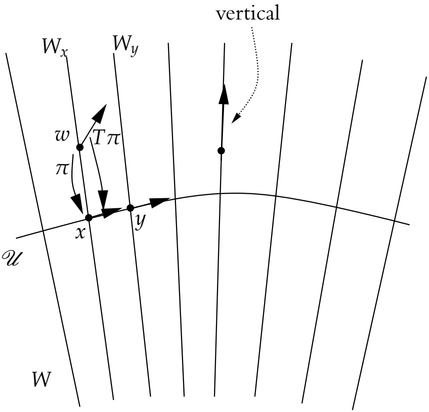
- Variational stress densities are sections of the vector bundle $L\big(J^1(W), \bigwedge^m(T^*\mathscr{U})\big)$, i.e, at each point, is assigns an $m$-covector to a jet at that point, linearly.
- If $S$ is a variational stress density, then the power of the force $F$ it represents over the body $\mathscr{B}$, while the the generalized velocity is $w$, is given by

$$F_{\mathscr{B}}(w) = \int_{\mathscr{B}} S(j^1(w)).$$

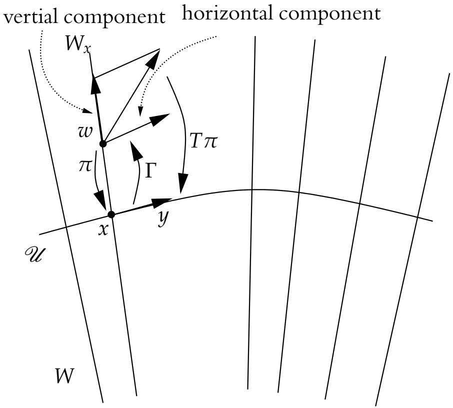
  This expression makes sense as $S(j^1(w))$, is an $m$-form whose value at a point $x \in \mathscr{B}$ is $S(x)\big(j^1(w)(x)\big)$.
- The local representation of $S$ is through the arrays $S_\alpha$ and $S^j_\beta$. The single component of the $m$-form $S\big(j^1(w)\big)$ in this representation is

$$S_\alpha w^\alpha + S^j_\beta w^\beta_{,j}.$$

# Linear Connections




no connection




$\Gamma$—the connection mapping

# The Case where a Connection is Given:

- If a connection is given on the vector bundle $W$, the jet bundle is isomorphic with the Whitney sum $W \oplus_{\mathscr{B}} L(T\mathscr{B}, W)$ by $j^1(w) \mapsto (w, \nabla w)$, where $\nabla$ denotes covariant derivative.
- A variational stress may be represented by sections $(S_0, S_1)$ of

$$L(W, \bigwedge^m (T^*\mathscr{U})) \oplus_{\mathscr{B}} L(L(T\mathscr{U}, W), \bigwedge^m (T^*\mathscr{B}))$$

  so the power is given by (see Segev (1986))

$$F_{\mathscr{B}}(w) = \int_{\mathscr{B}} S_0(w) + \int_{\mathscr{B}} S_1(\nabla w).$$

  We will refer to the section $S_1$ of $L(L(T\mathscr{U}, W), \bigwedge^m (T^*\mathscr{B}))$ as the *variational stress tensor*.
- With an appropriate definition of the divergence, a force may be written in terms of a body force and a surface force.

# Problem: Relation Between Variational and Cauchy Stresses

- *Can we extract the generalized Cauchy stress $\sigma$ from the variational stress $S$ invariantly*?
- There is a linear $p_\sigma \colon L\big(J^1(W), \bigwedge^m(T^*\mathscr{B})\big) \to L\big(W, \bigwedge^{m-1}(T^*\mathscr{B})\big)$ that gives a Cauchy stress $\sigma = p_\sigma(S)$ to any given variational stress $S$.
- Locally, if $\sigma$ is represented by $\sigma_{\beta\hat{\imath}}$ such that $\sigma_{\beta\hat{\imath}} w^\beta$ is the $i$-th component of the $(m-1)$-from $\sigma(w)$, locally $p_\sigma$ is given by

$$(x^i, S_\alpha, S^j_\beta) \longmapsto (x^i, \sigma_{\beta\hat{\imath}})$$

*where,*

$$\sigma_{\beta\hat{\imath}} = (-1)^{i-1} S^{+i}_{\ \beta}, \quad \textit{(no sum over i).}$$

- *Can you write a generalized definition of the divergence that applies even without a connection?* ✔ Locally, the divergence $\operatorname{Div} S$ is given by $\big(S^i_{\alpha,i} - S_\alpha\big)$.

# The Vertical Subbundle of the Jet Bundle:

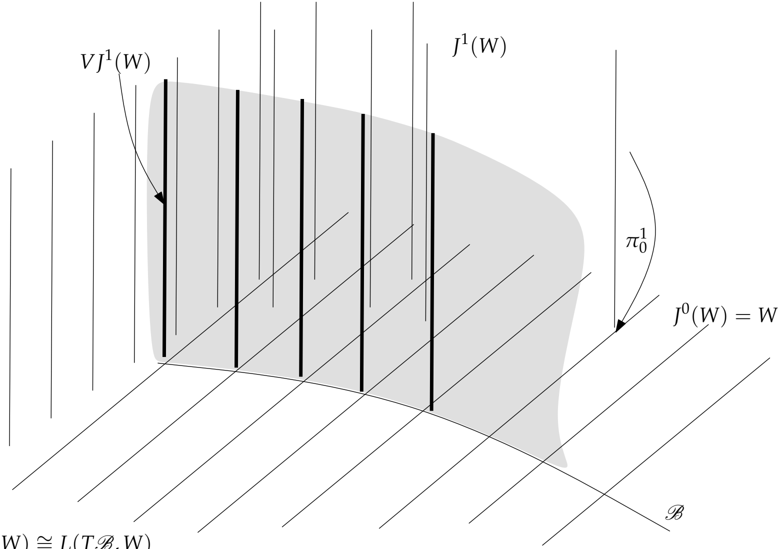
- Let $\pi_0^1 \colon J^1(W) \to W$ be the *natural projection* on the jet bundle that assign to any 1-jet at $x \in \mathscr{B}$ the value of the corresponding 0-jet, i.e., the value of the section at $x$.
- We define $VJ^1(W)$, the *vertical sub-bundle* of $J^1(W)$, to be the vector bundle over $\mathscr{B}$ such that

$$VJ^1(W) = (\pi_0^1)^{-1}(0),$$

  where 0 is the zero section of $W$.
- There is a natural isomorphism

$$I^+ \colon VJ^1(W) \to L\big(T\mathscr{U}, W\big).$$

# The Vertical Subbundle $VJ^1(W)$:




$VJ^1(W) \cong L(T\mathscr{B}, W)$

# The Vertical Component of a Variational Stress:

- Let $\iota_V \colon VJ^1(W) \to J^1(W)$ be the *inclusion mapping* of the sub-bundle.
- Consider the linear injection, $\iota_n = \iota_V \circ (I^+)^{-1} \colon L(T\mathscr{U}, W) \to J^1(W)$.
- Thus we have a linear surjection

$$\iota_n^* \colon L\big(J^1(W), \bigwedge^m (T^*\mathscr{B})\big) \to L\big(L(T\mathscr{U}, W), \bigwedge^m (T^*\mathscr{B})\big)$$

  given by $\iota_n^*(S) = S \circ \iota_n$.
- For a variational stress $S$, we will refer to

$$S^+ = \iota_n^*(S) \in L\big(L(T\mathscr{U}, W), \bigwedge^m (T^*\mathscr{B})\big)$$

  as the *vertical component* of $S$. (The *symbol* of the variational stress).
- If the variational stress is represented locally by $(S_\alpha, S^j_\beta)$, then, $S^+$ is represented locally by ${S^+}^i_\alpha = S^i_\alpha$.
- Clearly, one cannot define invariantly (without a connection) a *"horizontal"* component to the stress.

# Variational Fluxes:

- Since the jet of a real valued function $\varphi$ on $\mathscr{B}$ can be identified with a pair $(\varphi, d\varphi)$ in the trivial case where $W = \mathscr{B} \times \mathbb{R}$, the jet bundle can be identified with the Whitney sum $W \oplus_{\mathscr{B}} T^*\mathscr{U}$.
- $VJ^1(W)$ can be identified with $T^*\mathscr{U}$ and the vertical component of the variational stress is valued in $L\big(T^*\mathscr{U}, \bigwedge^m(T^*\mathscr{B})\big)$. We will refer to sections of $L\big(T^*\mathscr{U}, \bigwedge^m(T^*\mathscr{B})\big)$ as *variational fluxes*.
- There is a natural isomorphism

$$i_\wedge : \bigwedge^{m-1}(T^*\mathscr{B}) \to L\Big(T^*\mathscr{U}, \bigwedge^{m}(T^*\mathscr{B})\Big)$$

given by $i_\wedge(\omega)(\phi) = \phi \wedge \omega$.

# The Cauchy Stress Induced by a Variational Stress:

- Consider the *contraction* natural vector bundle morphism

$$c\colon L\big(L\big(T\mathscr{U},W\big),\bigwedge^{m}(T^*\mathscr{B})\big)\oplus_{\mathscr{B}} W\to L\big(T^*\mathscr{U},\bigwedge^{m}(T^*\mathscr{B})\big)$$

  given by

$$c(B,w)(\phi)=B(w\otimes\phi),$$

  for $B\in L\big(L\big(T\mathscr{U},W\big),\bigwedge^m(T^*\mathscr{B})\big)$, $w\in W$, and $\phi\in T^*\mathscr{U}$, where $(w\otimes\phi)(v)=\phi(v)w$. We also write $w\lrcorner B$ for $c(B,w)$.
- For a section $S^+$ of $L\big(L\big(T\mathscr{U},W\big),\bigwedge^m(T^*\mathscr{B})\big)$ and a vector field $w$, $w\lrcorner S^+$ is a *variational flux*.
- Consider the mapping

$$i_\sigma\colon L\big(L\big(T\mathscr{U},W\big),\bigwedge^{m}(T^*\mathscr{B})\big)\to L\big(W,\bigwedge^{m-1}(T^*\mathscr{B})\big)$$

  such that $i_\sigma\circ S^+(w)=i_\wedge^{-1}(w\lrcorner S^+)$. It is linear and injective.

## Cauchy Stresses and Variational Stresses (Contd.)

- $p_\sigma = i_\sigma \circ \iota^* : L\big(J^1(W), \bigwedge^m (T^*\mathscr{B})\big) \to L\big(W, \bigwedge^{m-1}(T^*\mathscr{B})\big)$ is a linear mapping (no longer injective) that gives a Cauchy stress to any given variational stress.
- Locally, $\sigma$ is represented by $\sigma_{\beta\hat{\imath}}$ such that $\sigma_{\beta\hat{\imath}} w^\beta$ is the $i$-th component of the $(m-1)$-from $\sigma(w)$.
- Locally $p_\sigma$ is given by

$$(x^i, S_\alpha, S^j_\beta) \mapsto (x^i, \sigma_{\beta\hat{\imath}})$$

*where,*

$$\sigma_{\beta\hat{\imath}} = (-1)^{i-1} S^{+}{}^{i}_{\beta}, \quad (\textit{no sum over } i).$$

# The Divergence of a Variational Stress:

- For a given variational stress $S$ and a generalized velocity $w$, consider the difference, an $m$-form,

$$d\big(p_\sigma(S)(w)\big) - S\big(j^1(w)\big).$$

- Locally, the difference is represented by

$$\big(S^i_{\alpha,i} - S_\alpha\big)w^\alpha$$

- This shows that the difference depends only on the values of $w$ and not its derivative.
- Define the *generalized divergence* of the variational stress $S$ to be the section $\mathrm{Div}(S)$ of the vector bundle $L\big(W, \bigwedge^m(T^*\mathscr{B})\big)$ satisfying

$$\mathrm{Div}(S)(w) = d\big(p_\sigma(S)(w)\big) - S\big(j^1(w)\big) = d\sigma(w) - S\big(j^1(w)\big),$$

$\sigma = p_\sigma(S)$, for every generalized velocity field $w$.

# The Principle of Virtual Power:

- Given a variational stress $S$, the expression for the power is

$$F_{\mathscr{B}}(w) = \int\limits_{\mathscr{B}} S\big(j^1(w)\big).$$

- Using the previous constructions and Stokes' theorem we have

$$F_{\mathscr{B}}(w) = \int\limits_{\partial\mathscr{B}} \iota^*_{\mathscr{B}}\big(\sigma(w)\big) - \int\limits_{\mathscr{B}} \mathrm{Div}(S)(w),$$

  where, $\sigma = p_\sigma(S)$ is the the Cauchy stress induced by the variational stress $S$, and $\iota^*_{\mathscr{B}}$ is the restriction of $(m-1)$-forms on $\mathscr{B}$ to $\partial\mathscr{B}$.
- Thus we have for $t_{\mathscr{B}}(w) = \iota^*_{\mathscr{B}}\big(\sigma(w)\big) = \iota^*_{\mathscr{B}}\big(p_\sigma(S)(w)\big)$ and $\mathrm{Div}\, S + b_{\mathscr{B}} = 0$, a force for each subbody $\mathscr{B}$ of the form

$$F_{\mathscr{B}}(w) = \int\limits_{\partial\mathscr{B}} b_{\mathscr{B}}(w) + \int\limits_{\mathscr{B}} t_{\mathscr{B}}(w).$$

# Conclusions:

- *The mapping relating the values of variational stress fields and Cauchy stresses*

$$p_\sigma \colon L\Big(J^1(W), \bigwedge^{m}(T^*\mathscr{U})\Big) \to L\Big(W, \bigwedge^{m-1}(T^*\mathscr{U})\Big),$$

  *is linear, surjective, but not injective.*

- *However, the mapping between the fields*

$$p \colon S \mapsto (\sigma, b), \quad \sigma = p_\sigma \circ S, \quad b = -\operatorname{Div} S,$$

  *is injective.*

- *The inverse, $p^{-1} \colon (\sigma, b) \mapsto S$, is given by*

$$S(x)(A) = b_x(w_x) + d\sigma(w)_x,$$

  *for any vector field $w$ whose jet at $x$ is $A$.*