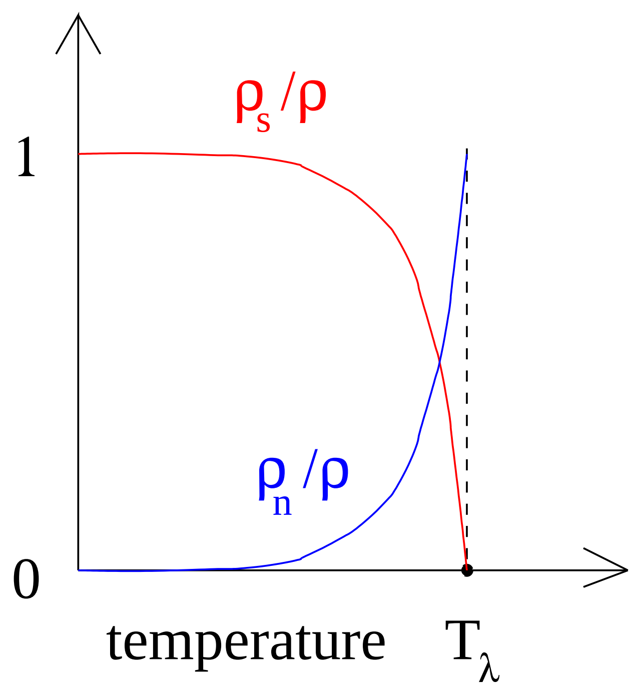
Lecture 1: The vortex filament method

1. **Helium II and quantised vortex lines**
2. **Vortex line as space curves**
3. **Applications**

Normal fluid and superfluid

Helium II: $\rho = \rho_n + \rho_s$

- Normal fluid:
  density ρ_n, velocity $\mathbf{v}_n$,
  entropy S, viscosity η
- Super fluid:
  density ρ_s, velocity $\mathbf{v}_s$,
  no entropy, no viscosity

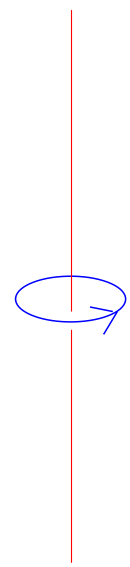


Superfluid vortex lines and quantization of the circulation

Onsager (1948), Feynman (1955), Vinen (1961)

$$\oint_C \mathbf{v}_s \cdot \mathbf{d}\boldsymbol{\ell} = \kappa$$

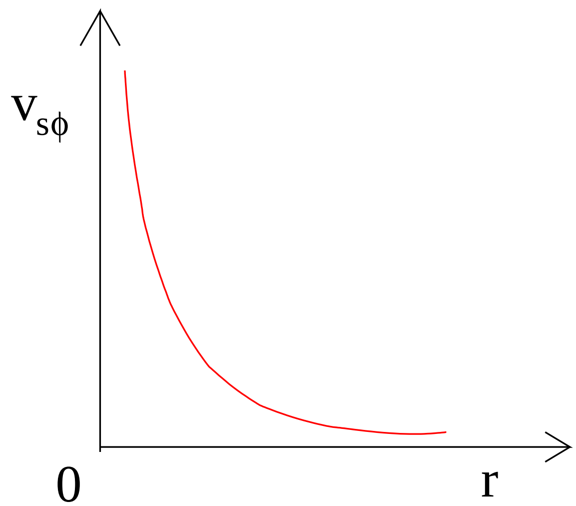
$$\kappa = \frac{h}{m} = 9.97 \times 10^{-4}\ \mathrm{cm}^2/\mathrm{sec}$$



Let the path C be a circle of radius r; then the superfluid velocity around the line is

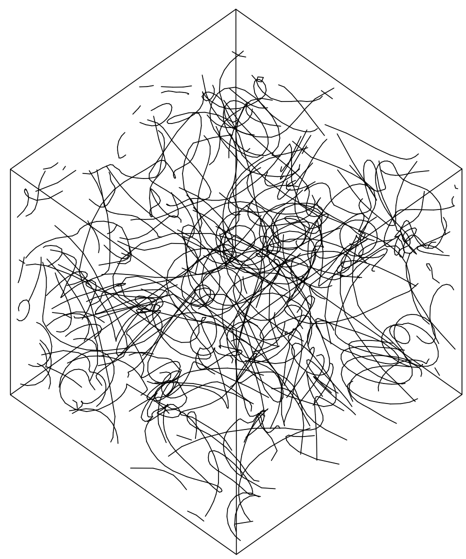
$$v_{s\phi} = \frac{\kappa}{2\pi r}$$

for $r \geq a \approx 10^{-8}$ cm (vortex core radius).



Quantum turbulence

L = Vortex line density = Vortex length/volume
Typical intervortex spacing $\delta \approx L^{-1/2}$

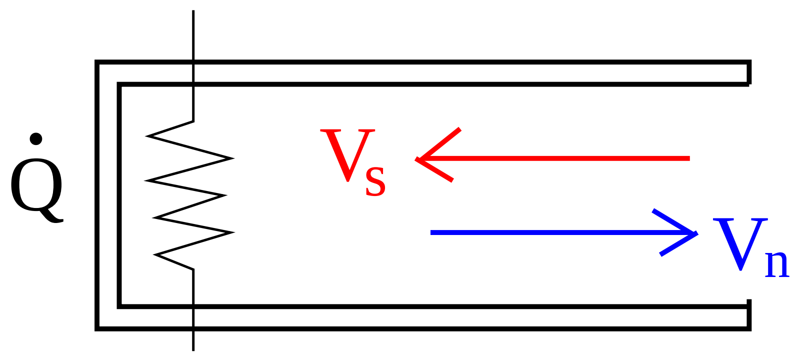


Quantum turbulence can be generated in many ways. Perhaps the most studied form is counterflow turbulence, which is relevant to heat transfer applications.

Laminar counterflow

A channel has a resistor at the closed end and is open to the helium bath at the other end. The heat flux $\dot{Q}$ is carried away by the normal fluid, $v_n = \dot{Q}/(\rho S T)$. Being the channel closed, the mass flux is zero, $\rho_n v_n + \rho_s v_s = 0$, hence the superfluid moves towards the resistor: $v_s = (\rho_n/\rho_s) v_n$, setting up a counterflow velocity proportional to the applied heat flux:

$$v_{ns} = v_n - v_s = \frac{\dot{Q}}{\rho_s S T}$$

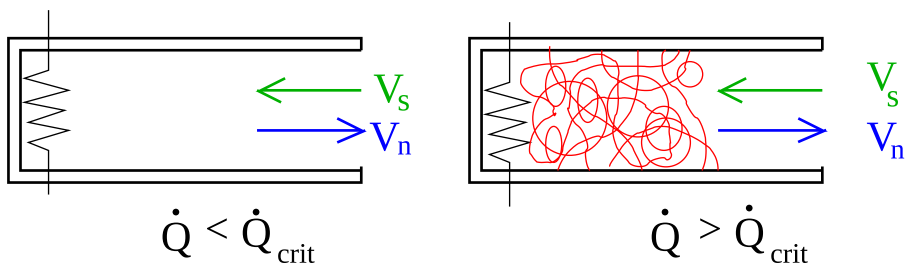


Turbulent counterflow

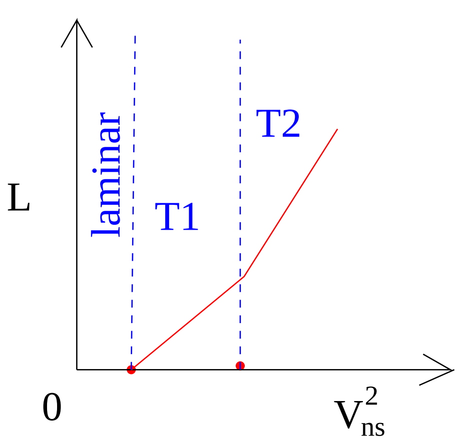
If $\dot{Q}$ (hence $V_{ns} = V_n - V_s$) exceeds a critical value then a tangle of vortex lines appears (superfluid turbulence).

Vortex line density

$$L = \gamma^2 V_{ns}^2$$



- Laminar
- (Weak) T–1 turbulent state
- (Strong) T–2 turbulent state



Vortex lines as space curves

Vortex core $a \approx 10^{-8}$ cm $\ll$ any other length scale, hence we can model vortex lines as space curves, an approach pioneered by Klaus Schwarz.

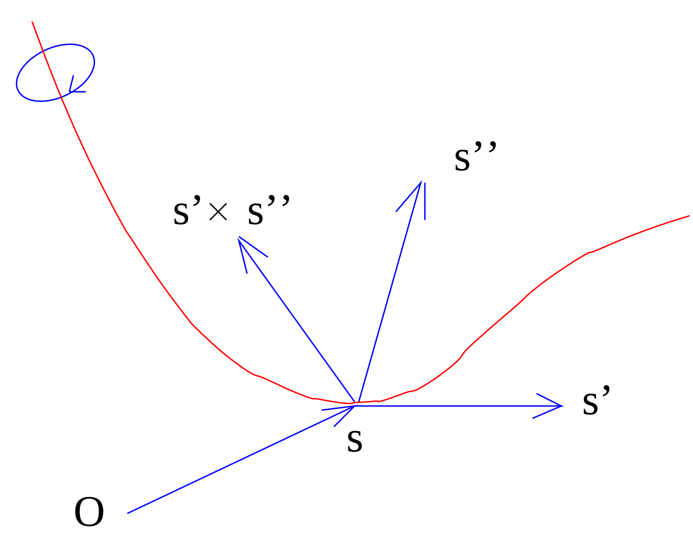
Tangent $\hat{\mathbf{T}}$, normal $\hat{\mathbf{N}}$ and binormal $\hat{\mathbf{B}}$

$\mathbf{s} = \mathbf{s}(\xi) =$ position, $\xi =$ arclength,
$c = |\mathbf{s}''| =$ curvature, $R = 1/c$ radius of curvature

$$\mathbf{s}' = \frac{d\mathbf{s}}{d\xi} = \hat{\mathbf{T}}$$

$$\frac{d\hat{\mathbf{T}}}{d\xi} = c\hat{\mathbf{N}}$$

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}$$



The Biot–Savart law and the LIA

Velocity $\mathbf{v}$, vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$
vector potential $\mathbf{v} = \nabla \times \mathbf{A}$

Poisson equation $\nabla^2 \mathbf{A} = -\boldsymbol{\omega}$
Solution

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi} \int \frac{\boldsymbol{\omega}(\mathbf{x}')d^3x'}{r}$$

where $r = |\mathbf{x} - \mathbf{x}'|$. If $\boldsymbol{\omega}(\mathbf{x}')d^3x' = \kappa \mathbf{d}\boldsymbol{\ell}(\mathbf{x}')$ then

$$\mathbf{A} = \frac{\kappa}{4\pi} \oint \frac{1}{r} \mathbf{d}\boldsymbol{\ell}'$$

hence the Biot–Savart law

$$\mathbf{v}(\mathbf{x}) = -\frac{\kappa}{4\pi} \oint \frac{(\mathbf{x} - \mathbf{x}')}{r^3} \times \mathbf{d}\boldsymbol{\ell}'$$

Local induction approximation (LIA)

$$\mathbf{v}(\mathbf{x}) \approx \beta \mathbf{s}' \times \mathbf{s}''$$

$$\beta = \frac{\kappa}{4\pi R} \ln (R/a_{eff})$$

where $R = 1/|\mathbf{s}''|$.

Problem: use LIA or BS ?

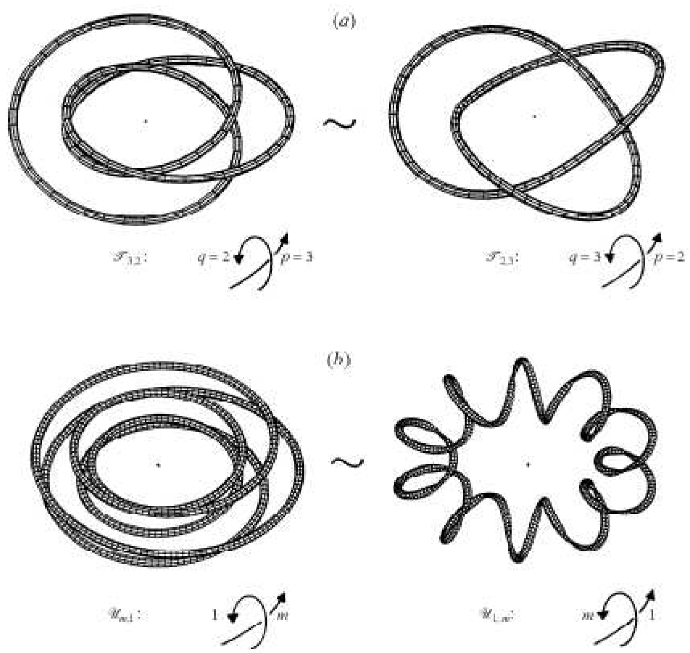
Vortex knots

Motivations:

- not as trivial as a vortex ring (unknot) but not as difficult as a vortex tangle
- Lord Kelvin and vortex theory of matter

Trefoil $\mathcal{T}_{p,q}$: a closed non–self–intersecting curve which cuts the meridian at $p > 1$ points and a longitude at $q > 1$ points (p and q being relatively prime integers). The winding number $w = q/p$ is a topological invariant. Top left: $\mathcal{T}_{3,2}$. Top right: $\mathcal{T}_{2,3}$

Special cases: if $p = 1$ or $q = 1$ we have an unknot. $\mathcal{T}_{m,1}$ is a toroidal coil (bottom left), and $\mathcal{T}_{1,m}$ is a poloidal coil (bottom right).



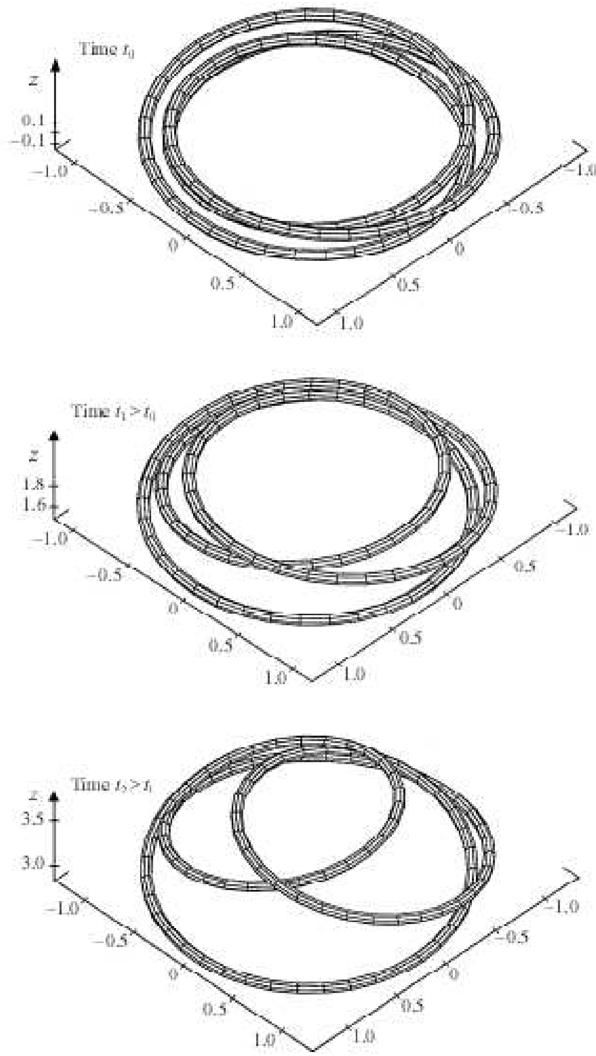
Linear stability theory

Ricca and Kida proved that $\mathcal{T}_{p,q}$ is stable under LIA if and only if $q > p$, that is if and only if $w > 1$.

We expect $\mathcal{T}_{2,3}$ to be stable and $\mathcal{T}_{3,2}$ to be unstable.

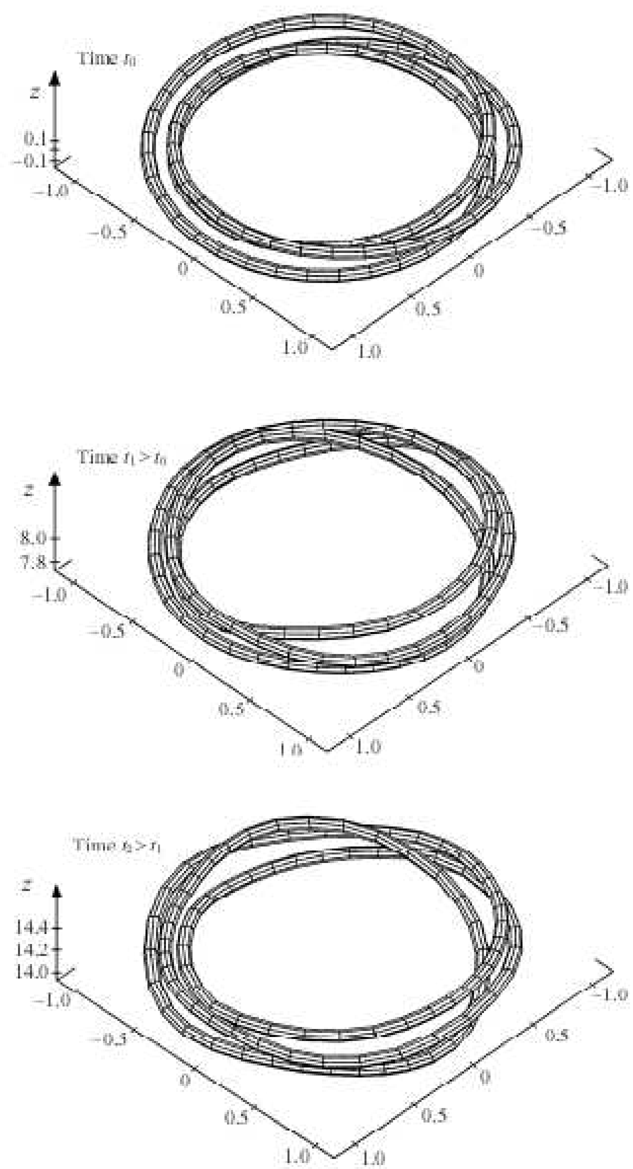
Numerical evolution of $\mathcal{T}_{3,2}$ under LIA

Unstable as predicted.



Numerical evolution of $\mathcal{T}_{3,2}$ under BS

Surprise: the vortex knot is structurally stable.



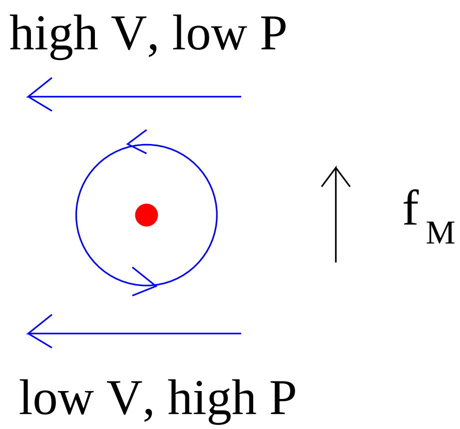
Schwarz's equation

$\mathbf{v}_L$ = vortex line velocity
Forces acting on unit length of vortex line:

- Magnus force:

$$\mathbf{f}_M = \rho_s \boldsymbol{\kappa} \times (\mathbf{v}_L - \mathbf{v}_s)$$

$(\boldsymbol{\kappa} = \kappa \mathbf{s}' = \kappa \hat{\boldsymbol{\omega}})$



- Drag force:

$$\mathbf{f}_D = \rho_s \kappa \alpha \mathbf{s}' \times [\mathbf{s}' \times (\mathbf{v}_s - \mathbf{v}_n)] + \alpha' \mathbf{s}' \times (\mathbf{v}_s - \mathbf{v}_n)$$

Mutual friction coefficients

$$\alpha = \frac{B\rho_n}{2\rho}, \qquad \alpha' = \frac{B'\rho_n}{2\rho}$$

No inertia, so $\mathbf{f}_M + \mathbf{f}_D = \mathbf{0}$ per unit length, hence

$$\mathbf{v}_L = \frac{d\mathbf{s}}{dt} = \mathbf{v}_s - \alpha \mathbf{s}' \times (\mathbf{v}_s - \mathbf{v}_n) + \alpha' \mathbf{s}' \times [\mathbf{s}' \times (\mathbf{v}_s - \mathbf{v}_n)]$$

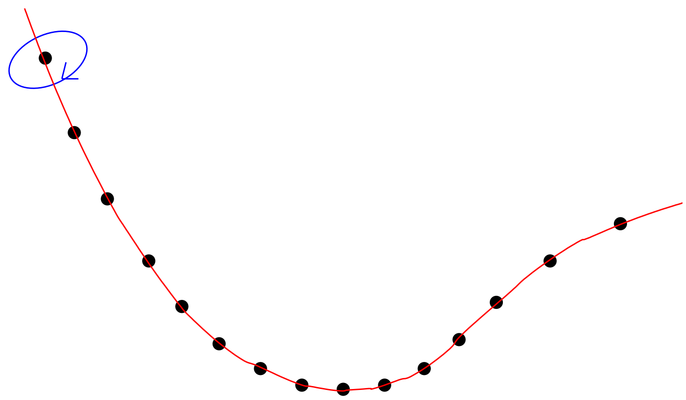
where
$\mathbf{v}_s = \mathbf{v}_s^{self} + \mathbf{v}_s^{ext}$
$\mathbf{v}_s^{self}$ self–induced velocity (Biot–Savart or LIA)
$\mathbf{v}_s^{ext}$ externally applied superflow.

Under LIA $\mathbf{v}_s^{self} = \beta \mathbf{s}' \times \mathbf{s}''$ and Schwarz's equation reduces to

$$\frac{d\mathbf{s}}{dt} = \mathbf{v}_s^{ext} + \beta \mathbf{s}' \times \mathbf{s}'' + \alpha \mathbf{s}' \times (\mathbf{v}_{ns}^{ext} - \beta \mathbf{s}' \times \mathbf{s}'') \\ - \alpha' \mathbf{s}' \times [\mathbf{s}' \times (\mathbf{v}_{ns}^{ext} - \beta \mathbf{s}' \times \mathbf{s}'')]$$

where $\mathbf{v}_{ns}^{ext} = \mathbf{v}_n^{ext} - \mathbf{v}_s^{ext}$
and $\mathbf{v}_n^{ext}$ is the externally applied normal flow.

Numerical discretization



Space discretization: variable meshing along filaments to resolve regions of high curvature and same computational power if filaments are less curved.

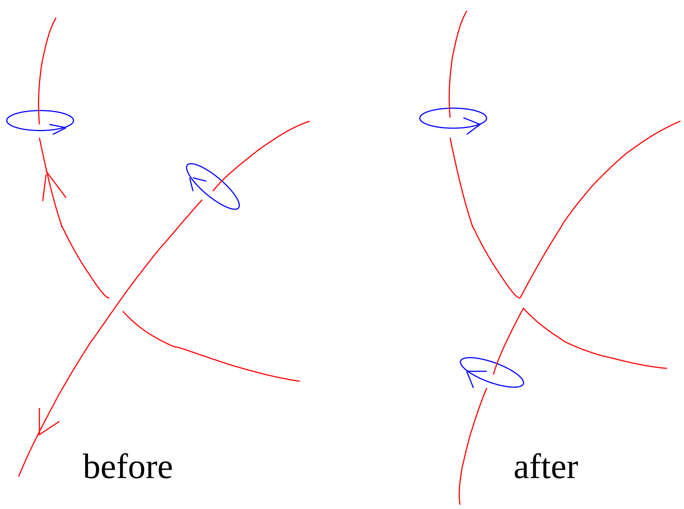
Time discretization: explicit. Can be adaptive.

Computer cost increases with N^2, where N is the number of vortex points.

Reconnections

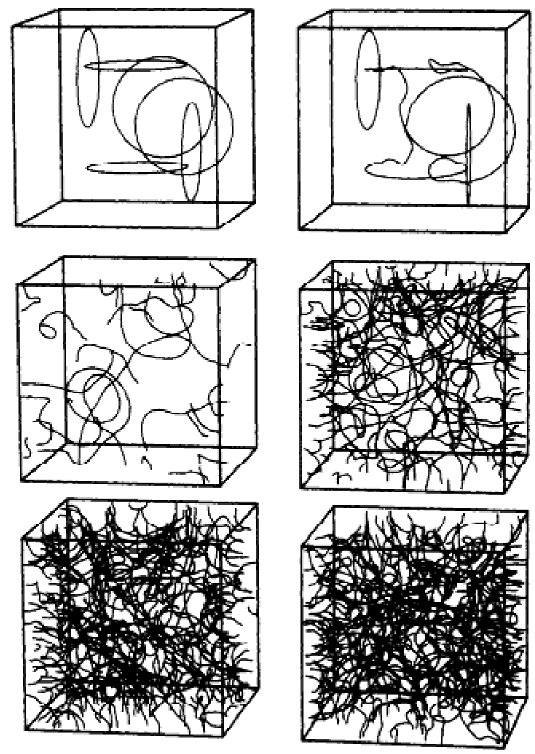
Schwarz recognized that they are essential for turbulence

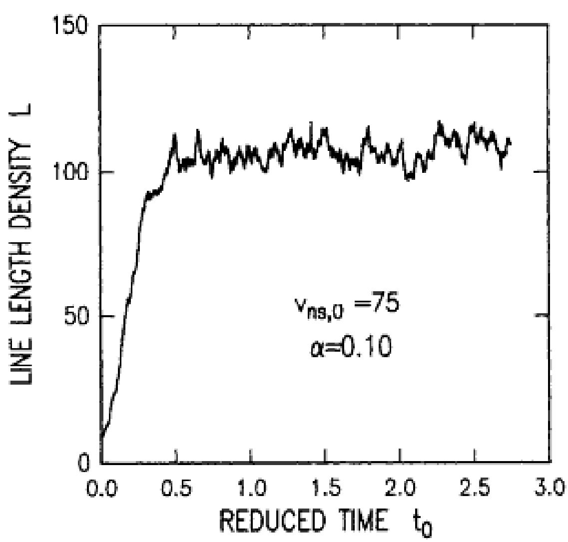
Reconnections are performed ad hoc by the numerical algorithm



Later Koplik and Levine proved vortex reconnections using the NLSE method

KW Schwarz, PRB 38, 2398, 1988: First self–sustaining reconnecting vortex tangle, $L \sim V_{ns}^2$





Limitations of the approach of Schwarz

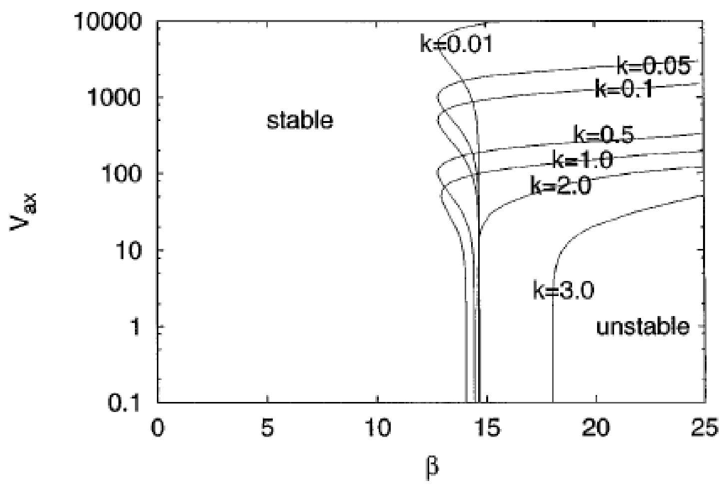
- The normal fluid velocity is imposed externally (bad if the normal fluid is turbulent)
- The model is uncompressible (bad at low T)

Stability of the normal fluid

Normal flow with parabolic profile of amplitude V_{ax} in cylindrical pipe of radius R forced by vortex line density L_0

$$\rho_n \left(\frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n \cdot \nabla)\mathbf{v}_n \right) = -\frac{\rho_n}{\rho} \nabla p - \rho_s S \nabla T$$
$$+\eta \nabla^2 \mathbf{v}_n - \left(\frac{B \rho_n \rho_s}{2\rho} \right) \left(\frac{2}{3} \right) \kappa L_0 (\mathbf{v}_n - \mathbf{v}_s)$$

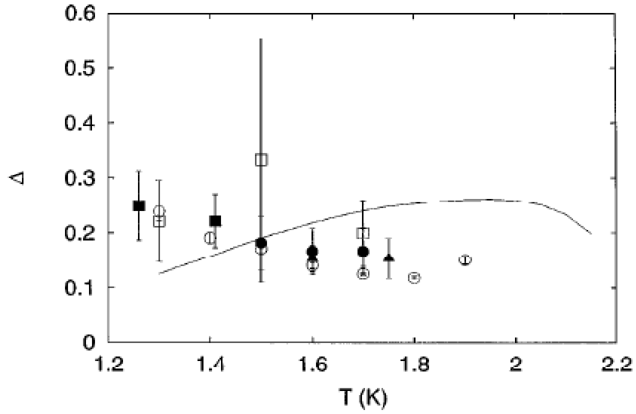
Stability boundary for $m = 1$ azimuthal mode and various axial wavelengths k (Melotte and Barenghi, PRL 80, 4181, 1998):



$$\beta = \frac{B \rho_s \kappa L_0 R^2}{3 \rho \nu_n}$$

The fact that many modes becomes unstable at the same L_0 suggests the onset of turbulence.

Comparison with experiments

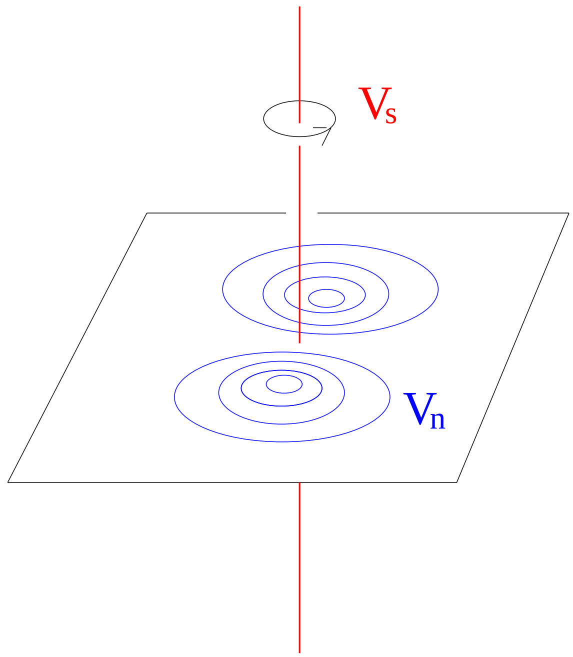


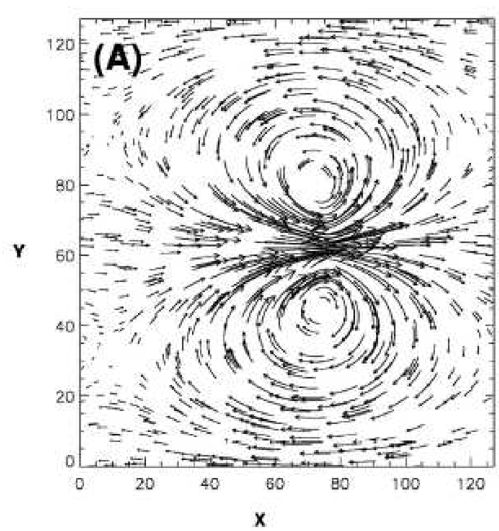
$$\Delta = L_0^{-1/2} R^{-1}$$

Conclusion: in the T-2 state both superfluid and normal fluid are turbulent

Perturbation of the normal fluid

2–dim action of a vortex line on the normal fluid





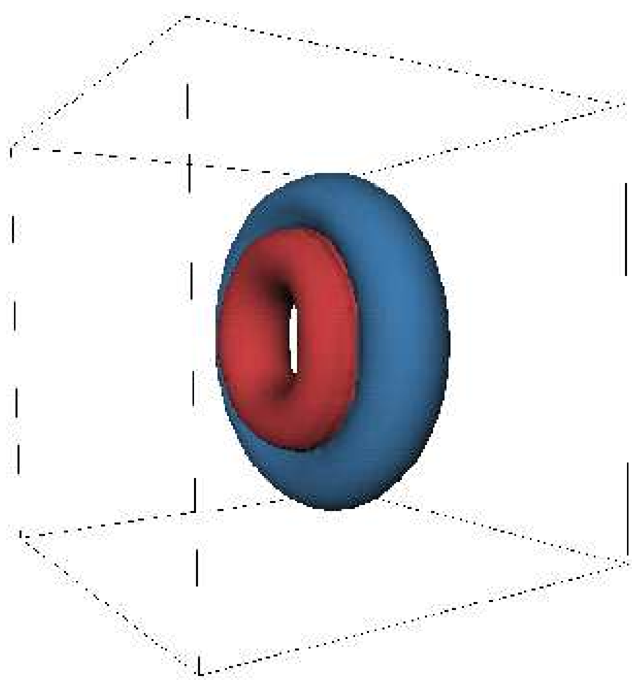
Idowu, Willis, Barenghi and Samuels, PRB 62, 3409, 2004

Self–consistent vortex dynamics

$$\begin{tikzcd} V_s \arrow[r, bend left, "\text{Navier−Stokes}"] & V_n \arrow[l, bend left, "\text{Schwarz}"] \end{tikzcd}$$

Implement Schwarz's equation with a Navier–Stokes solver for $\mathbf{v}_n$.

A superfluid vortex ring carries along two normal fluid rings of opposite polarity (Kivotides, Barenghi and Samuels, Science 290, 777, 2000)



Scaling

$$\mathbf{v}_s^{self} \approx \beta \mathbf{s}' \times \mathbf{s}'' \text{under LIA}$$

$$\beta = \frac{\kappa}{4\pi} \ln\left(\frac{R}{a_{eff}}\right) \qquad R \approx 1/|\mathbf{s}''|$$

Consider a homogeneous vortex tangle. Assume that the only variables which matter are β and the counterflow V_{ns} which creates the tangle.

$[V_{ns}] = \text{cm/sec} \qquad [\beta] = \text{cm}^2/\text{sec} \qquad [\beta/V_{ns}] = \text{cm}$

The vortex line density

$$L = \frac{1}{V} \int d\xi = \frac{\Lambda}{V}$$

(where Λ = length V = volume) has dimension $1/\text{cm}^2$, thus

$$L = C_L \left(\frac{V_{ns}}{\beta}\right)^2$$

for some dimensionless C_L, hence $L \sim V_{ns}^2$, as observed in the experiments.

Similarly the average radius of curvature R scales as $R = 1/|\mathbf{s}''| \sim L^{-1/2}$.

Vinen's equation

Vinen (1957) argued that the steady–state vortex line density L_0 in counterflow turbulence is due to the balance of production and destruction processes, which he modelled as the growth of vortex rings and the annihilation of opposite oriented vortex lines, obtaining

$$\frac{dL}{dt} = \frac{\chi_1 B \rho_n}{2\rho} V_{ns} L^{3/2} - \frac{\chi_2 \kappa}{2\pi} L^2$$

where χ_1 and χ_2 are dimensionless and $\mathcal{O}(1)$.

Steady state solution:

$$L_0 = \gamma^2 V_{ns}^2$$

where

$$\gamma = \frac{\pi B \rho_n \chi_1}{\kappa \rho \chi_2}$$

It can be shown that Vinen's equation arises from vortex dynamics. Infact

$$\frac{dL}{dt} = \frac{1}{V} \int d\xi \mathbf{s}' \cdot \frac{d\mathbf{s}'}{dt}$$

Use Schwarz equation (LIA) and assume isotropic configuration:

$$\begin{aligned}
\frac{dL}{dt} = \frac{\alpha}{V} \mathbf{v}_{ns}^{ext} \cdot \int d\xi \mathbf{s}' \times \mathbf{s}'' - \frac{\alpha\beta}{V} \int d\xi |\mathbf{s}''|^2 - \frac{\alpha'}{V} \mathbf{v}_{ns}^{ext} \cdot \int d\xi \mathbf{s}'' \\
= \frac{\alpha}{V} \mathbf{v}_{ns}^{ext} \cdot \int d\xi \mathbf{s}' \times \mathbf{s}'' - \frac{\alpha\beta}{V} \int d\xi |\mathbf{s}''|^2 \\
= \alpha \mathbf{v}_{ns}^{ext} \cdot \mathbf{I}_\ell L^{3/2} - \alpha\beta c_2^2 L^2
\end{aligned}$$

which has the form of Vinen equation, where

$$\mathbf{I}_\ell = \frac{1}{V L^{3/2}} \int d\xi \mathbf{s}' \times \mathbf{s}''$$

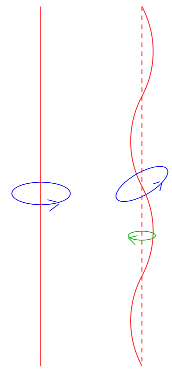
$$c_2^2 = \frac{1}{V L^2} \int d\xi |\mathbf{s}''|^2$$

Note that the second term at the RHS is always negative, so

$$\int d\xi \mathbf{s}' \times \mathbf{s}'' \neq 0$$

hence some anisotropy is actually necessary to maintain a steady state.

Kelvin waves



Consider a vortex line with the shape of a helix (Kelvin wave) of amplitude ϵ:

$$\mathbf{s} = (\epsilon \cos\phi; \epsilon \sin\phi; z), \qquad \phi = kz - \omega t$$

If $\epsilon \ll 1$ then $z \approx \xi$. Using LIA, the self–induced velocity is

$$\mathbf{v}_s^{self} = \beta \mathbf{s}' \times \mathbf{s}'' \approx \beta k^2 \epsilon (\sin\phi; -\cos\phi; 0)$$

In the absence of friction, the equation of motion is

$$\mathbf{v}_L = d\mathbf{s}/dt = \mathbf{v}_s^{self}$$

Assuming ϵ constant, we have

$$\omega = \beta k^2$$

which is the dispersion relation of Kelvin waves.

Glaberson instability

In the presence of friction (neglecting α') the equation of motion is

$$\frac{d\mathbf{s}}{dt} = \mathbf{v}_s^{self} + \alpha \mathbf{s}' \times (\mathbf{v}_{ns}^{ext} - \mathbf{v}_s^{self})$$

Assuming $\mathbf{v}_{ns}^{ext} = (0; 0; V_{ns})$ and $\epsilon = \epsilon(t)$ we have

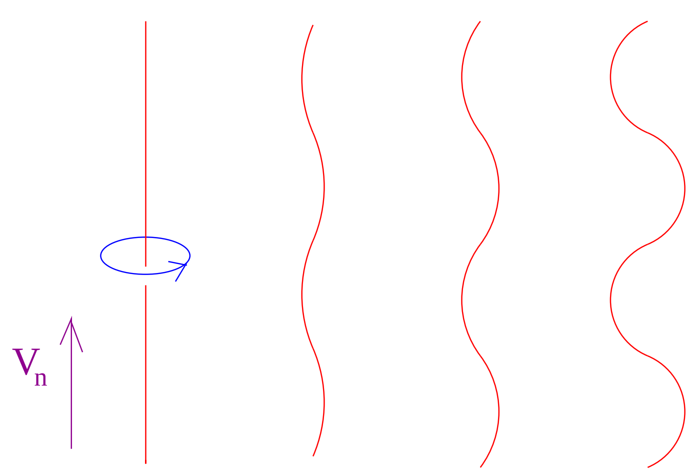
$$\frac{d\epsilon}{dt} = \alpha(kV_{ns} - \beta k^2)\epsilon$$

hence the amplitude of the wave is

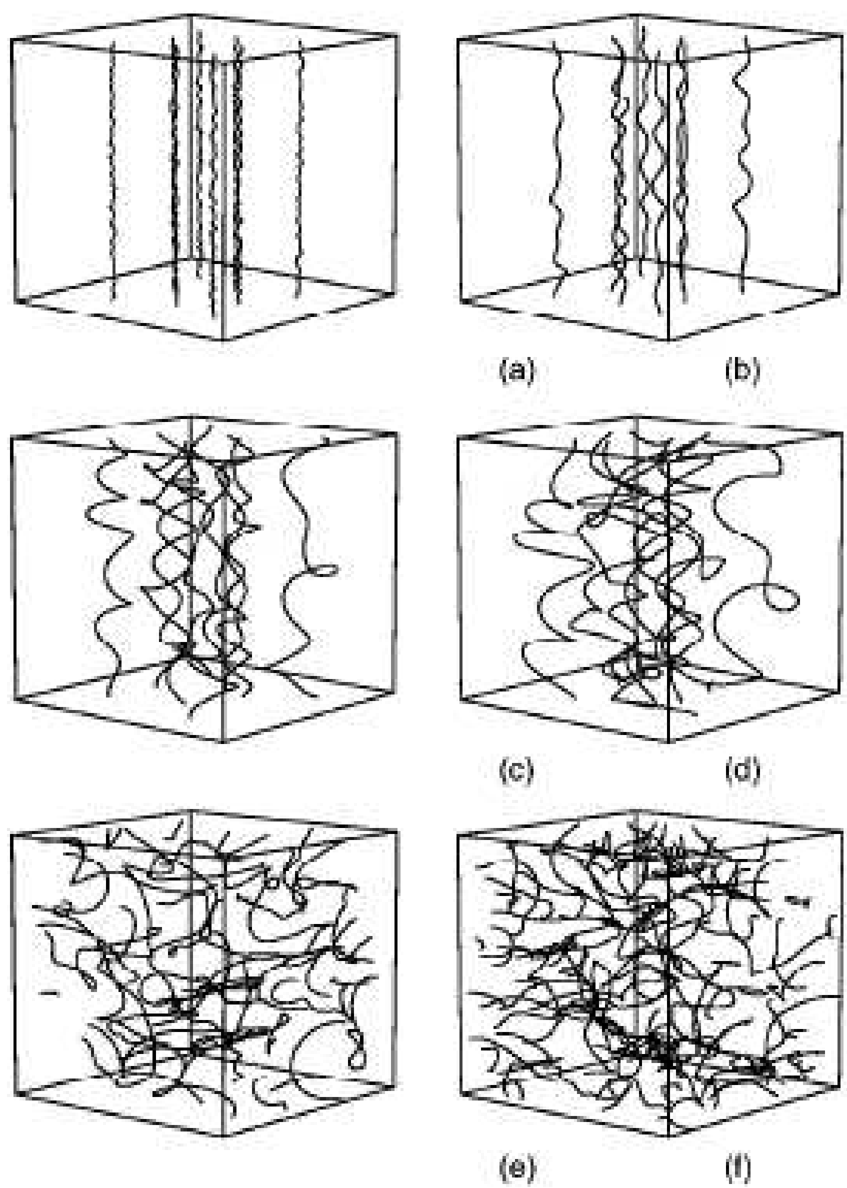
$$\epsilon(t) = \epsilon(0)e^{\sigma t}$$

where $\sigma = \alpha(kV_{ns} - \beta k^2)$ is the growth rate.

If $\sigma > 0$ the Kelvin wave is unstable and grows exponentially with time. Given V_{ns}, the largest growth rate is $\sigma_{max} = \alpha V_{ns}^2/(4\beta)$ and occurs at $k_{max} = V_{ns}/(2\beta)$



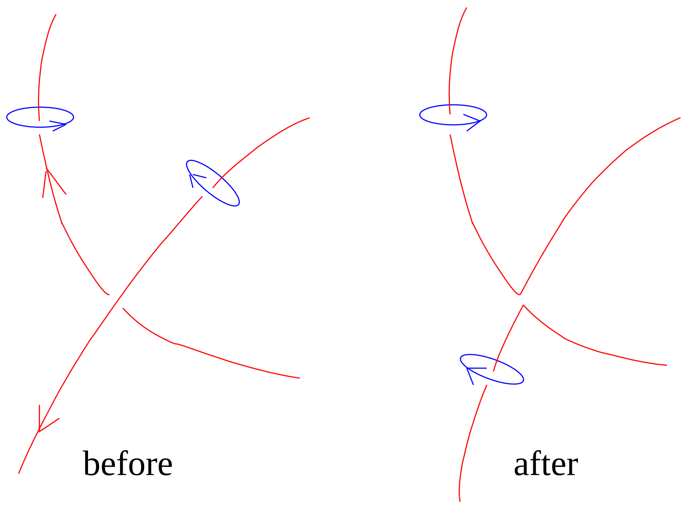
Tsubota, Araki and Barenghi, PRL 90, 205301, 2003. Rotating vortex array in the presence of an axial flow parallel to the vortices. Note how Kelvin waves grow, until $\epsilon \approx \delta$, at which point reconnections occur, and a tangle is formed.



Energy dissipation near absolute zero

Experiments show that at temperatures of few mK's, so low that the normal fluid is virtually absent, vorticity decays (Davis, Hendry and McClintock, Physica B 280, 43, 2000). It is thought that, in the absence of viscosity at such low T, the energy sink is sound. Classically, rotating vortices radiate sound, provided that the wavenumber k is large enough for this effect to be important (Vinen, PRB 61, 1410, 2000). What creates the necessary large k ?

Kelvin wave cascade

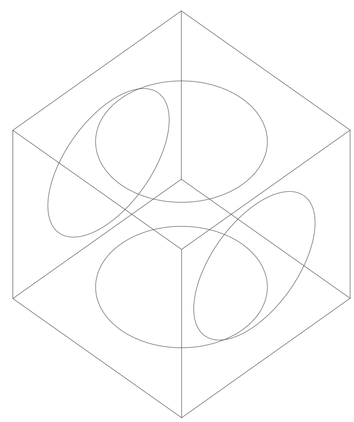


A vortex reconnection creates a cusp, which launches Kelvin waves. The nonlinear interaction of the waves generates higher and higher k, until k is big enough that energy can be efficiently radiated away.

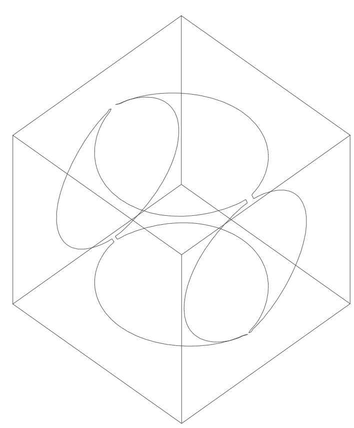
Kivotides, Vassilicos, Samuels and Barenghi (PRL, 86, 3080, 2001) studied collision of vortex rings using the vortex filament model and found evidence for this Kelvin wave cascade. The resulting spectrum scales as $E(k) \sim k^{-1}$.

Time sequence:

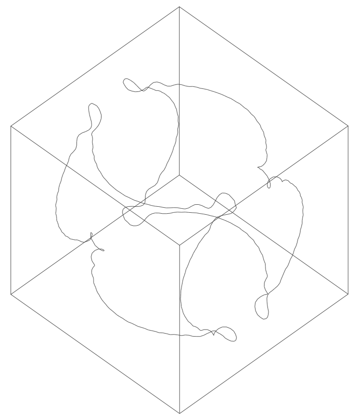
t=0.000



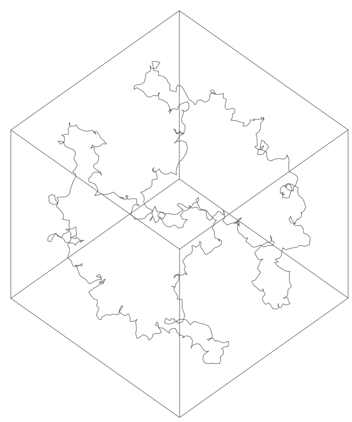
t=0.059



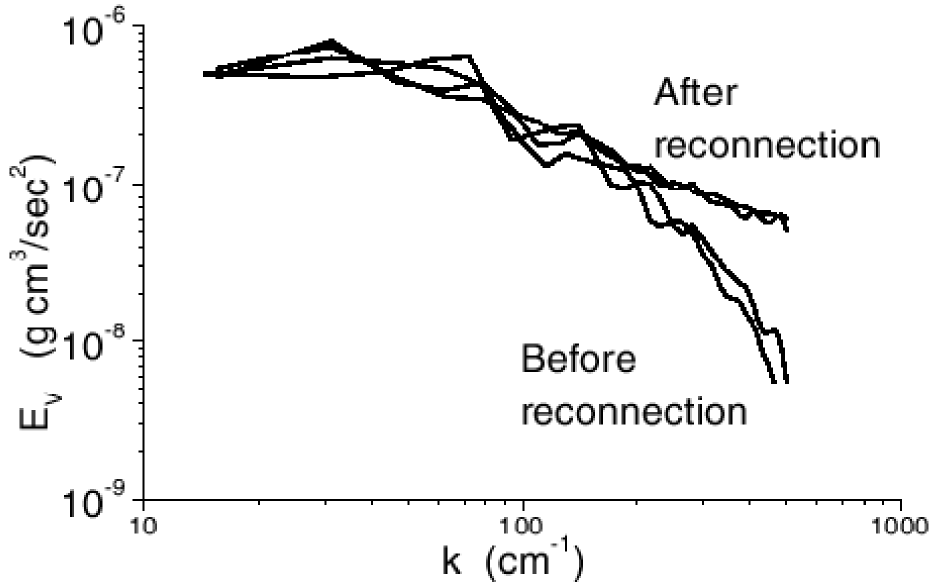
t=0.069



t=0.129



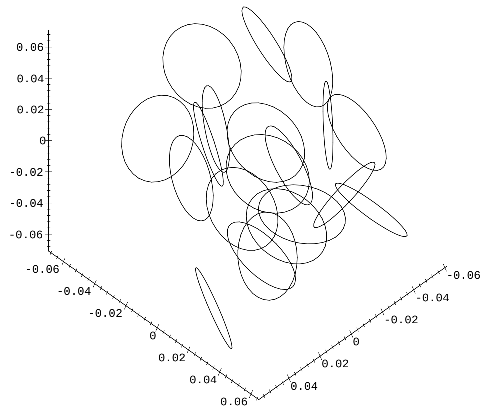
Energy spectrum before and after the reconnections.



Diffusion of quantised vorticity

In an classical viscous Navier–Stokes fluid vorticity can diffuse in space. In a classical inviscid Euler fluid there is no viscous diffusion and reconnections cannot occur, so a packet of vorticity initially localised in a region of space cannot undo the links. In a superfluid vortex lines can reconnect and change the topology, so the packet can "diffuse" away by evaporation of small loops (Barenghi and Samuels, PRL 89, 155302, 2002).

Note that in the following time sequence the scale of the region containing the vortex loops becomes larger and larger.



0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.06
-0.04
-0.02
0
0.02
0.04
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-0.04
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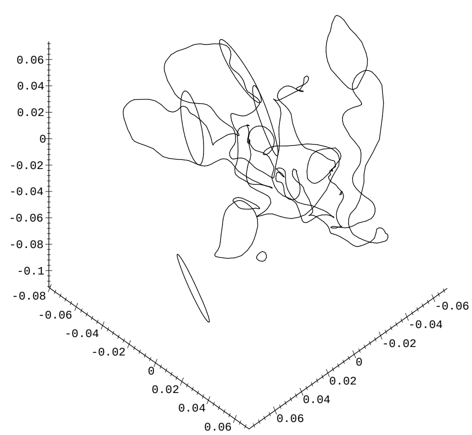
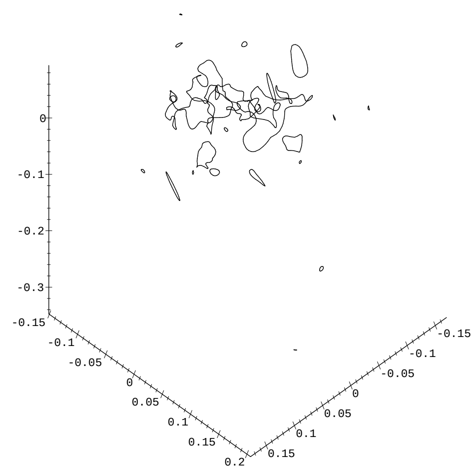
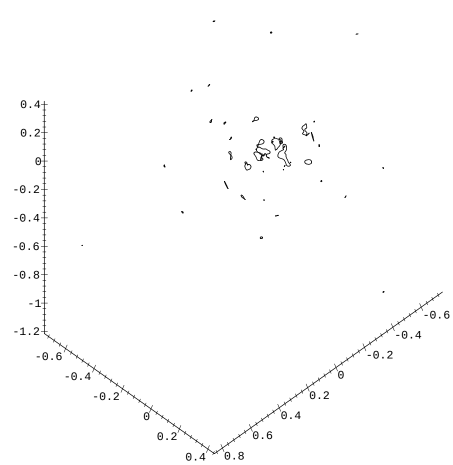
0
0.02
0.04



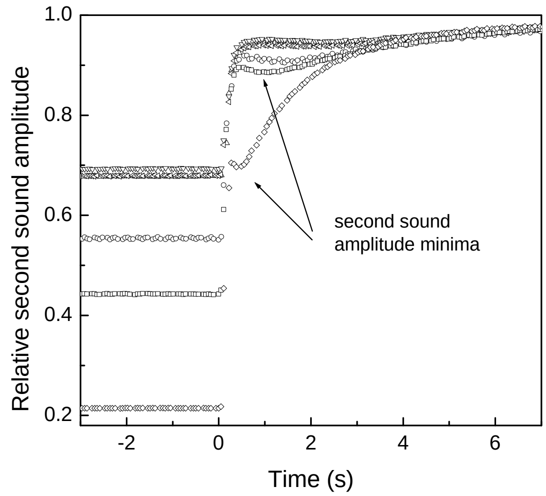
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
-0.06
-0.04
-0.02
0
0.02
0.04
0.06

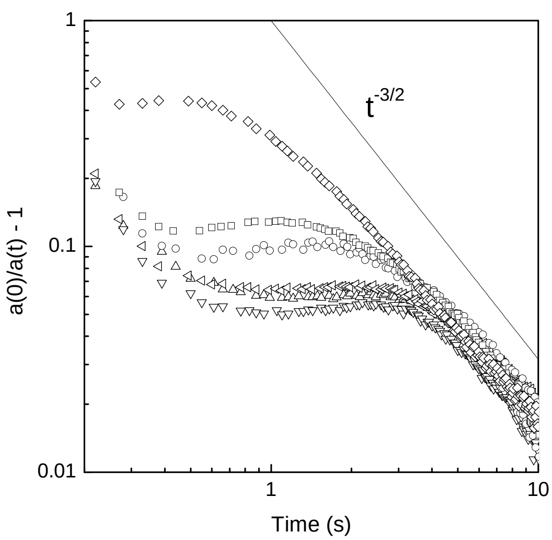

0
-0.1
-0.2
-0.3
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
0.2
-0.15
-0.1
-0.05
0
0.05
0.1
0.15


0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-0.6
-0.4
-0.2
0
0.2
0.4
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8

Decay of counterflow turbulence

Second sound attenuation data





The second sound wave equation is

$$\ddot{\mathbf{q}} + (2 - B')\mathbf{\Omega} \times \dot{\mathbf{q}} - B\hat{\mathbf{\Omega}} \times (\mathbf{\Omega} \times \dot{\mathbf{q}}) = c^2\nabla(\nabla \cdot \mathbf{q})$$

where $\mathbf{q} = \mathbf{v}_n - \mathbf{v}_s$ and $c =$ second sound speed

Assume sound propagation in the x direction:
$\mathbf{q} = e^{i\omega t - ikx}(q_x, q_y, 0)$

Vorticity: $\mathbf{\Omega} = (\Omega \sin(\theta), 0, \Omega \cos(\theta))$

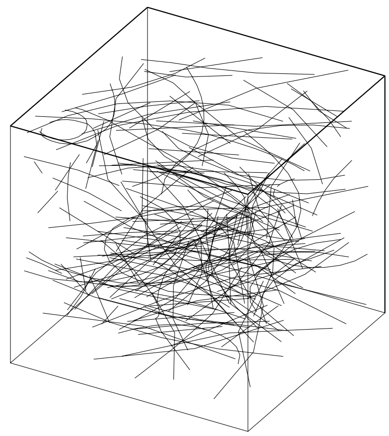
$\theta =$ angle from the z direction.

If $\Omega/\omega \ll 1$, then

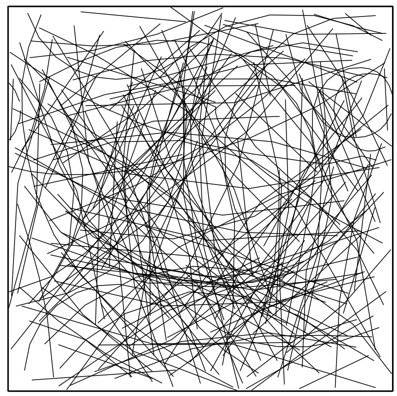
$$k = \left(\frac{\omega}{c}\right)[1 - \frac{i\Omega B \cos^2(\theta)}{2\omega}]$$

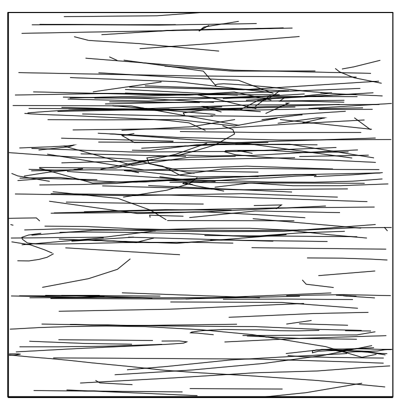
The negative imaginary part of k is the attenuation coefficient, where $\gamma = \pi/2 + \theta$ is the angle between the direction of the vorticity and that of the second sound.

Tangle at beginning of decay stage

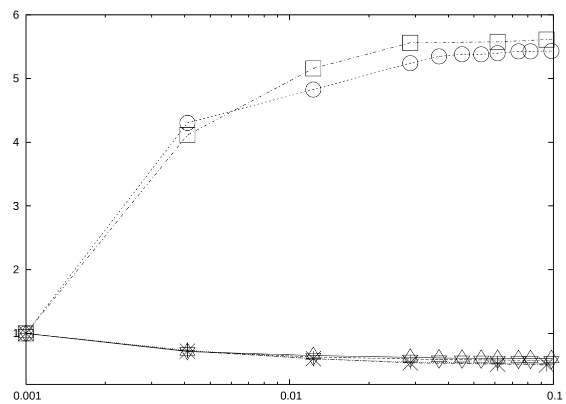


xy and xz projections.



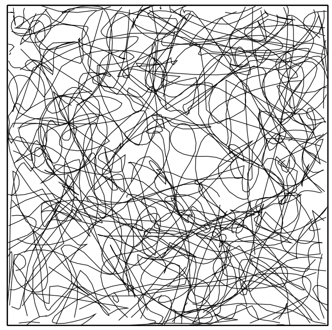


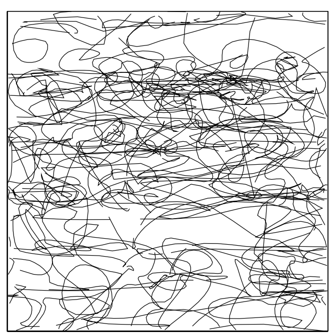
Relative projected lengths $\Lambda_x(t)/\Lambda_x(0)$ (a, triangles pointing up), $\Lambda_y(t)/\Lambda_y(0)$ (b, triangles pointig down) and $\Lambda_z(t)/\Lambda_z(0)$ (c, squares) vs time. Also plotted are the same quantities $\Lambda_x(t)/\Lambda_x(0)$ (d, crosses), $\Lambda_y(t)/\Lambda_y(0)$ (e, diagonal crosses) and $\Lambda_z(t)/\Lambda_z(0)$ (f, circles) but computed using the LIA starting from the same initial state.



Decaying tangle

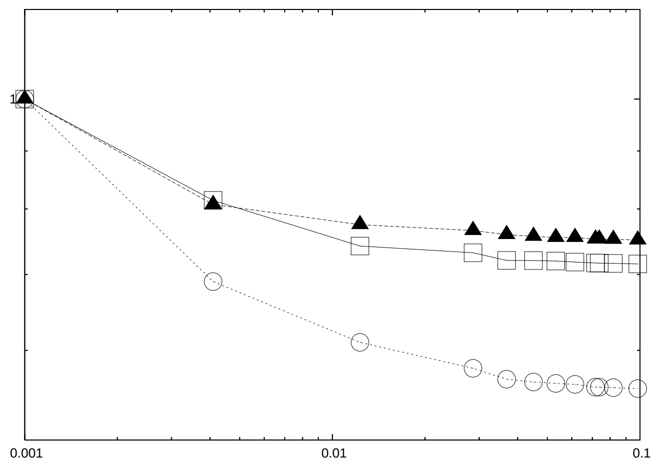
xy and xz projections.





Define $\Delta S_x = \Lambda \sin^2(\gamma_1) = (\Lambda_y^2 + \Lambda_z^2)/\Lambda$ and similarly for ΔS_y, ΔS_z, then integrate over the tangle to get S_x, S_y, S_z, which are proportional to the the observed second sound signal propagating along x, y and z.

Plot of S_x/S_x^0 (triangles pointing up), S_y/S_y^0 (squares) and S_z/S_z^0 (circles) vs time. Note the relatively slower decay of the transverse signals S_x and S_y.



The same but for longer time using LIA. Compare to experimental data.