Some Extensions and Analysis of Flux and Stress Theory

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Structures of the Mechanics of Complex Bodies

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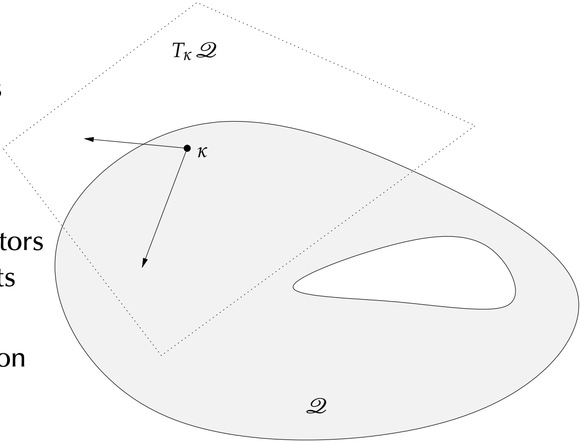
The Global Point of View

C^n-Functionals

Review of Basic Kinematics and Statics on Manifolds

- The mechanical system is characterized by its configuration space—a manifold $\mathscr{Q}$.
- *Velocities* are tangent vectors to the manifold—elements of $T\mathscr{Q}$.
- A *Force* at the configuration κ is a linear mapping $F\colon T_\kappa\mathscr{Q} \to \mathbb{R}$.



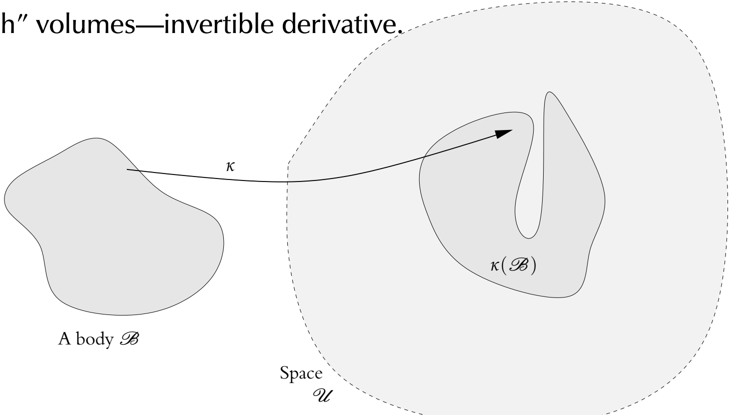
Can we apply this framework to Continuum Mechanics?

Problems Associated with the Configuration Space in Continuum Mechanics

- What is a configuration?
- Does the configuration space have a structure of a manifold?
- The configuration space for continuum mechanics is infinite dimensional.

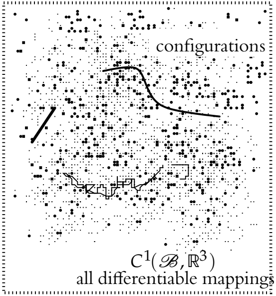
Configurations of Bodies in Space

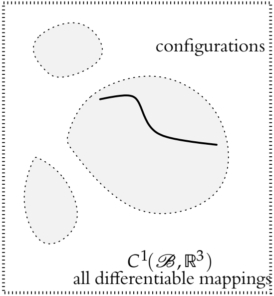
- A mapping of the body into space;
- material impenetrability—one-to-one;
- continuous deformation gradient (derivative);
- do not “crash” volumes—invertible derivative.



Manifold Structure for Euclidean Geometry

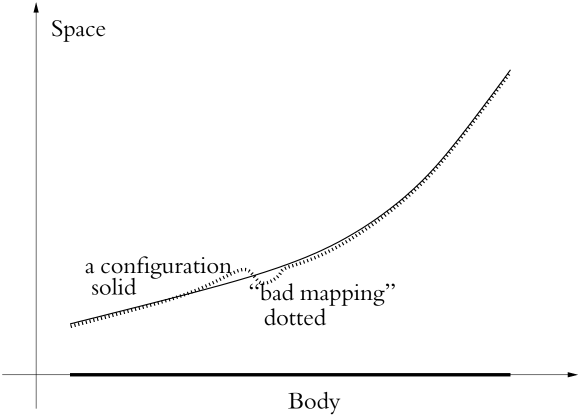
- If the body is a subset of $\mathbb{R}^3$ and space is modeled by $\mathbb{R}^3$, the collection of differentiable mappings $C^1(\mathscr{B}, \mathbb{R}^3)$ is a vector space
- However, the subset of "good" configurations is not a vector space, e.g., $\kappa - \kappa = 0$—not one-to-one.
- We want to make sure that the subset of configurations $\mathscr{Q}$ is an open subset of $C^1(\mathscr{B}, \mathbb{R}^3)$, so it is a trivial manifold.





The C^0-Distance Between Functions

- The C^0-distance between functions measures the maximum difference between functions.
- A configuration is arbitrarily close to a “bad” mapping.

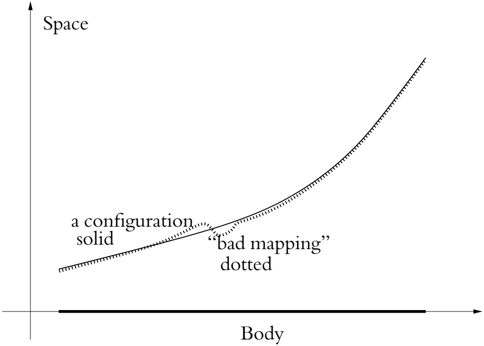


The C^1-Distance Between Functions

- The C^1 distance between functions measures the maximum difference between functions and their derivative

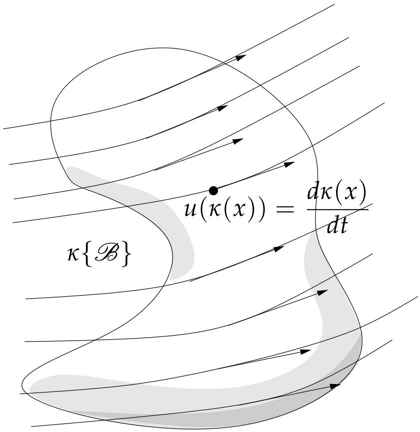
$$|u - v|_{C^1} = \sup\{|u(x) - v(x)|, |Du(x) - Dv(x)|\}.$$

- A configuration is always a finite distance away from a "bad" mapping.



Conclusions for $\mathbb{R}^3$

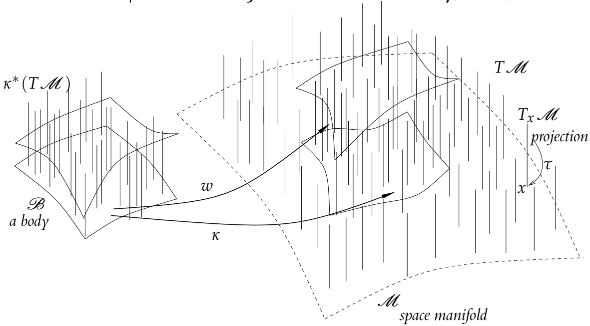
- If we use the C^1-norm, the configuration space of a continuous body in space is an open subset of $C^1(\mathscr{B}, \mathbb{R}^3)$-the vector space of all differentiable mapping.
- $\mathscr{Q}$ is a trivial infinite dimensional manifold and its tangent space at any point may be identified with $C^1(\mathscr{B}, \mathbb{R}^3)$.
- A tangent vector is a velocity field.



For Manifolds

- Both the body $\mathscr{B}$ and space $\mathscr{U}$ are differentiable manifolds.
- The configuration space is the collection $\mathscr{Q} = \text{Emb}(\mathscr{B}, \mathscr{U})$ of the embeddings of the body in space. This is an open submanifold of the infinite dimensional manifold $C^1(\mathscr{B}, \mathscr{U})$.
- The tangent space $T_\kappa \mathscr{Q}$ may be characterized as

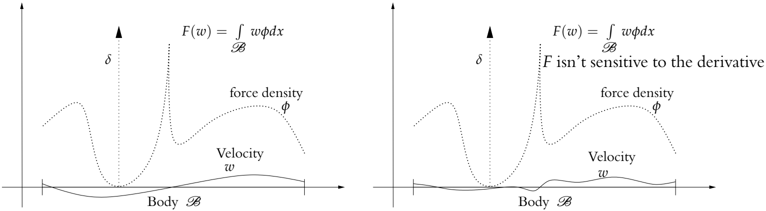
$$T_\kappa \mathscr{Q} = \{w \colon \mathscr{B} \to T\mathscr{Q} | \tau \circ w = \kappa\}, \quad \text{or alternatively,} \quad T_\kappa \mathscr{Q} = C^1(\kappa^* T\mathscr{U}).$$



Representation of C^0-Functionals by Integrals

- Assume you measure the size of a function using the C^0-distance, $\|w\| = \sup\{|w(x)|\}$.
- A linear functional $F\colon w \mapsto F(w)$ is continuous with respect to this norm if $F(w) \to 0$ when $\max|w(x)| \to 0$.
- *Riesz representation theorem:* A continuous linear functional F with respect to the C^0-norm may be represented by a unique measure μ in the form

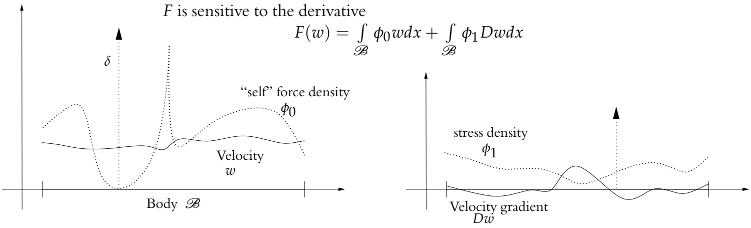
$$F(w) = \int_{\mathscr{B}} w \, d\mu.$$



Representation of C^1-Functionals by Integrals

- Now, you measure the size of a function using the C^1-distance, $\|w\| = \sup\{|w(x)|, |Dw(x)|\}$.
- A linear functional $F\colon w \mapsto F(w)$ is continuous with respect to this norm if $F(w) \to 0$ when both $\max|w(x)| \to 0$ and $\max|Dw(x)| \to 0$.
- Representation theorem: A continuous linear functional F with respect to the C^1-norm may be represented by measures $\sigma_0,\ \sigma_1$ in the form

$$F(w) = \int_{\mathscr{B}} w\, d\sigma_0 + \int_{\mathscr{B}} Dw\, d\sigma_1.$$

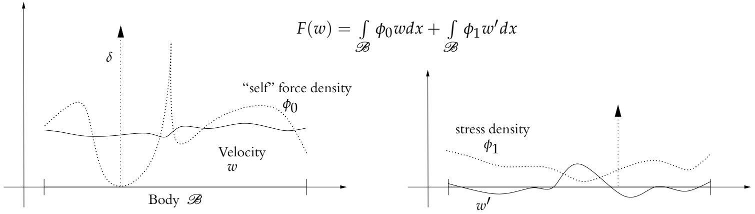


Non-Uniqueness of C^1-Representation by Integrals

- We had an expression in the form

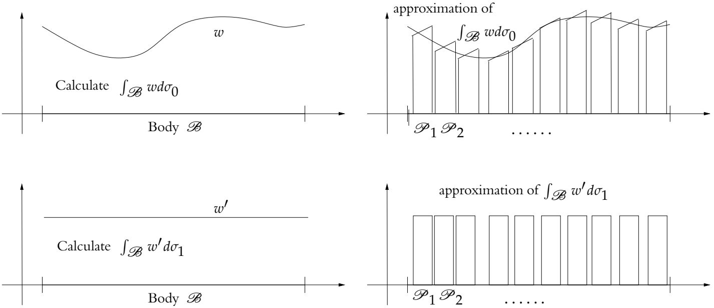
$$F(w) = \int_{\mathscr{B}} w \, d\sigma_0 + \int_{\mathscr{B}} w' \, d\sigma_1.$$

- If we were allowed to vary w and w' independently, we could determine σ_0 and σ_1 uniquely.
- This cannot be done because of the condition $w' = Dw$.



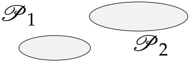
Unique Representation of a Force System

- Assume we have a force system, i.e., a force $F_{\mathscr{P}}$ for every subbody $\mathscr{P}$ of $\mathscr{B}$.
- We can approximate pairs of non-compatible functions w and w', i.e., $w' \neq Dw$, by piecewise compatible functions.



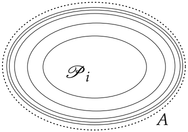
- This way the two measures are determined uniquely.
- One needs consistency conditions for the force system.

Generalized Cauchy Consistency Conditions



- *Additivity:*

$$F_{\mathscr{P}_1\cup\mathscr{P}_2}(w|_{\mathscr{P}_1\cup\mathscr{P}_2}) = F_{\mathscr{P}_1}(w|_{\mathscr{P}_1}) + F_{\mathscr{P}_2}(w|_{\mathscr{P}_2}).$$



- *Continuity:* If $\mathscr{P}_i \to A$, then $F_{\mathscr{P}_i}(w|_{\mathscr{P}_1})$ converges and the limit depends on A only.

- *Uniform Boundedness:* There is a $K > 0$ such that for every subbody $\mathscr{P}$ and every w,

$$|F_{\mathscr{P}}(w|_{\mathscr{P}}) \leq K\|w_{\mathscr{P}}\|.$$

Main Tool in Proof: Approximation of measurable sets by bodies with smooth boundaries.

Generalizations

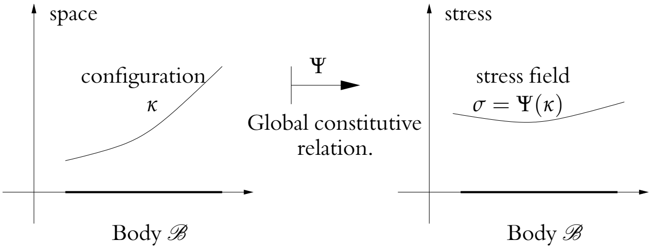
- All the above may be formulated and proved for differentiable manifolds.
- This formulation applies to continuum mechanics of order $k > 1$ (stress tensors of order k). One should simply use the C^k-norm instead of the C^1-norm.
- The generalized Cauchy conditions also apply to continuum mechanics of order $k > 1$. This is the only formulation of Cauchy conditions for higher order continuum mechanics.

Locality and Continuity in Constitutive Theory

Global Constitutive Relations

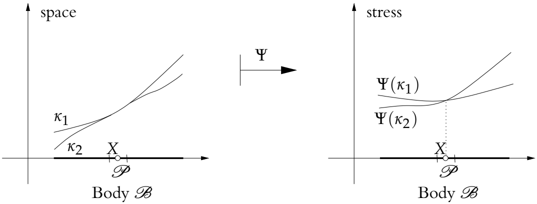
(Elasticity for Simplicity)

- $\mathscr{Q}$, the configuration space of a body $\mathscr{B}$.
- $C^0\big(\mathscr{B}, L(\mathbb{R}^3, \mathbb{R}^3)\big)$, the collection of all stress fields over the body.
- $\Psi\colon \mathscr{Q} \to C^0\big(\mathscr{B}, L(\mathbb{R}^3, \mathbb{R}^3)\big)$, a global constitutive relation.

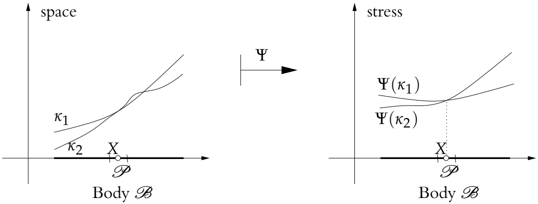


Locality and Materials of Grade-n

Germ Locality: If two configurations κ_1 and κ_2 are equal on a subbody containing X, then the resulting stress fields are equal at X.



Material of Grade-n or *n-Jet Locality:* If the first n derivatives of κ_1 and κ_2 are equal at X, then, $\Psi(\kappa_1)(X) = \Psi(\kappa_2)(X)$. (Elastic = grade 1.)



n-Jet Locality and Continuity

Basic Theorem: If a constitutive relation $\Psi : \mathscr{Q} \to C^0(\mathscr{B}, L(\mathbb{R}^3, \mathbb{R}^3))$ is local and continuous with respect to the C^n-norm, then, it is n-jet local. In particular, if Ψ is continuous with respect to the C^1-topology, the material is elastic.