Load Capacity of Bodies and Structures

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Load Capacity Ratio

Notation

Ω – a given homogeneous elastic-plastic body or a structure,

σ_Y – the yield stress,

t – a loading traction field given on the boundary $\partial\Omega$,

$t_{\max}$ – the maximum of the external loading,

$$t_{\max} = \operatorname{ess\,sup}_{y \in \partial\Omega} |t(y)| = \|t\|_\infty$$

Result

There is a minimimal number C such that the body will not collapse as long as

$$t_{\max} \leqslant C\sigma_Y$$

independently of the distribution of the external traction t.

The Expression for the Load Capacity Ratio

The number C, a purely geometric property of the body Ω, is given by

$$\frac{1}{C} = \sup_{w} \frac{\int_{\Gamma_t} |w| \, \mathrm{d}A}{\int_{\Omega} |\varepsilon(w)| \, \mathrm{d}V} = \|\gamma_D\|,$$

where,

w – *an isochoric (incompressible) vector field,*

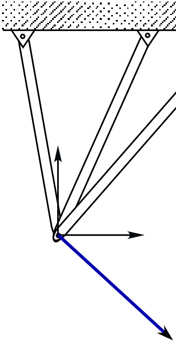
$\varepsilon(w)$ – *the linear strain associated with w,*

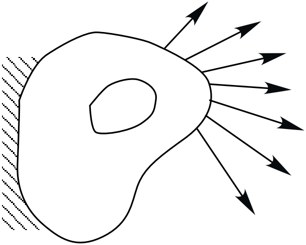
$$\varepsilon(w)_{ij} = \tfrac{1}{2}(w_{i,j} + w_{j,i}),\ \varepsilon(w)_{ii} = 0;$$

γ_D – *the trace mapping taking a vector field on Ω and giving its boundary value (restriction to the boundary of continuous vector fields).*

Introduction

Statically Indeterminate Problems:





- *An infinite number of solutions.*

  Find the optimal. (A generalized inverse)

  - *Find the worst ratio between the optimal stress*

    and the maximum of the external loading.

Related Work

- R. Segev, 2003, Generalized stress concentration factors, *Mathematics and Mechanics of Solids*, first published on June 10, 2005 as doi: 10.1177/1081286505044131.
- R. Segev, 2004, Generalized stress concentration factors for equilibrated forces and stresses, accepted for publication, *J. Elasticity*, arXiv:physics/0407136.
- R. Peretz and R. Segev, 2005, Bounds on the trace mapping of *LD*-fields, accepted for publication, *Computers and Mathematics with Applications*, arXiv:math.AP/0505006.
- R. Segev, 2005, Stress optimization for supported bodies, submitted for publication, arXive:math.AP/0511014.
- R. Segev, 2007, "Load capacity of bodies", *International Journal of Non-Linear Mechanics*, **42**, 250 - 257, doi:10.1016/j.ijnonlinmec.2006.10.012.

The Setting for the Continuum Problem

Definitions of the Main Variables

Ω – a given body (bounded), $\Gamma = \partial\Omega$ – its boundary,

Γ_0 – the part of the boundary where the body is fixed,

t – a surface traction field given on $\Gamma_t \subset \Gamma$,

ν – the unit normal to the boundary

σ – a stress field that is in equilibrium with t,

$\sigma_{\max}$ – the maximal magnitude of the stress

$$\sigma_{\max} = \operatorname{ess\,sup}_{x\in\Omega} |\sigma(x)| = \|\sigma\|_\infty.$$

Remark: The treatment may be generalized to include body forces.

- There is a class of stress fields that are in equilibrium with t.
- We denote this class of stress fields by Σ_t.

The Problem

- Find the least value S_t of $\sigma_{\max}$, i.e.,

$$S_t = \inf_{\sigma \in \Sigma_t} \{\sigma_{\max}\} = \inf_{\sigma \in \Sigma_t} \{\|\sigma\|_\infty\}.$$

  - *Question*: Is there an optimal stress field σ_{opt} such that

$$S_t = \|\sigma_{\text{opt}}\|_\infty\,?$$

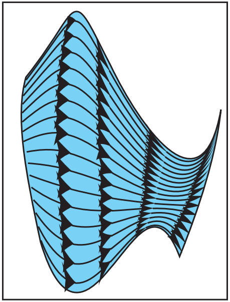
- Find the *generalized stress concentration factor*

$$K = \sup_t \frac{S_t}{\operatorname{ess\,sup}_y |t(y)|}.$$

The Corresponding Scalar Problem: the Junction Problem

- Given the flux density ϕ on the boundary of Ω with $\int_{\partial\Omega} \phi \mathrm{d}A = 0$.
- Set $V_\phi = \{v\colon \Omega \to \mathbb{R}^3,\ v_{i,i} = 0 \text{ in } \Omega,\ ,v_i \nu_i = \phi \text{ on } \partial\Omega\}$
  —compatible velocity fields.
- For each $v \in V_\phi$, set $v_{\max} = \operatorname{ess\,sup}_{x\in\Omega} |v(x)|$.
- Find the least value v_ϕ^{opt} of $v_{\max}$, i.e.,

$$v_\phi^{\mathrm{opt}} = \inf_{v \in V_\phi} \{v_{\max}\}.$$



The optimal velocity field for the junction Ω.

The Results

Theorem (Segev 2004, [2])

- *The optimal value S_t is given by*

$$S_t = \sup_{w \in C^\infty(\overline{\Omega},\mathbb{R}^3)} \frac{\left|\int_{\partial\Omega} t \cdot w \,\mathrm{d}A\right|}{\int_\Omega |\varepsilon(w)|\,\mathrm{d}V} = \sup_{w \in C^\infty(\overline{\Omega},\mathbb{R}^3)} \frac{|t(w)|}{\|\varepsilon(w)\|_1},$$

  $|\varepsilon(w)|$ *is the norm of the value of the stretching* $\varepsilon(w) = \frac{1}{2}(\nabla w + \nabla w^T)$.
- *The optimum is attained for some* $\sigma_{\mathrm{opt}} \in \Sigma_t$.
- *Mathematically:*

$$S_t = \|\textit{Force Functional}\|.$$

Stress Concentration for Engineers

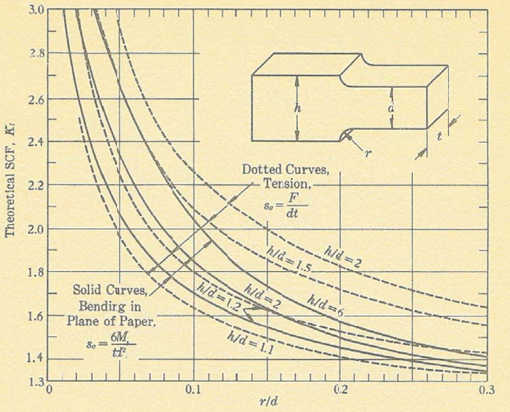


FIGURE AF 9 Flat Plate with Fillets. The tensile load is central. For $h/d = 1.1$, tensile and bending curves are very close together down to $r/d = 0.04$. (After R. E. Peterson)[4.21]

Generalized Stress Concentration Factors:

- Assume a body Ω is given (open, regular with smooth boundary).
- Assume a surface traction t is given and let σ be a stress field that is in equilibrium with t.
- The *stress concentration factor* associated with the pair t, σ is

$$K_{t,\sigma} = \frac{\operatorname{ess\,sup}_x \{|\sigma(x)|\}}{\operatorname{ess\,sup}_y \{|t(y)|\}}, \quad x \in \Omega, \quad y \in \partial\Omega.$$

- Denote by Σ_t the collection of all possible stress fields that are in equilibrium with t. (There are many such stress fields because material properties are not specified.)

- The *optimal stress concentration factor* for the force t is defined by

$$K_t = \inf_{\sigma \in \Sigma_t} \{K_{t,\sigma}\}.$$

- The *generalized stress concentration factor* K—a purely geometric property of Ω—is defined by

$$K = \sup_t \{K_t\} = \sup_t \inf_{\sigma \in \Sigma_t} \left\{ \frac{\operatorname{ess\,sup}_x \{|\sigma(x)|\}}{\operatorname{ess\,sup}_y \{|t(y)|\}} \right\}.$$

Concerning the Generalized Stress Concentration Factor

Theorem (Segev 2004, [2])

- *Define the generalized stress concentration factor K by*

$$K = \sup_t \frac{S_t}{\operatorname{ess\,sup}_{y\in\partial\Omega} |t(y)|} \,.$$

- *Then,*

$$K = \|\gamma\| = \sup_{w\in C^\infty(\overline{\Omega},\mathbb{R}^3)_0} \frac{\int_{\Gamma_t} |w| \, \mathrm{d}A}{\int_\Omega |\varepsilon(w)| \, \mathrm{d}V}.$$

Relation to Limit Analysis in Plasticity

Notation for plasticity:

Deviatoric projection – $\pi_D(m) = m - \frac{1}{3} m_{ii} I$ for every matrix m.

$\pi_D \colon \mathbb{R}^6 \longrightarrow D \subset \mathbb{R}^6$, the space of traceless matrices.

Yield function Y – a semi-norm on the space of matrices

$$Y(m) = \left| m - \tfrac{1}{3} m_{ii} I \right|, \quad |\cdot| \text{ is a norm on the space of matrices.}$$

Yield condition – $Y(m) = \sigma_Y$.

Semi-norms – $\|\chi\|^Y = \|Y \circ \chi\|$, $\|\sigma\|_\infty^Y = \|Y \circ \sigma\|_\infty$

are norms on the subspaces of trace-less fields.

Collapse – $\|\sigma\|_\infty^Y \geqslant \sigma_Y$.

Thus, in the previous definitions of the optimal stress we have to use the semi-norms or restrict ourselves to the appropriate subspaces containing trace-less fields.

Optimal Stresses and Limit Analysis

- *The limit analysis problem:* Given t and σ_Y, find

$$\lambda_t^* = \sup \lambda, \quad \text{such that } \exists \sigma,\ \|\sigma\|_\infty^Y \leqslant \sigma_Y,\ \sigma \in \Sigma_{\lambda t}$$

- Christiansen and Temam & Strang:

$$\lambda_t^* = \sup_{\|\sigma\|_\infty^Y \leqslant \sigma_Y} \inf_{t(w)=1} \int_\Omega \sigma_{ij}\varepsilon(w)_{ij} \mathrm{d}V = \inf_{t(w)=1} \sup_{\|\sigma\|_\infty^Y \leqslant \sigma_Y} \int_\Omega \sigma_{ij}\varepsilon(w)_{ij} \mathrm{d}V$$

- Limit design $\quad \Leftrightarrow \quad S_t = \sigma_Y.$

  Easy to see that

$$\frac{\sigma_Y}{S_t} = \lambda_t^*.$$

- Our expression for S_t is equivalent to the theorem.

The Load Capacity Ratio

- Given σ_Y, consider the *collapse manifold*

$$\Psi = \{t \mid S_t = \sigma_Y\}\,.$$

- Find the *load capacity ratio*

$$C = \frac{1}{\sigma_Y}\inf_{t\in\Psi}\|t\|_\infty, \quad \Rightarrow \text{no collapse for } t \text{ with } \|t\|_\infty \leqslant C\sigma_Y$$

- Easy to see that

$$C = \frac{1}{K}\,.$$

- The expression for K using the yield norms

$$K = \sup_{t\in L^\infty(\Gamma_t,\mathbb{R}^3)} S_t = \sup_{w\,\text{incomp}} \frac{\int_{\Gamma_t}|w|\,\mathrm{d}A}{\int_\Omega|\varepsilon(w)|\,\mathrm{d}V} = \|\gamma_D\|.$$

General Mathematical Structure

$$
\begin{array}{ccccc}
L^1(\Gamma_t, \mathbb{R}^3) & \xleftarrow{\gamma_0} & LD(\Omega)_0 & \xrightarrow{\varepsilon_0} & L^1(\Omega, \mathbb{R}^6) \\
\Vert & & \uparrow \iota & & \iota \uparrow \downarrow \pi_D^\circ \\
L^1(\Gamma_t, \mathbb{R}^3) & \xleftarrow{\gamma_D} & LD(\Omega)_D & \xrightarrow{\varepsilon_D} & L^1(\Omega, D)
\end{array}
$$

$$
\begin{array}{ccccc}
\text{boundary velocities} & \xleftarrow{\text{boundary value}} & \text{velocity fields} & \xrightarrow{\varepsilon_0} & \text{strain fields} \\
\Vert & & \uparrow \text{inclusion} & & \text{inclusion} \uparrow \downarrow \pi_D^\circ \\
L^1(\Gamma_t, \mathbb{R}^3) & \xleftarrow{\gamma_D} & \begin{array}{l}\text{incompressible} \\ \text{velocity fields}\end{array} & \xrightarrow{\varepsilon_D} & \begin{array}{l}\text{incompressible} \\ \text{strain fields}\end{array}
\end{array}
$$

General General Mathematical Structure - Continued

$$
\begin{array}{ccccc}
L^\infty(\Gamma_t,\mathbb{R}^3) & \xrightarrow{\gamma_0^*} & LD(\Omega)_0^* & \xleftarrow{\varepsilon_0^*} & L^\infty(\Omega,\mathbb{R}^6) \\
\Big\| & & \Big\downarrow \iota^* & & \iota^* \Big\downarrow \Big\uparrow \pi_D^{\circ *} \\
L^\infty(\Gamma_t,\mathbb{R}^3) & \xrightarrow{\gamma_D^*} & LD(\Omega)_D^* & \xleftarrow{\varepsilon_D^*} & L^\infty(\Omega,D).
\end{array}
$$

$$
\begin{array}{ccccc}
\text{boundary tractions} & \xrightarrow{\gamma_0^*} & \text{forces} & \xleftarrow{\varepsilon_0^*} & \text{stress fields} \\
\Big\| & & \Big\downarrow \text{inclusion} & & \text{restriction} \Big\downarrow \Big\uparrow \pi_D^{\circ *} \\
\text{boundary tractions} & \xrightarrow{\gamma_D^*} & \text{forces with deviatoric stresses} & \xleftarrow{\varepsilon_D^*} & \text{deviatoric stress fields}
\end{array}
$$

Properties of the Mappings

$$
\begin{array}{ccccc}
\text{boundary velocities} & \xleftarrow{\text{boundary value}} & \text{velocity fields} & \xrightarrow{\varepsilon_0} & \text{strain fields} \\
\Big\| & & \Big\uparrow \text{inclusion} & & \text{inclusion} \Big\uparrow \Big\downarrow \pi_D^{\circ} \\
L^1(\Gamma_t, \mathbb{R}^3) & \xleftarrow{\gamma_D} & \begin{array}{l}\text{incompressible}\\ \text{velocity fields}\end{array} & \xrightarrow{\varepsilon_D} & \begin{array}{l}\text{incompressible}\\ \text{strain fields}\end{array}
\end{array}
$$

ε_0 – the strain mapping for velocity fields that satisfy the boundary conditions (zero on an open subset of the boundary). *Injective.*

γ – the trace mapping. *Sujective.*

Introducing $LD(\Omega)$ (Temam 85)

Recall: $\operatorname{ess\,sup}_x |\sigma(x)| = \|\sigma\|_\infty$ suggests:

$$\text{Stress Fields} = L^\infty(\Omega, \mathbb{R}^6) \quad \text{so} \quad \text{Stretching Fields} = L^1(\Omega, \mathbb{R}^6).$$

Conclusion:

$$\text{Body Velocities} = \left\{ w \colon \Omega \to \mathbb{R}^3; \varepsilon(w) \in L^1(\Omega, \mathbb{R}^6) \right\}.$$

Set

$$LD(\Omega) = \left\{ w \colon \Omega \to \mathbb{R}^3; w \in L^1(\Omega, \mathbb{R}^3), \varepsilon(w) \in L^1(\Omega, \mathbb{R}^6) \right\},$$

$$\|w\|_{LD} = \|w\|_1 + \|\varepsilon(w)\|_1.$$

Equivalent Norm for $LD(\Omega)$

- Let
$$\pi_{\mathscr{R}} \colon LD(\Omega) \longrightarrow \mathbb{R}^3 \times o(3)$$
be any projection on the space of rigid velocity fields on the body.
- An equivalent norm for $LD(\Omega)$:
$$\|w\|_{LD} = \|\pi_{\mathscr{R}}(w)\| + \|\varepsilon(w)\|_1.$$

- Displacement boundary conditions imply no rigid motion component:
$$\|w\| = \|\varepsilon(w)\|_1.$$

- $\varepsilon_0 \colon LD(\Omega)_0 \longrightarrow L^1(\Omega, \mathbb{R}^6)$ *is norm preserving.*

Properties of $LD(\Omega)$

- *Approximations:* $C^\infty(\overline{\Omega}, \mathbb{R}^3)$ is dense in $LD(\Omega)$.
- *Traces:* There is a unique, continuous, linear trace mapping

$$\gamma\colon LD(\Omega) \longrightarrow L^1(\partial\Omega, \mathbb{R}^3)$$

such that $\gamma(u|_\Omega) = u|_{\partial\Omega}$, $u \in C(\overline{\Omega}, \mathbb{R}^3)$.

Tools Used in the Proof:

- The Hahn-Banach Theorem,
- $\|\gamma\| = \|\gamma^*\|$.

Proof of The Expression for the GSCF

We had

$$S_t = \sup_{w \in LD(\Omega)_0} \frac{\left|\int_{\partial\Omega} t \cdot w \, \mathrm{d}A\right|}{\int_\Omega |\varepsilon(w)| \, \mathrm{d}V} = \sup_{w \in LD(\Omega)_0} \frac{|t(\gamma_0(w))|}{\|\varepsilon(w)\|_1},$$

$$= \sup_{w \in LD(\Omega)_0} \frac{|\gamma_0^*(t)(w)|}{\|w\|_{LD}} = \|\gamma_0^*(t)\|,$$

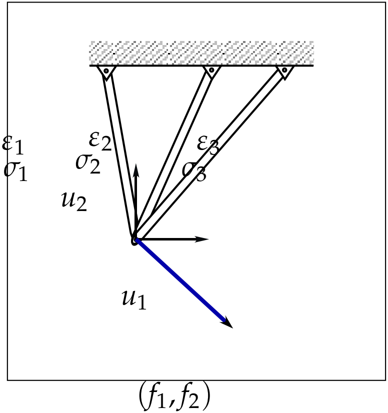
so,

$$K = \sup_{t \in L^\infty(\Gamma_t, \mathbb{R}^3)} \frac{S_t}{\|t\|} = \sup_{t \in L^\infty(\Gamma_t, \mathbb{R}^3)} \left\{ \frac{\|\gamma_0^*(t)\|}{\|t\|} \right\} = \|\gamma_0^*\| = \|\gamma_0\|$$

where the last equality is the standard equality between the norm of a mapping and the norm of its dual.

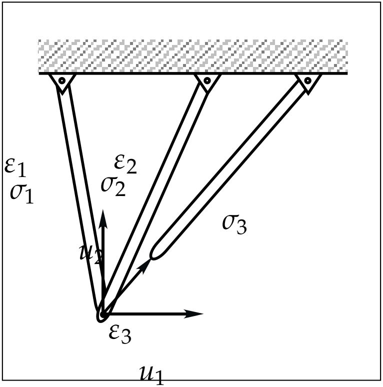
The Optimization Problem for Structures

- We have 2 equations with 3 unknowns.
- Having this one degree of freedom we may look for the solution such that the maximal stress in the bars is the least.



Notation

- $\mathbb{R}^D = \mathcal{W}$ — compatible virtual displacements of the structure.
- $\mathbb{R}^N = \mathcal{S}$ — not necessarily compatible local virtual displacements of all parts of the structure.
- The global forces $f = (f_1, f_2)$ are *dual* —perform work—for vectors in $\mathcal{W}$. $f \in \mathcal{W}^*$.
- The stresses (local forces) $(\sigma_1, \sigma_2, \sigma_3)$ are *dual*, perform work $\sigma_i \varepsilon_i$, for vectors in $\mathcal{S}$. $\sigma \in \mathcal{S}^*$.
- $A \colon \mathcal{W} \longrightarrow \mathcal{S}$ — *interpolation mapping*.
- *Equilibrium:* $\sigma(A(u)) = f(u)$ that we write as $f = A^*(\sigma)$.



- *Remark:* For simplicity, strains are replaced by changes in length and stresses are tensions in the bars.

Stress Optimization for Structures

The optimal maximal stress is given by

$$\boxed{S_f = \inf_{\sigma} \left\{ \max_j \left| \sigma_j \right| ; \ \sigma \textit{ in equilibrium with } f \right\} = \sup_u \frac{|f_i u_i|}{\sum_j \left| A_{jk} u_k \right|} \, .}$$

- *Methods of linear programming are applicable.*

Generalized Stress Concentration Factor for Structures

Theorem

Define the generalized stress concentration factor K by

$$K = \sup_f \left\{ \frac{S_f}{\max_k |f_k|} \right\} = \sup_f \frac{1}{\max_k |f_k|} \left\{ \sup_u \frac{|f_i u_i|}{\sum_j |\varepsilon_j(u)|} \right\}.$$

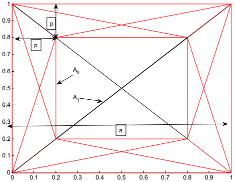
Then,

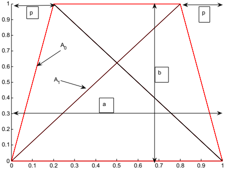
$$\frac{1}{C} = K = \sup_u \frac{\sum_i |u_i|}{\sum_j |A(u)_j|},$$

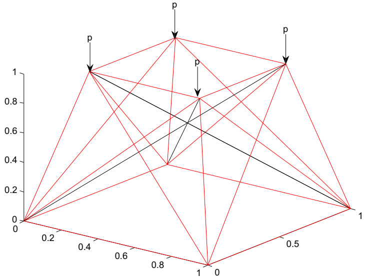
or,

$$\boxed{\frac{1}{C} = K = \|A^{-1}\|.}$$

Truss Examples







Truss Examples (continued)