Equilibrium and non-equilibrium physics of low dimensional quantum gases

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Inhomogeneous superfluids Pisa July 2007

1D is very special

- No single-particle excitations at low energies.
- Absence of long-range order. No BEC.
- Non-refractive scattering: Integrability.
- Absence of thermalization.

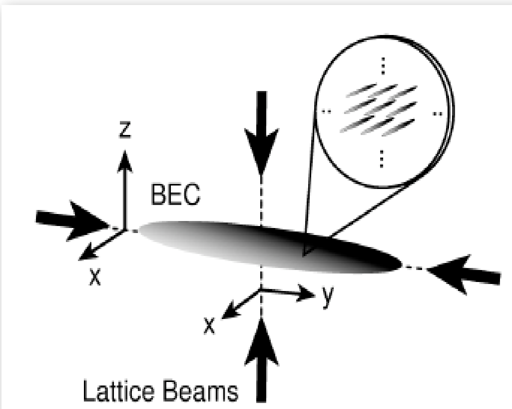
What happens when one goes from 1D to $D > 1$?

Second lecture

1. Competing phases in optical lattices: quasi-1D lattices.
2. 2D Bose gas: BKT phenomena. BKT in the presence of Josephson coupling.
3. Fast rotation: quantum Hall regime. Edge excitations and Topological order in vortex liquids.

Low-energy description of quasi-1D OL's

Very anisotropic lattice ($J_x >> J$)



[AF Ho, MAC & T Giamarchi, PRL 92 (2004)]

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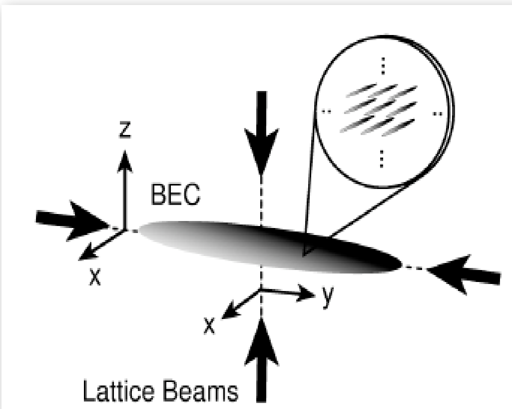

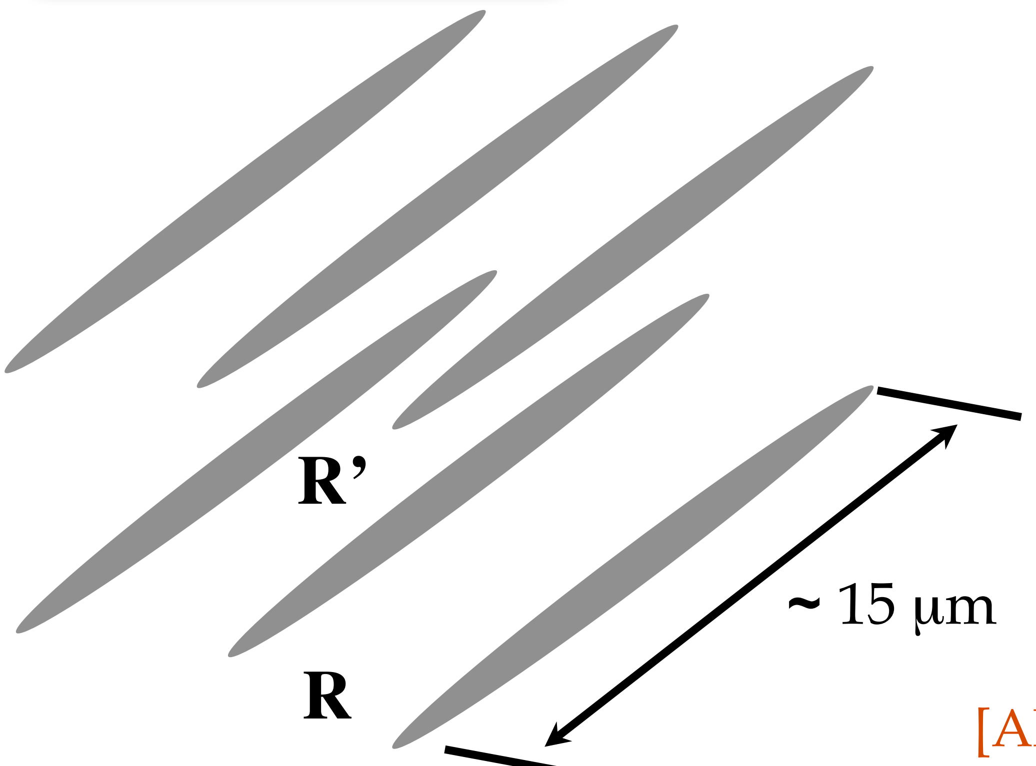

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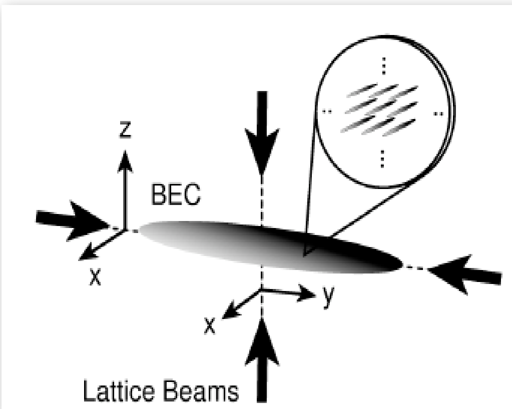

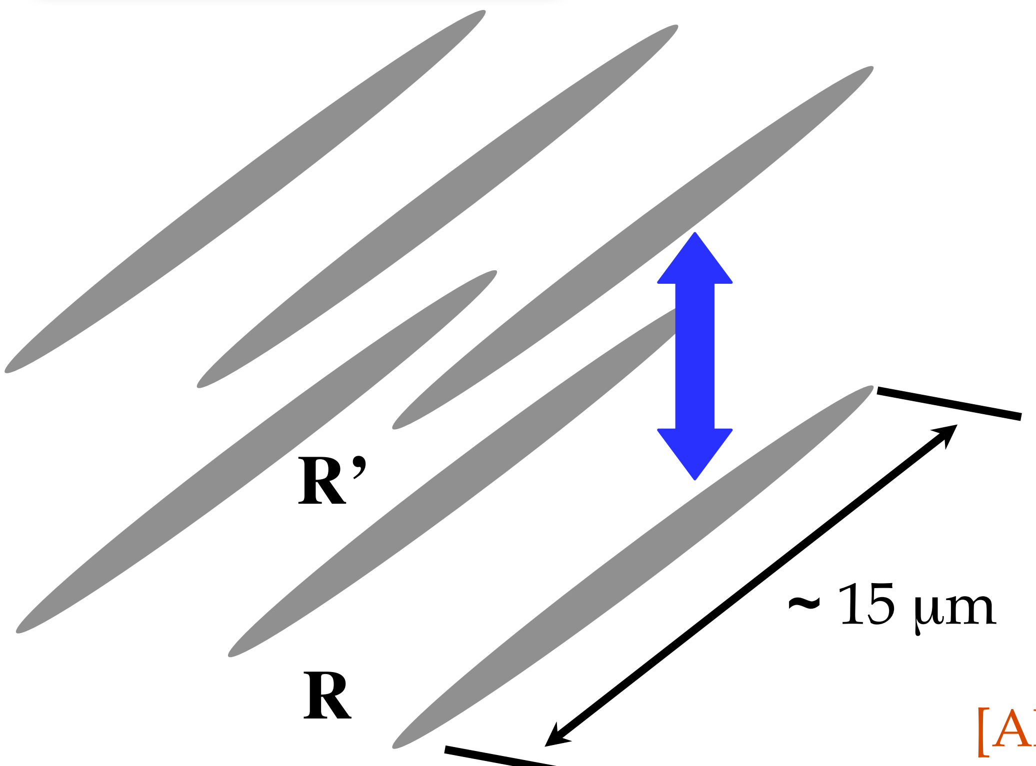

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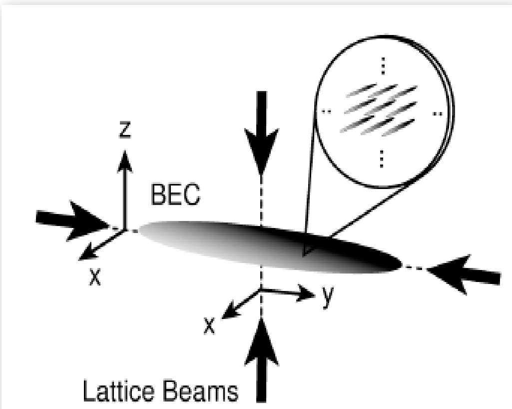

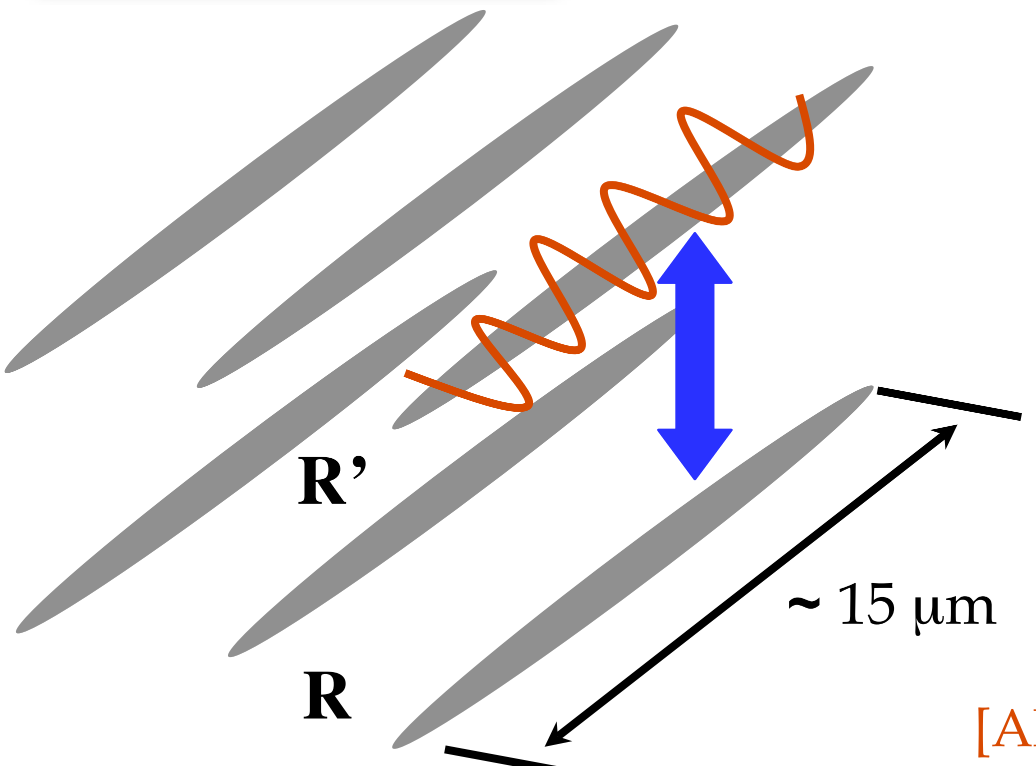

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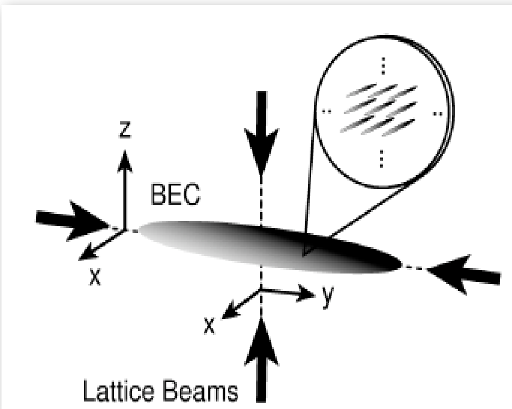

$$
\begin{aligned}
H_{\mathrm{eff}} &= \frac{\hbar v_s}{2\pi}\sum_{\mathbf{R}}\int_0^L dx\left[K\left(\partial_x\theta_{\mathbf{R}}\right)^2+K^{-1}\left(\partial_x\phi_{\mathbf{R}}\right)^2\right]\\
&+\frac{\hbar g_u}{2\pi a^2}\sum_{\mathbf{R}}\int_0^L dx\,\cos(2\phi_{\mathbf{R}})\\
&-\frac{\hbar g_J}{2\pi a^2}\sum_{\langle\mathbf{R},\mathbf{R}'\rangle}\int_0^L dx\,\cos\left(\theta_{\mathbf{R}}-\theta_{\mathbf{R}'}\right)
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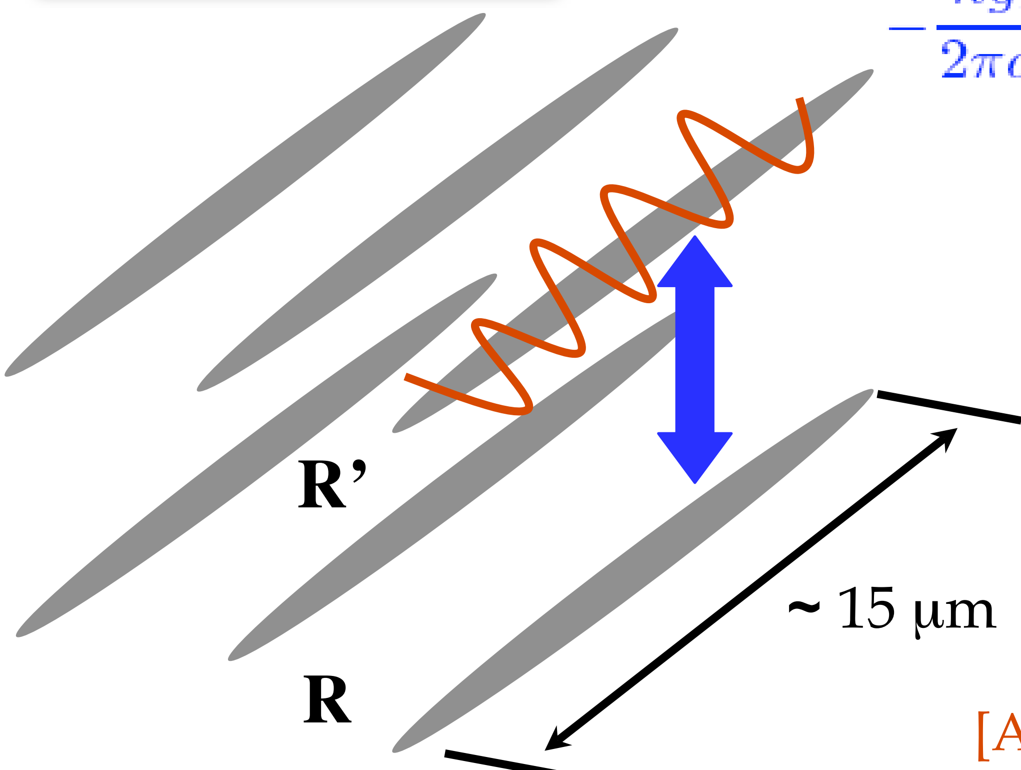
$$



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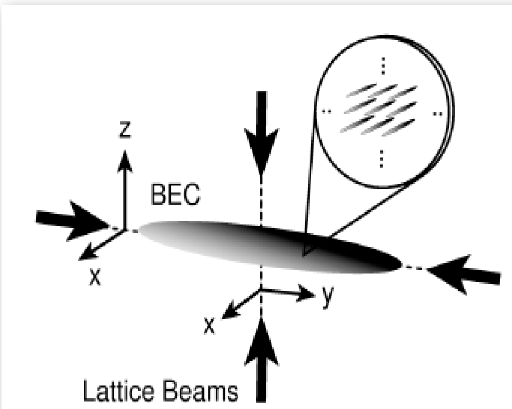

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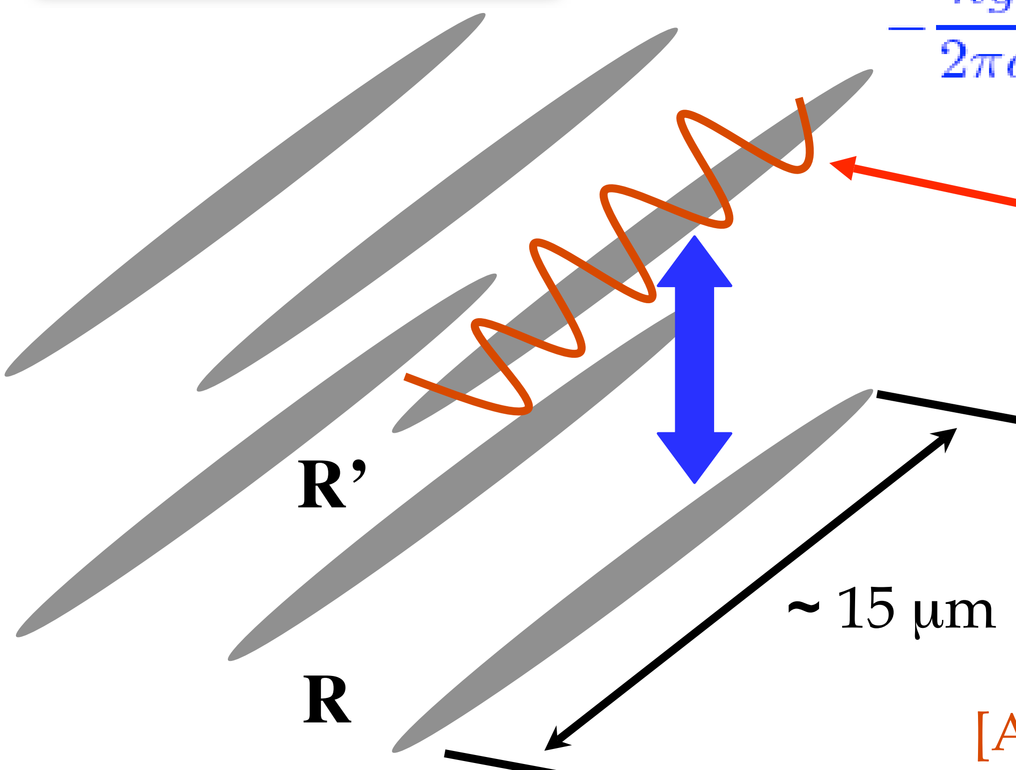

Periodic potential
localizes the atoms

[AF Ho, MAC & T Giamarchi, PRL 92 (2004)]

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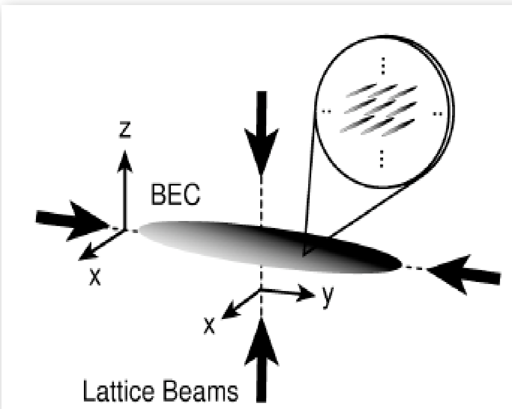

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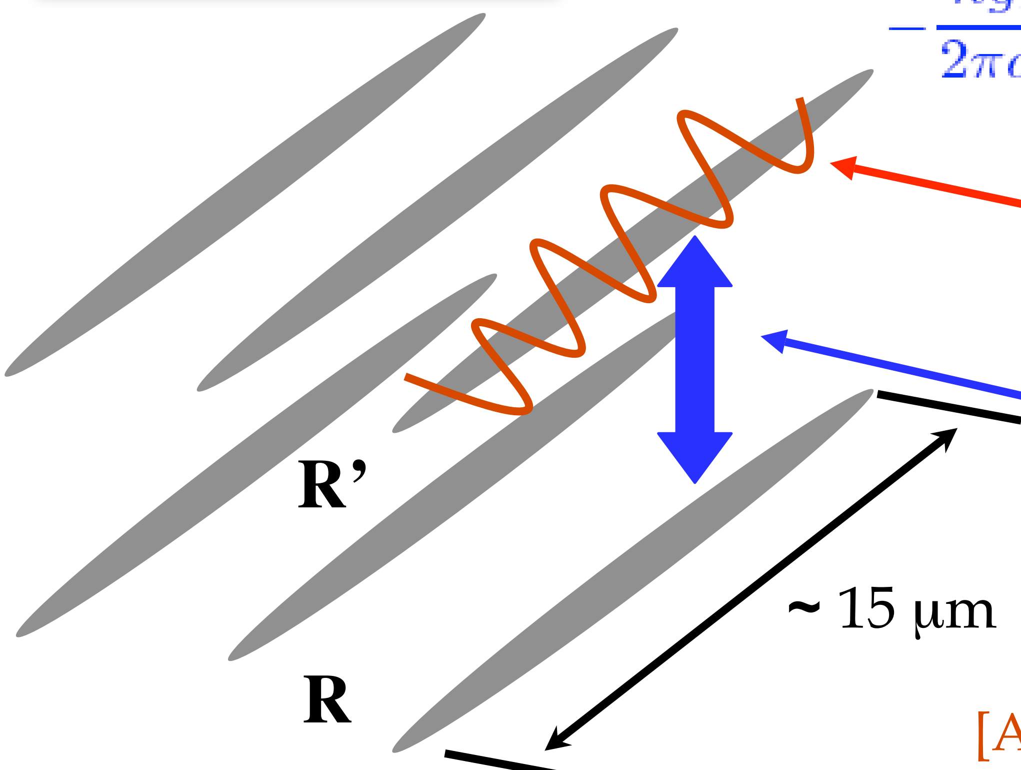

Periodic potential localizes the atoms

Josephson coupling Delocalizes the atoms

[AF Ho, MAC & T Giamarchi, PRL 92 (2004)]

Zero-temperature phase diagram

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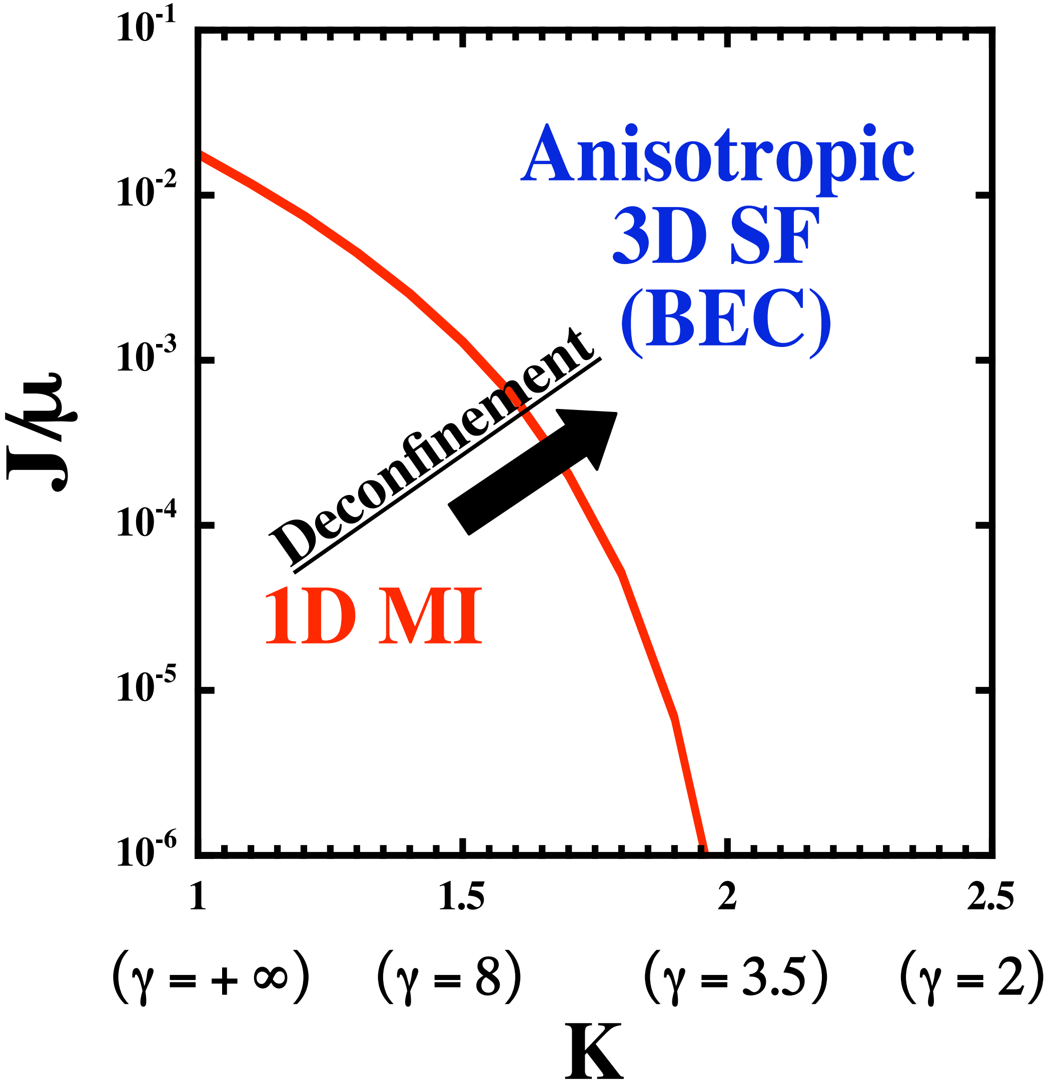
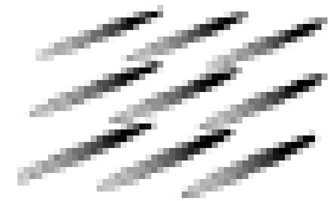


Zero-temperature phase diagram

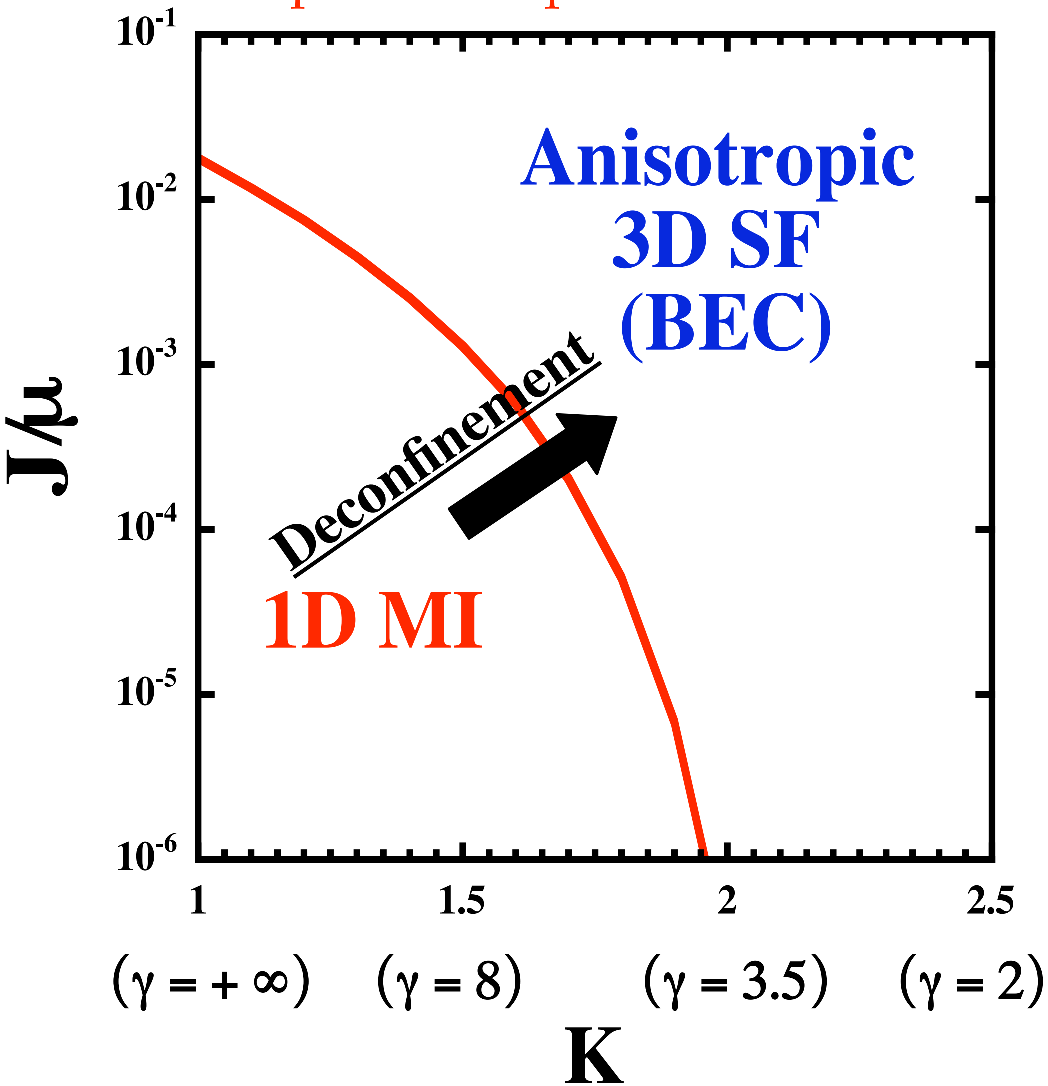
'Charging' energy [AF Ho, MAC & T Giamarchi, PRL 92 (2004)]

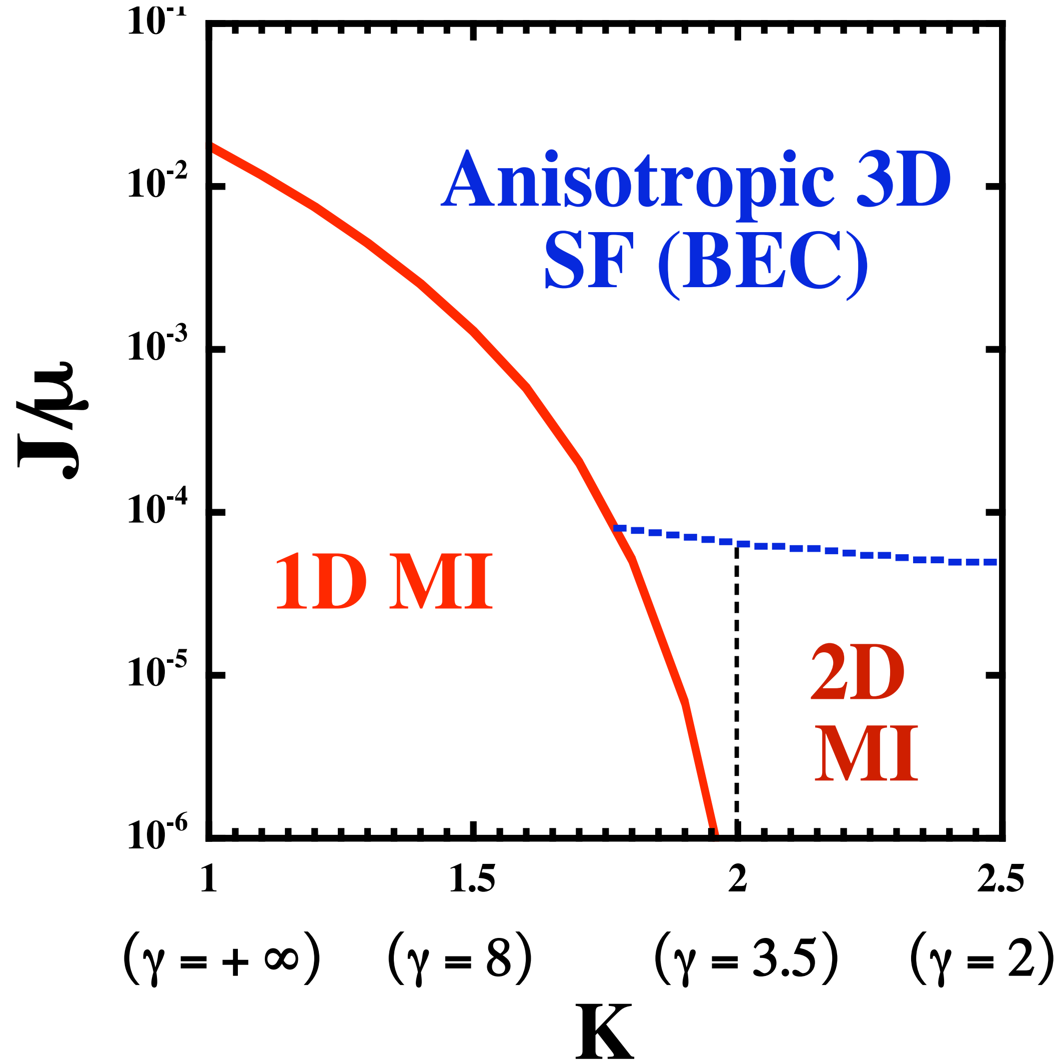
$$E_J \propto J N_0^{1-\frac{1}{2K}} \sim E_C = \frac{\hbar k_F v_F}{2K^2}\frac{1}{N_0}$$

Quantum self-trapping seen in expansion experiments



Array of *atomic* 'quantum dots'





2D optical lattices: 3D Superfluid (BEC) phase

[AF Ho, MAC & T Giamarchi, PRL 92 (2004)]

Mean-field theory: condensate fraction

$$\psi_0^2(T=0) \sim \rho_0 \left(\frac{J}{\mu}\right)^{1/(4K-1)} \qquad \left(\frac{2\pi T_c}{\hbar v_s \rho_0}\right)^{2-1/2K} = f(K)\frac{4J}{\hbar v_s \rho_0}$$

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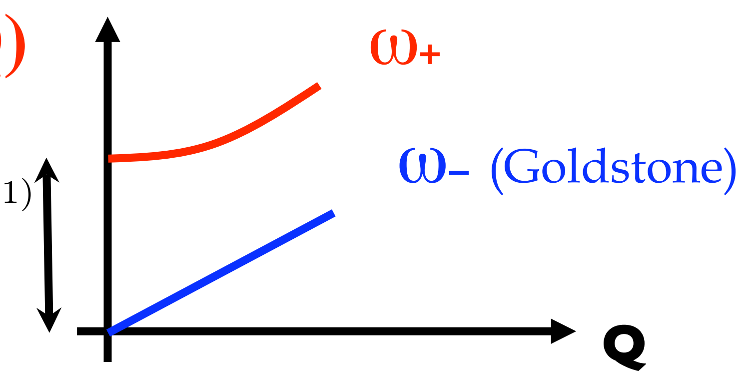
Excitation spectrum

$$\Delta_+ \sim \mu \left(\frac{J}{\mu}\right)^{2K/(4K-1)}$$



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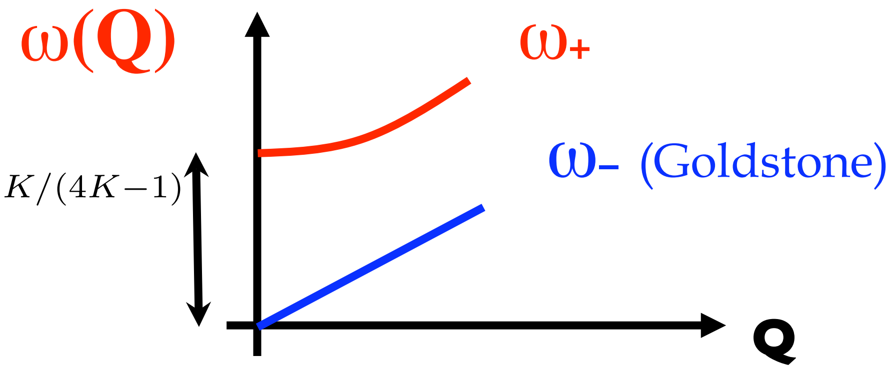


Variational approach: momentum distribution at T = 0

$$\frac{n(\mathbf{Q},q)}{|w(\mathbf{Q})|^2} \simeq \psi_0^2 \delta(\mathbf{Q})\delta(q) + \frac{\pi b^2 \psi_0^2/2K}{\left[q^2 + (v_\perp \mathbf{Q}/v_s)^2\right]^{1/2}}$$

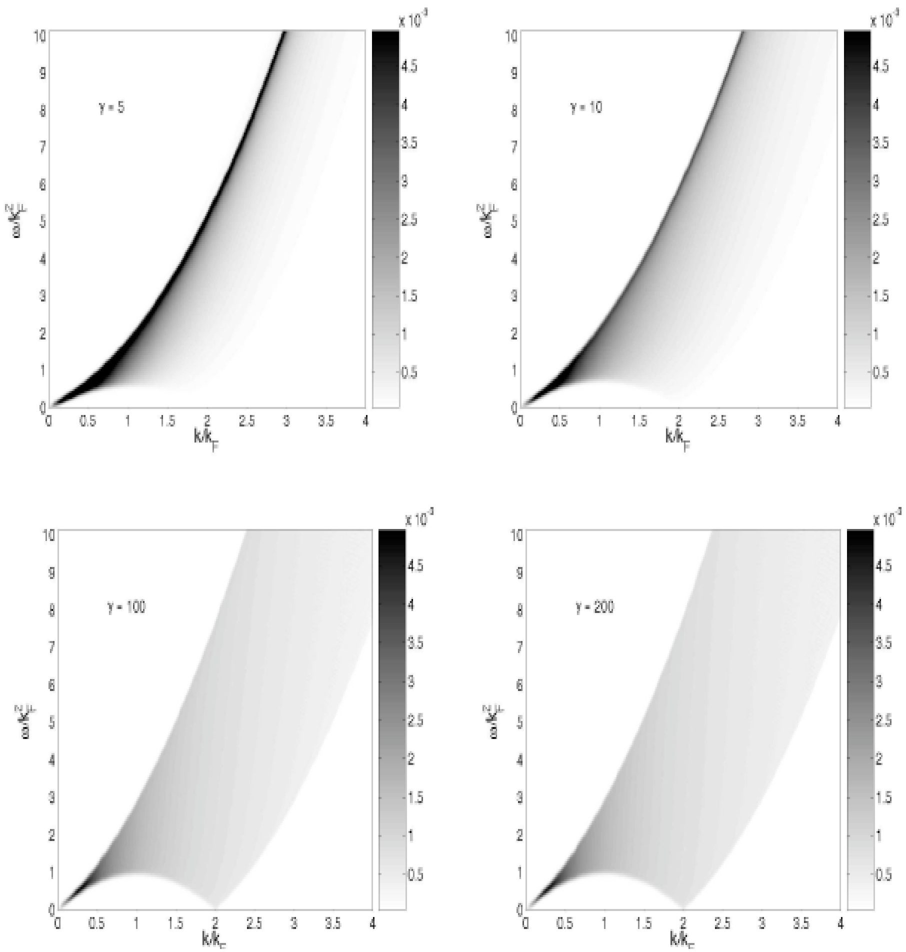
transverse velocity:

$$v_\perp \sim \mu b (J/\mu)^{2K/(4K-1)}/\hbar$$

A roton mimimum?

1D Regime: Lieb-Liniger model

[JS Caux & P Calabresse, PRA (2006)]



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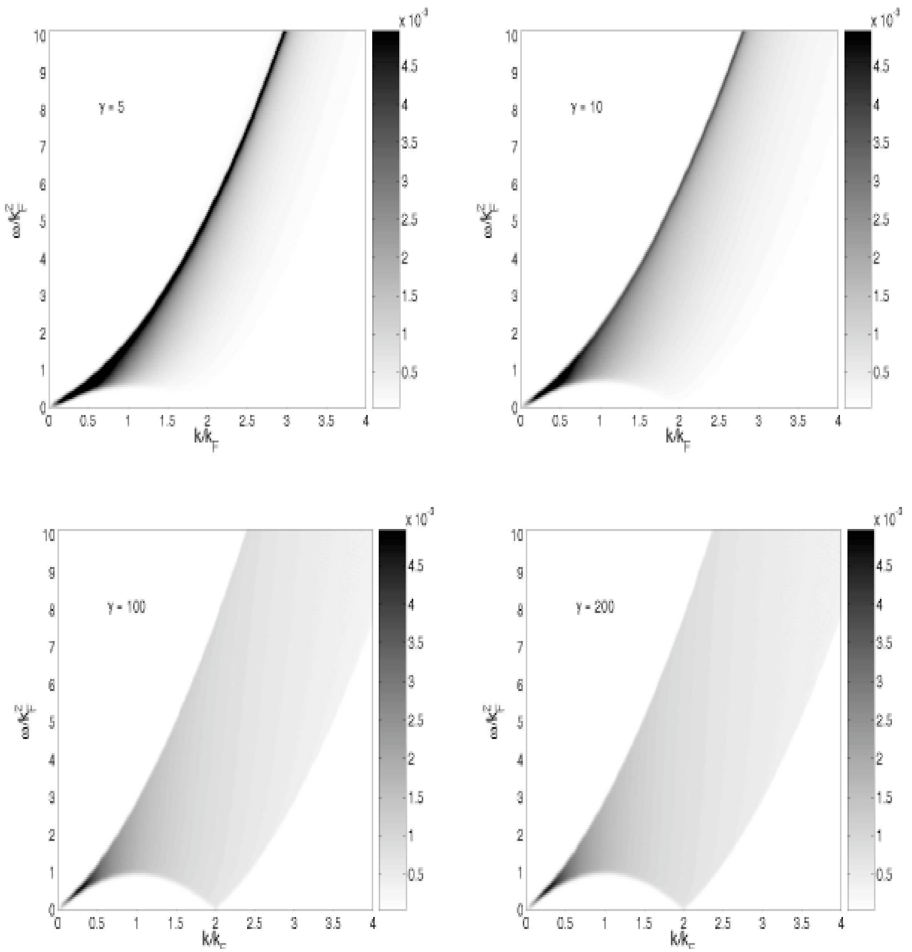
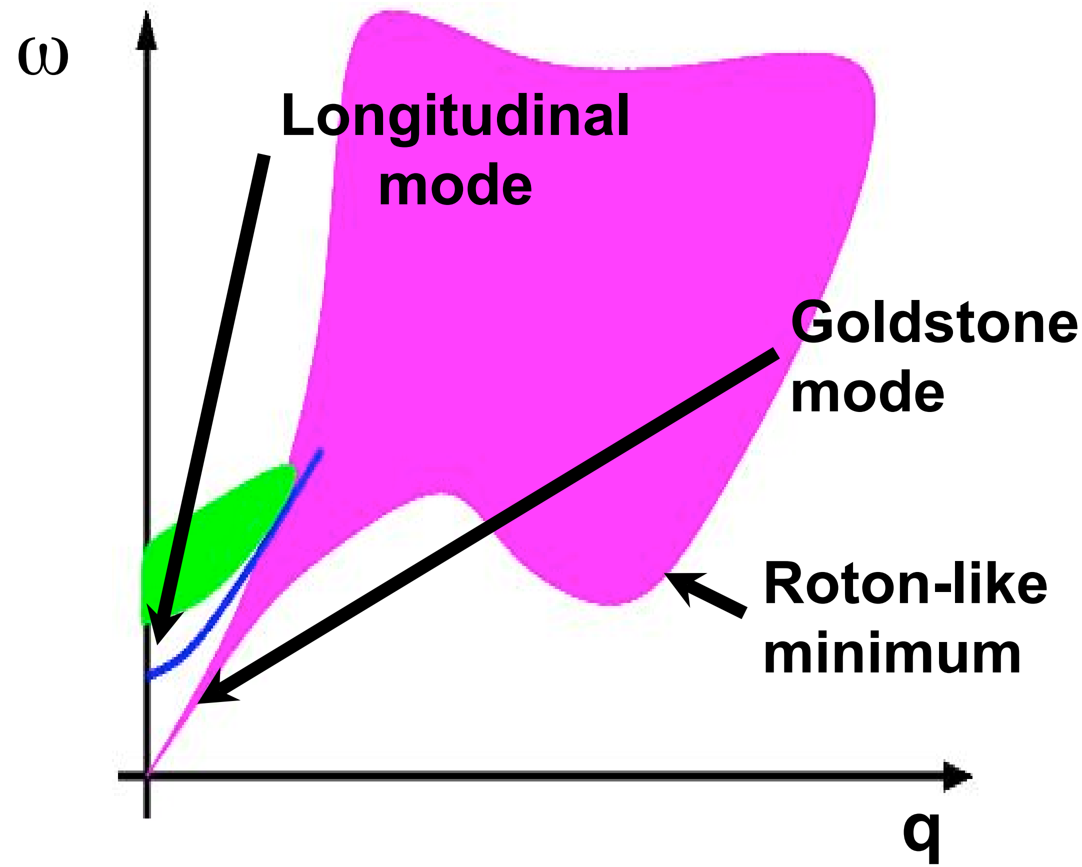


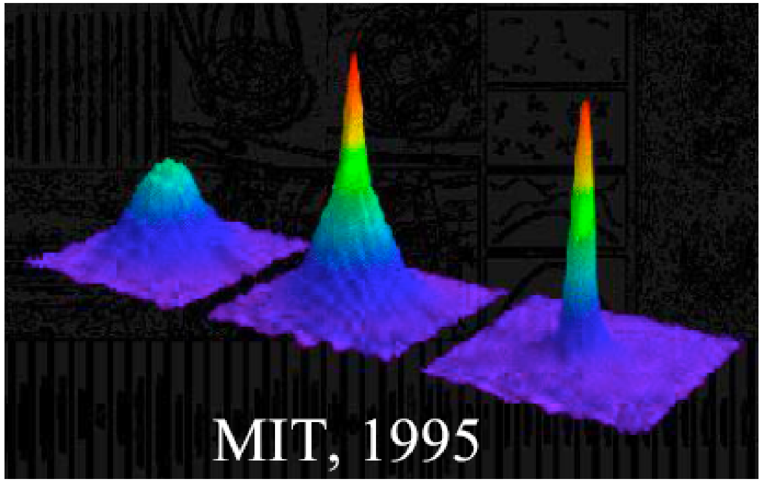
Anisotropic BEC (J << μ)



[A Iucci, MAC, AF Ho & T Giamarchi PRA 73 (2006)
MAC, AF Ho & T Giamarchi New J of Phys 8 (2006)]

When does the deconfinement transition occur?

Time-of-flight (TOF) = n(k)



Width of n(k) near k = 0

$$\text{3D BEC width} \sim \frac{1}{R}$$

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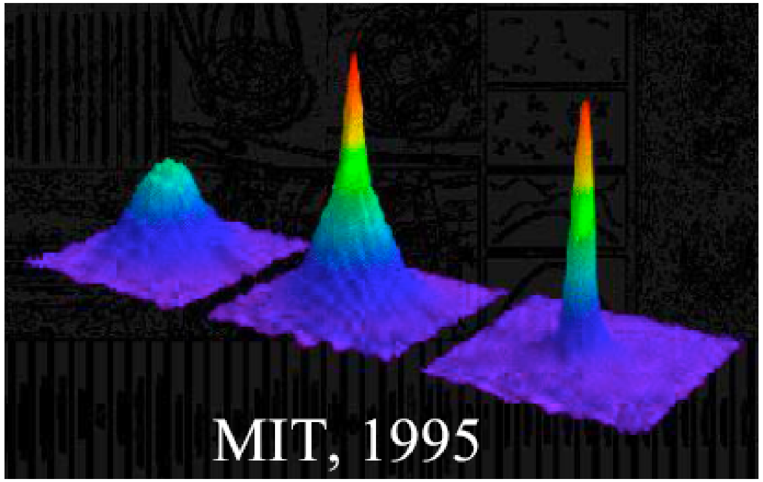

Width of n(k) near k = 0

3D BEC width $\sim \frac{1}{R}$

1D SF width $\sim \max\{\frac{1}{L}, \frac{\hbar v_s}{T}\}$

Mott Insulator width $\sim \frac{\Delta}{\hbar v_s}$

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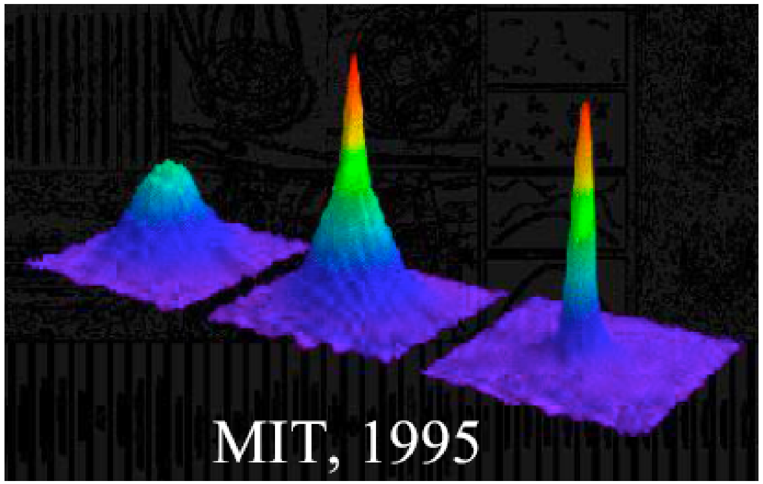

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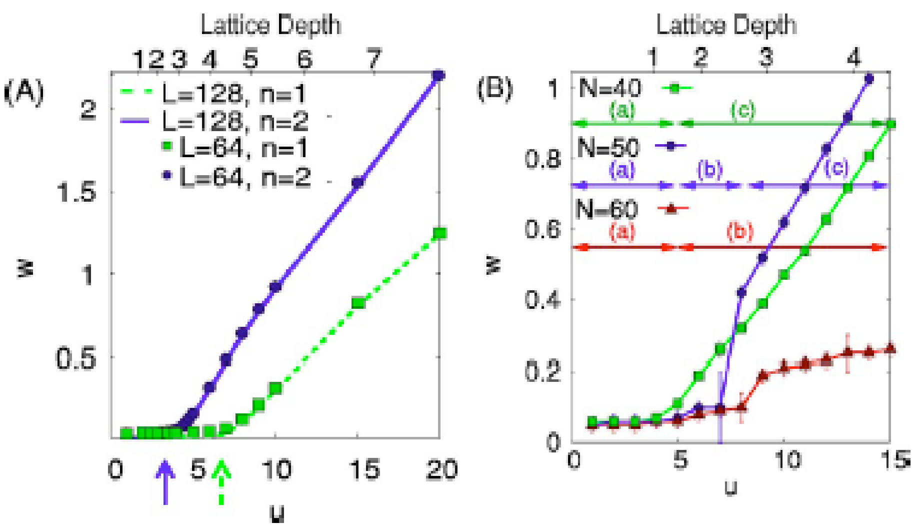

[C. Kollath et al. PRA 69 (2004)]

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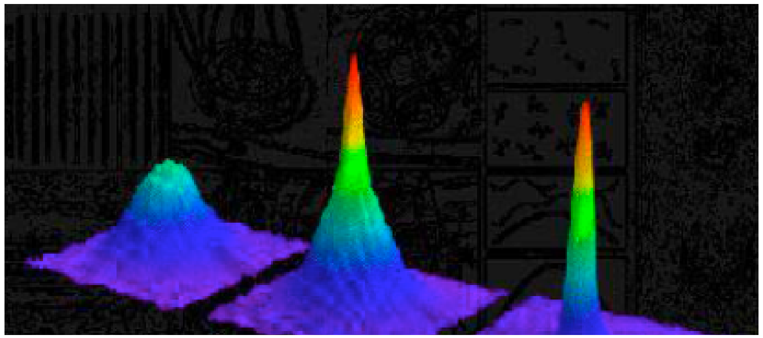

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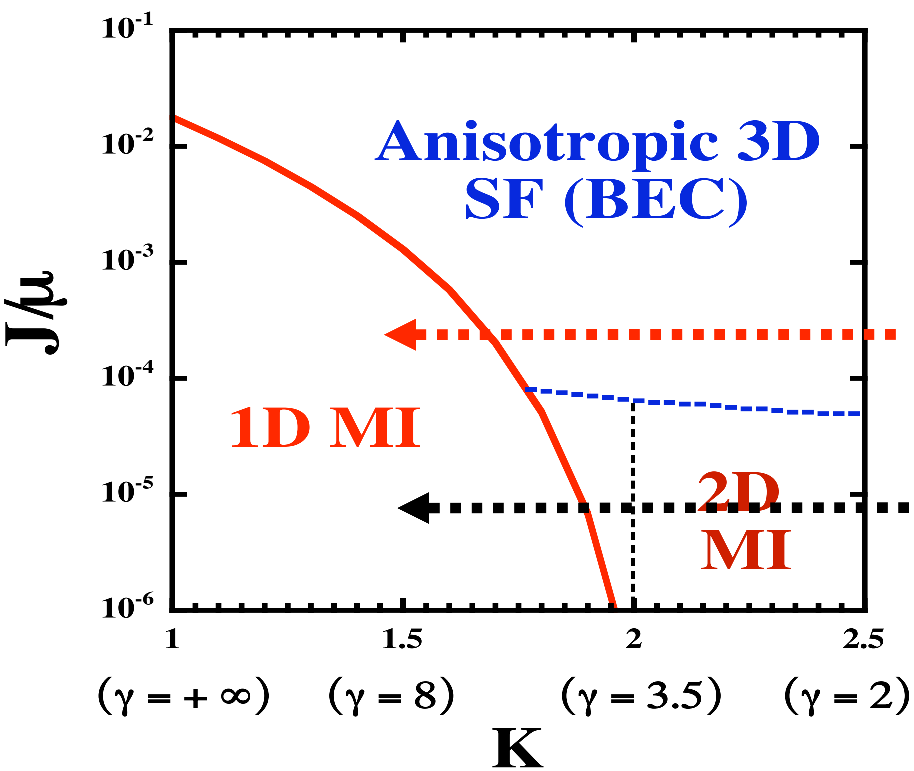
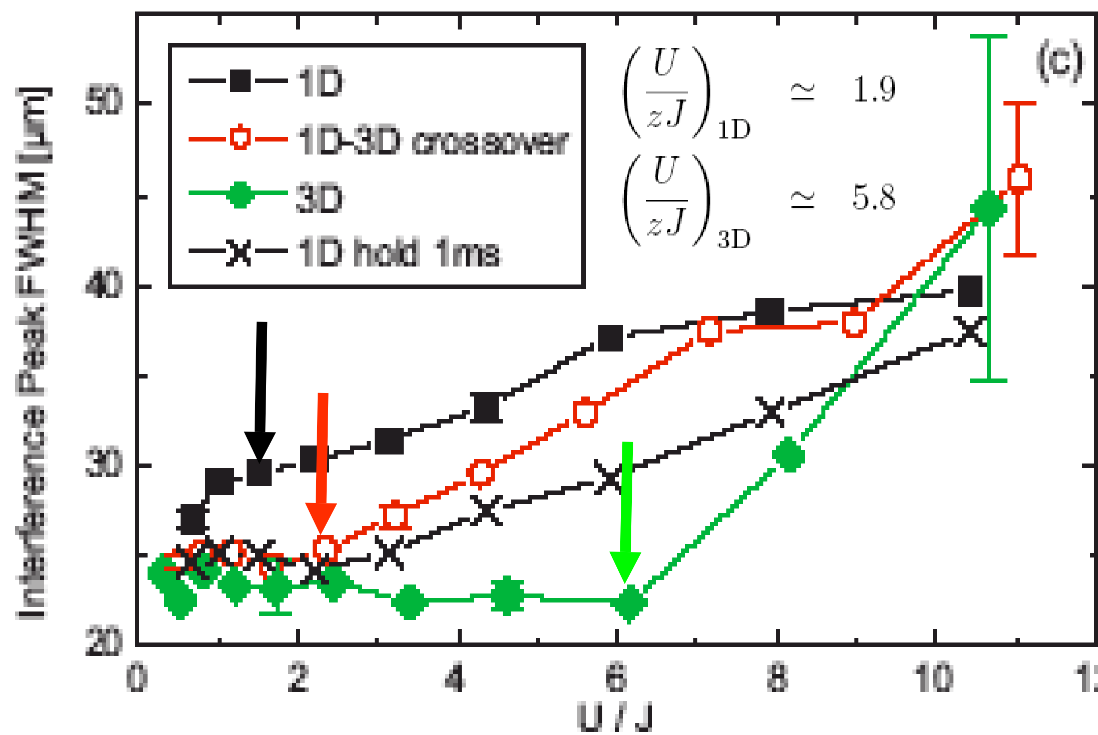
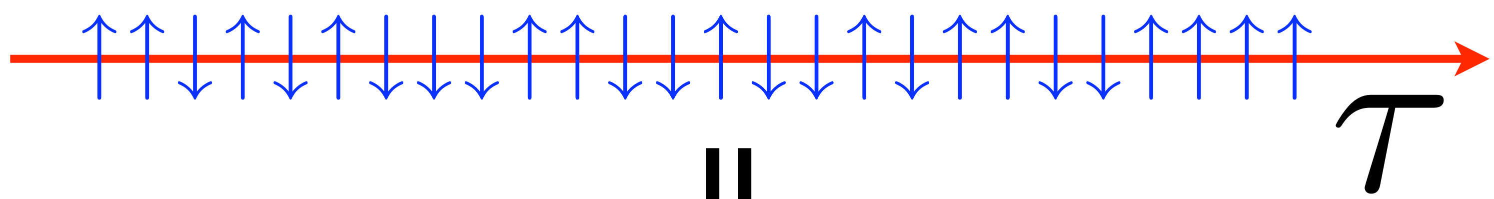




[T. Stöferle et al. PRL 92 (2004)]

When Classical maps onto Quantum

Classical Ising model (a typical configuration)



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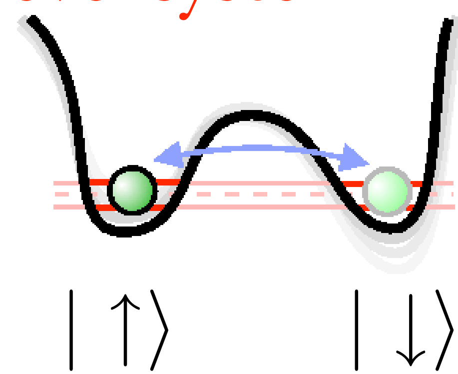
(Imaginary time evolution of) Quantum two-level system

$$Z = \mathrm{Tr}\, e^{-\frac{H_{\mathrm{Ising}}}{T}} = \mathrm{Tr}\, e^{-L\hat{H}_Q}$$



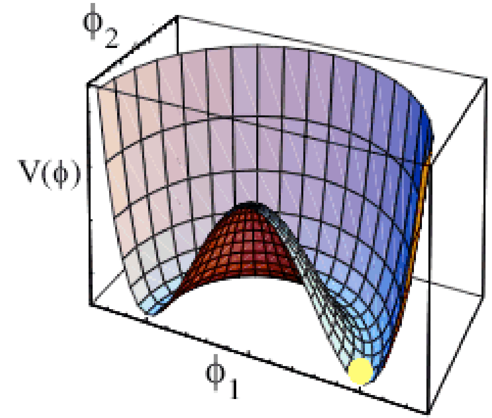
$$H_{\mathrm{Ising}} = -\sum_{i=1}^{L} \sigma_i^z (J\sigma_{i+1}^z - h) \qquad \hat{H}_Q = \epsilon(J,h)\hat{\sigma}^z + \Delta(J,h)\hat{\sigma}^x$$

2D XY = 1D Sine-Gordon

Low-T phase of the 2D Bose Gas

Classical phase fluctuations

$$\mu \ll T \quad \Rightarrow \quad \Psi(\mathbf{r}) \simeq \rho_0^{1/2}(T)\, e^{i\Theta(\mathbf{r})}$$



The XY model

$$\frac{\mathcal{H}}{T} \simeq \frac{K(T)}{2\pi} \int d\mathbf{r}\, (\nabla\Theta(\mathbf{r}))^2 = \frac{K(T)}{2\pi A} \sum_{\mathbf{q}} \mathbf{q}^2 |\Theta(\mathbf{q})|^2$$

Equipartition

$$\mathbf{q}^2|\Theta(\mathbf{q})|^2 \propto T \Rightarrow \langle \Theta^2(\mathbf{r}=\mathbf{0})\rangle \propto \int d^{D=2}\mathbf{q} \left(\frac{T}{\mathbf{q}^2}\right) \propto T \log\left(\frac{A}{\xi^2}\right) \to +\infty$$

Quasi-long range order

$$\langle \Psi^\dagger(\mathbf{r})\Psi(\mathbf{0})\rangle \propto \langle e^{i\Theta(\mathbf{r})} e^{-i\Theta(\mathbf{0})}\rangle \propto \left(\frac{\xi}{|\mathbf{r}|}\right)^{\frac{1}{2K(T)}}$$

But high-T expansions yield a disordered phase $\langle e^{i\Theta(\mathbf{r})} e^{-i\Theta(\mathbf{0})}\rangle \propto e^{-|\mathbf{r}|/\xi_c(T)}$

Not the whole story: enter vortices

$$\Psi(\mathbf{r}) \sim e^{i\Theta(\mathbf{r})}$$

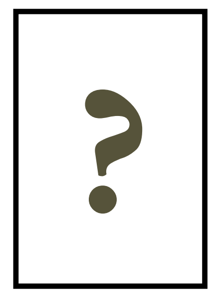
$$\mathbf{v}(\mathbf{r}) = -\frac{\hbar}{2m}\mathrm{Re}\left[\Psi^{\dagger}(\mathbf{r})i\nabla\Psi(\mathbf{r})\right] \propto \nabla\Theta(\mathbf{r}) \qquad \oint \mathbf{v}\cdot d\mathbf{l} = \frac{h}{m}\,n$$

$$E_{\text{vortex}} = \frac{m\rho_s}{2}\int d\mathbf{r}\,\left(\mathbf{v}_{\text{vortex}}(\mathbf{r})\right)^2 \simeq \frac{\hbar^2\pi\rho_s}{m}\log\left(\frac{\sqrt{A}}{\xi}\right) \qquad |\mathbf{v}_{\text{vortex}}(\mathbf{r})| \sim \frac{1}{|\mathbf{r}|}$$

$$S_{\text{vortex}} \simeq \log\left(\frac{A}{\xi^2}\right)$$

$$F_{\text{vortex}} = E_{\text{vortex}} - T\,S_{\text{vortex}}$$

$$T_c^{\text{KT}} = \frac{\hbar^2\pi\rho_s}{2m} \Rightarrow K(T_c^{\text{KT}}) = 2$$



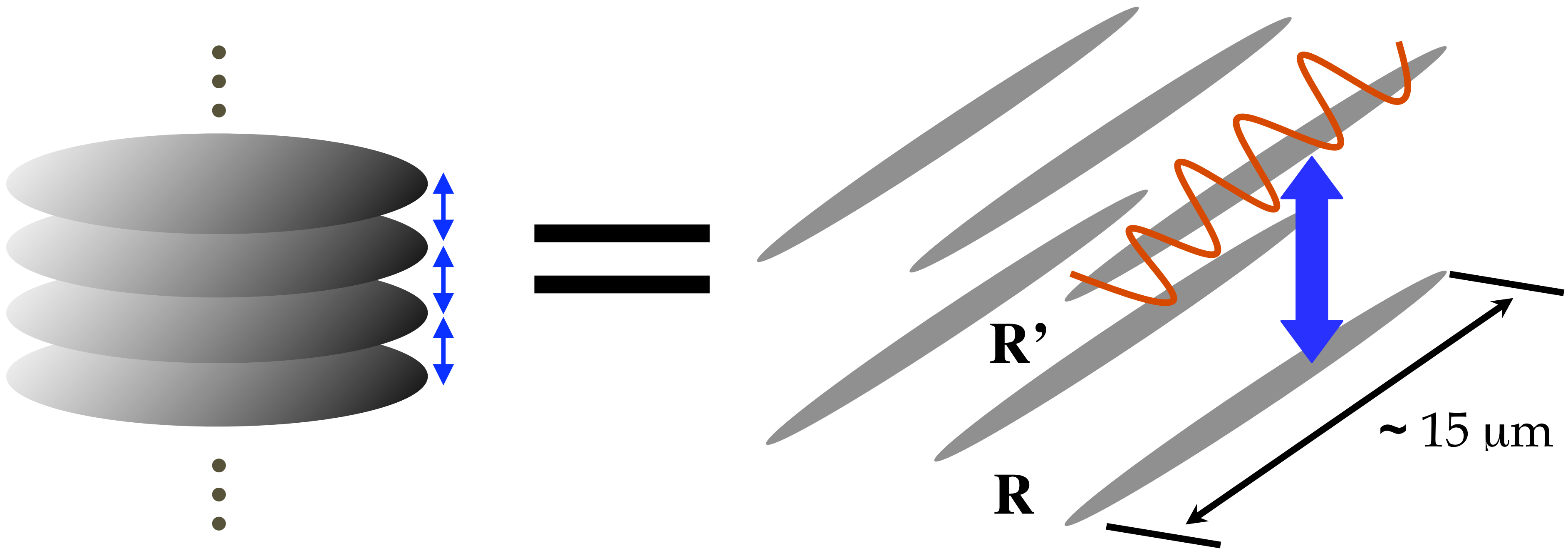
B **K** **T**

BKT predict a (non-Landau) continuous phase transition between a QLR ordered and a disordered phase mediated by vortex-anti-vortex unbinding

Coupling XY models

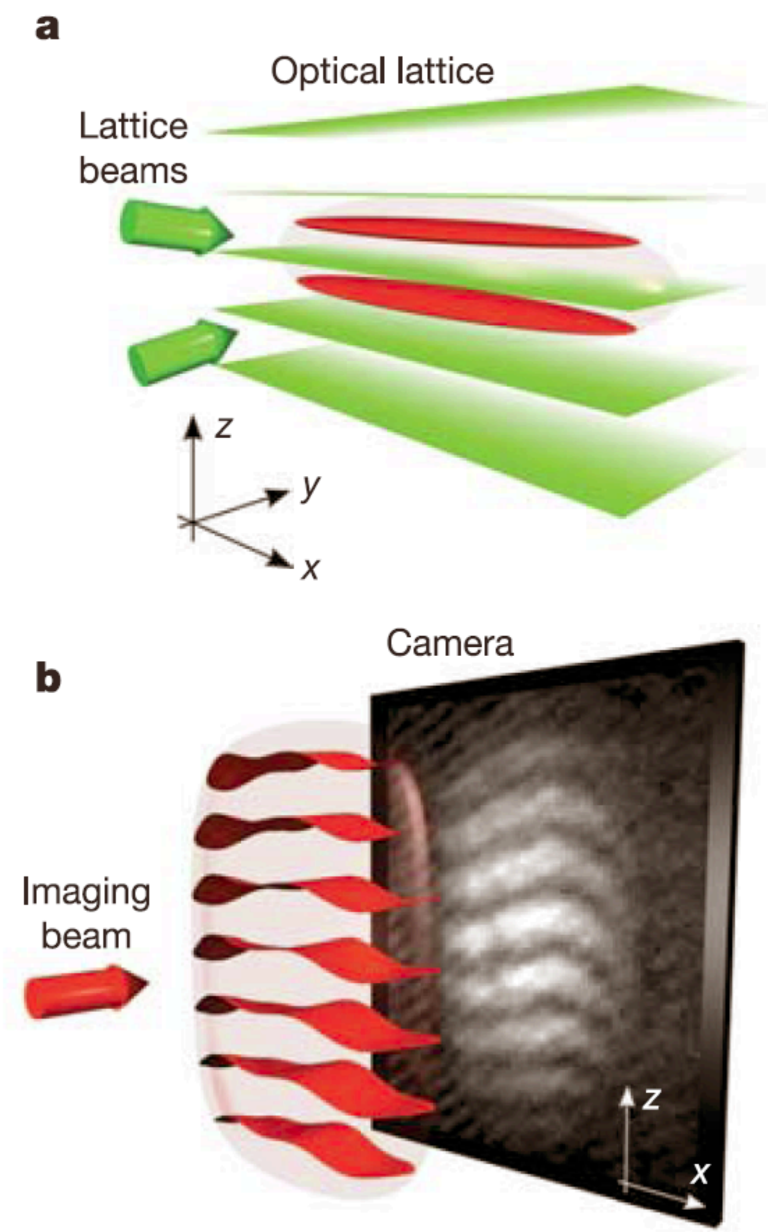
BKT-like physics in a stack of 2D Bose Gases

SF to MI in quasi-1D optical lattices [T Stöferle *et al* PRL 92 (2004)]



The ENS Experiment

[Z Hadzibabic *et al* Nature (2006)]

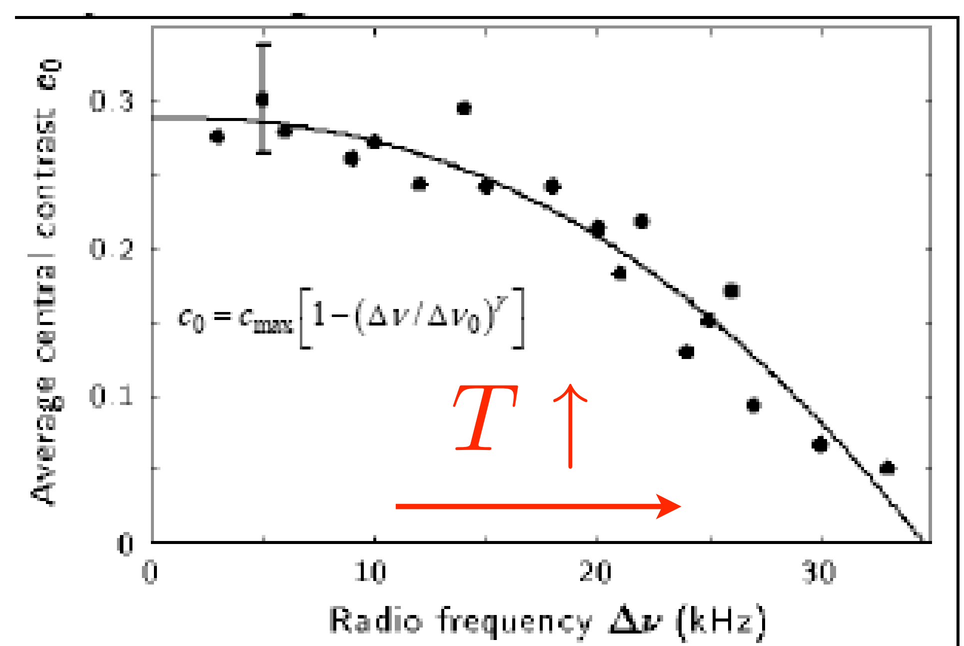


Interference pattern

$$F(x,z) = G(x,z)\left[1 + c(x)\cos\left(2\pi z / D + \varphi(x)\right)\right]$$

Random phase $\langle F(x,z)\rangle = f(x)$

c_0 = Average $c(0)$ as a thermometer

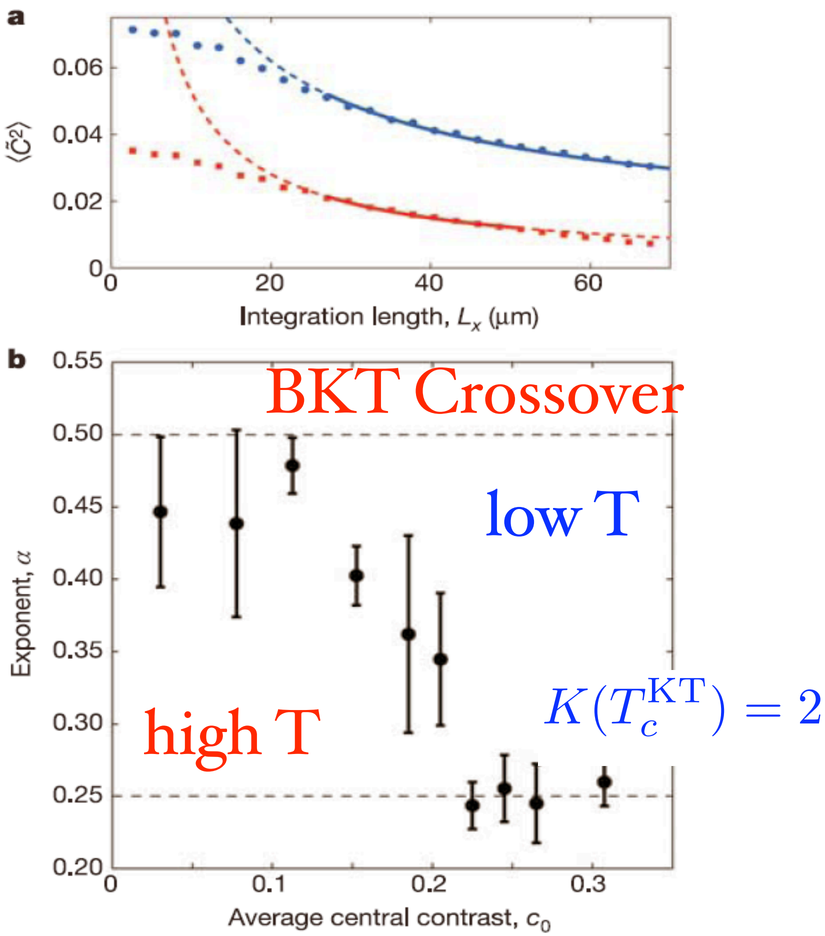


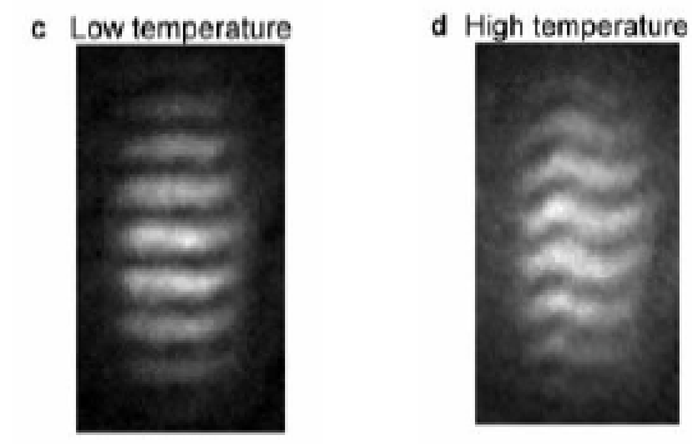
The ENS Experiment

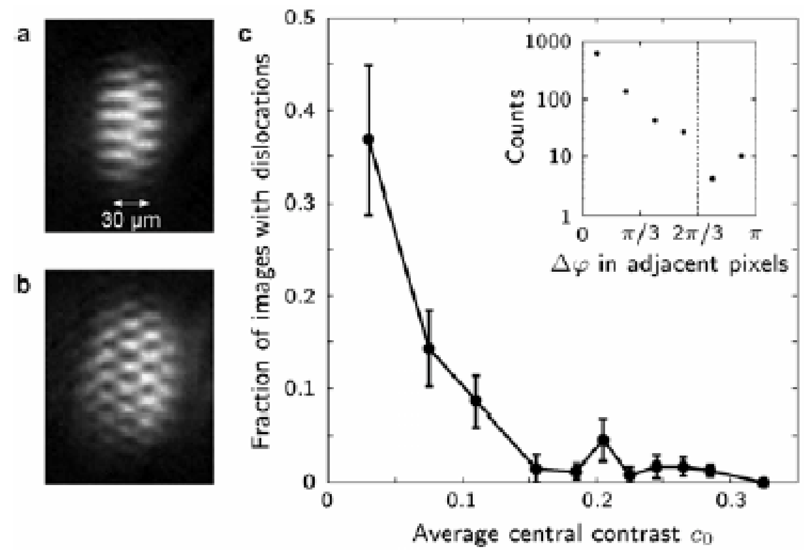
$$g_1(\mathbf{r}) = \langle \Psi^\dagger(\mathbf{r})\Psi(\mathbf{0})\rangle \sim |\mathbf{r}|^{-\alpha}, \quad \alpha = \frac{1}{2K(T)}$$

[Z Hadzibabic *et al* Nature (2006)]

$$\langle C^2(L_x)\rangle \approx \frac{1}{L_x}\int_0^{L_x} dx\,[g_1(x,0)]^2 \propto L_x^{-2\alpha}$$







Josephson coupled classical 2D Bose gases

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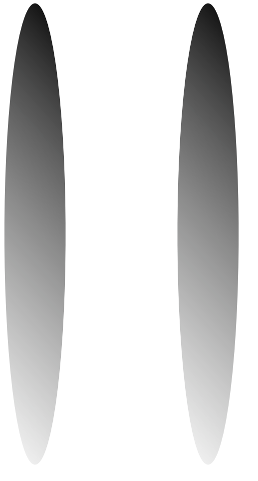

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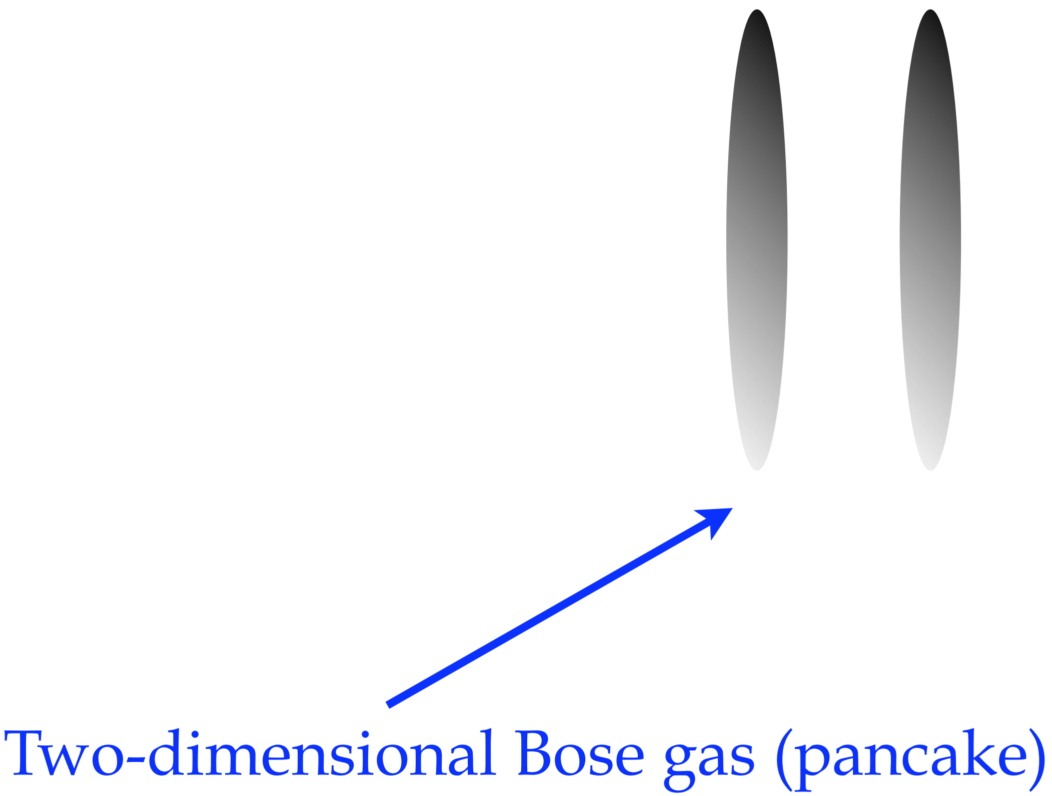

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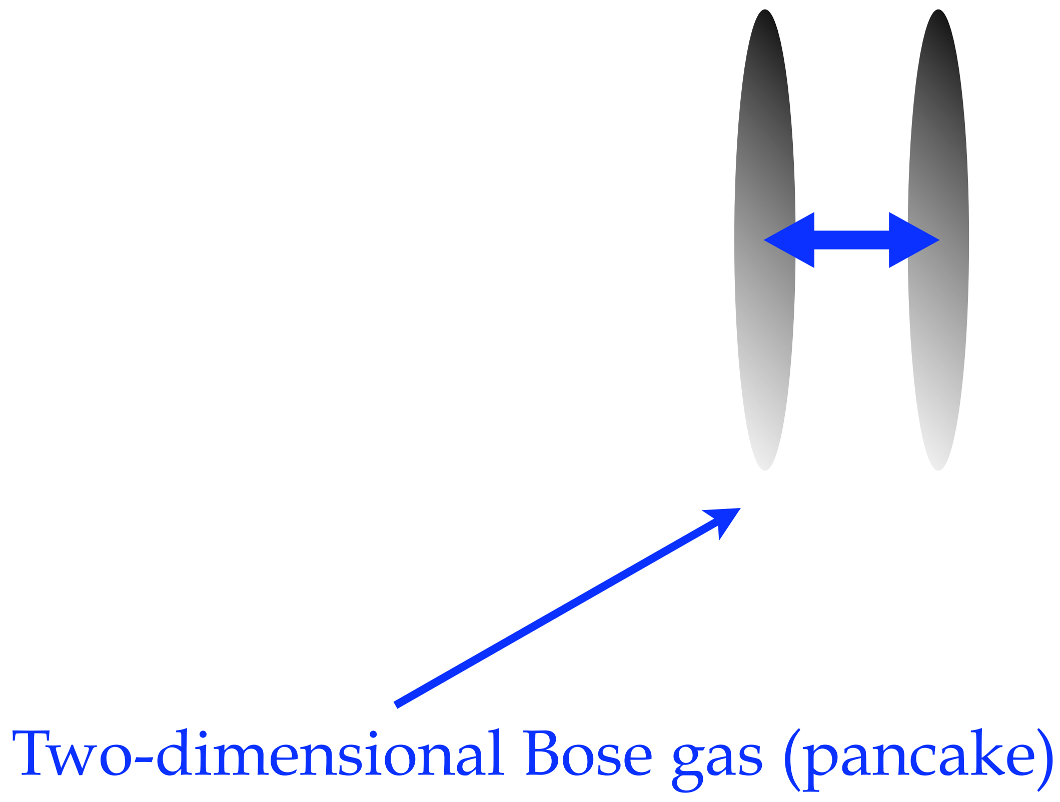

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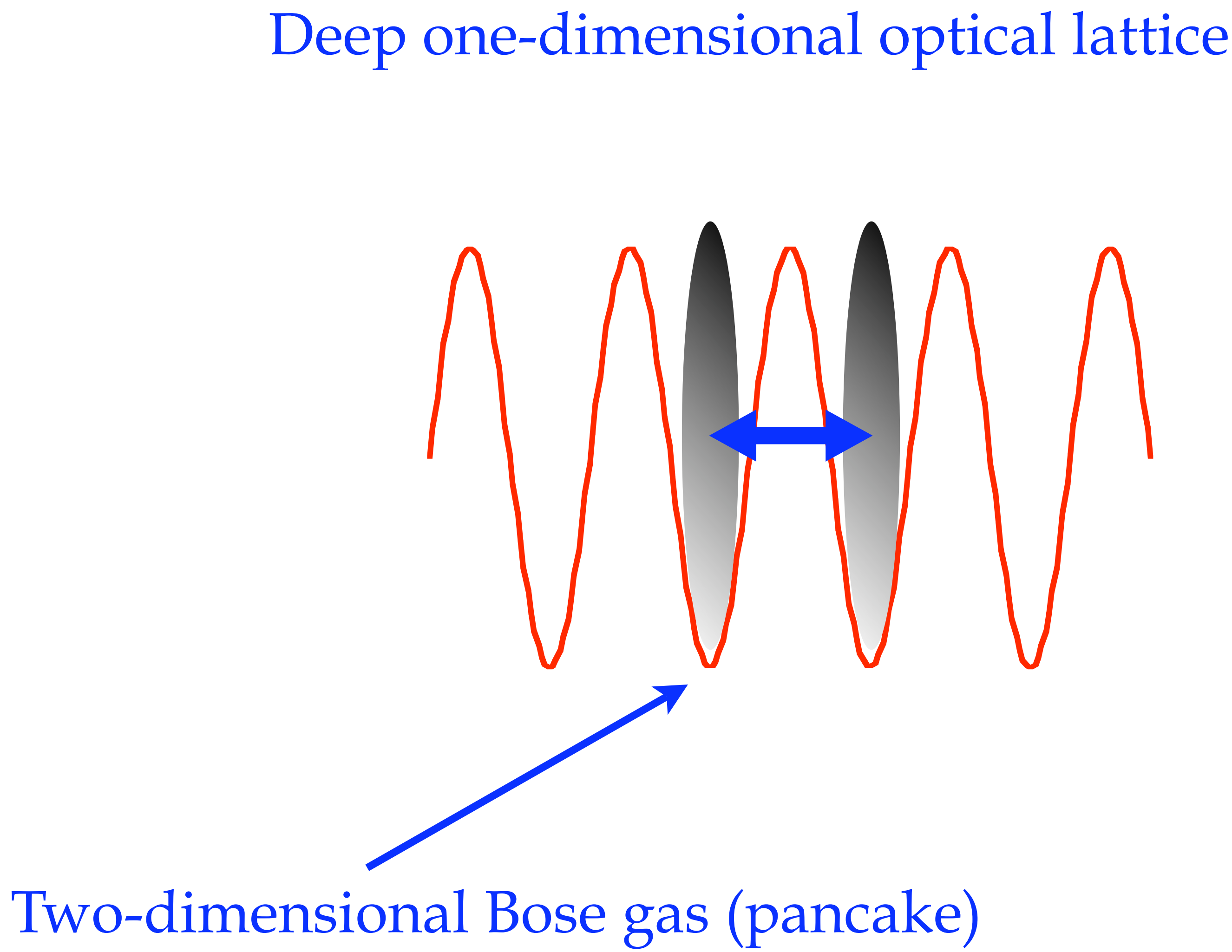
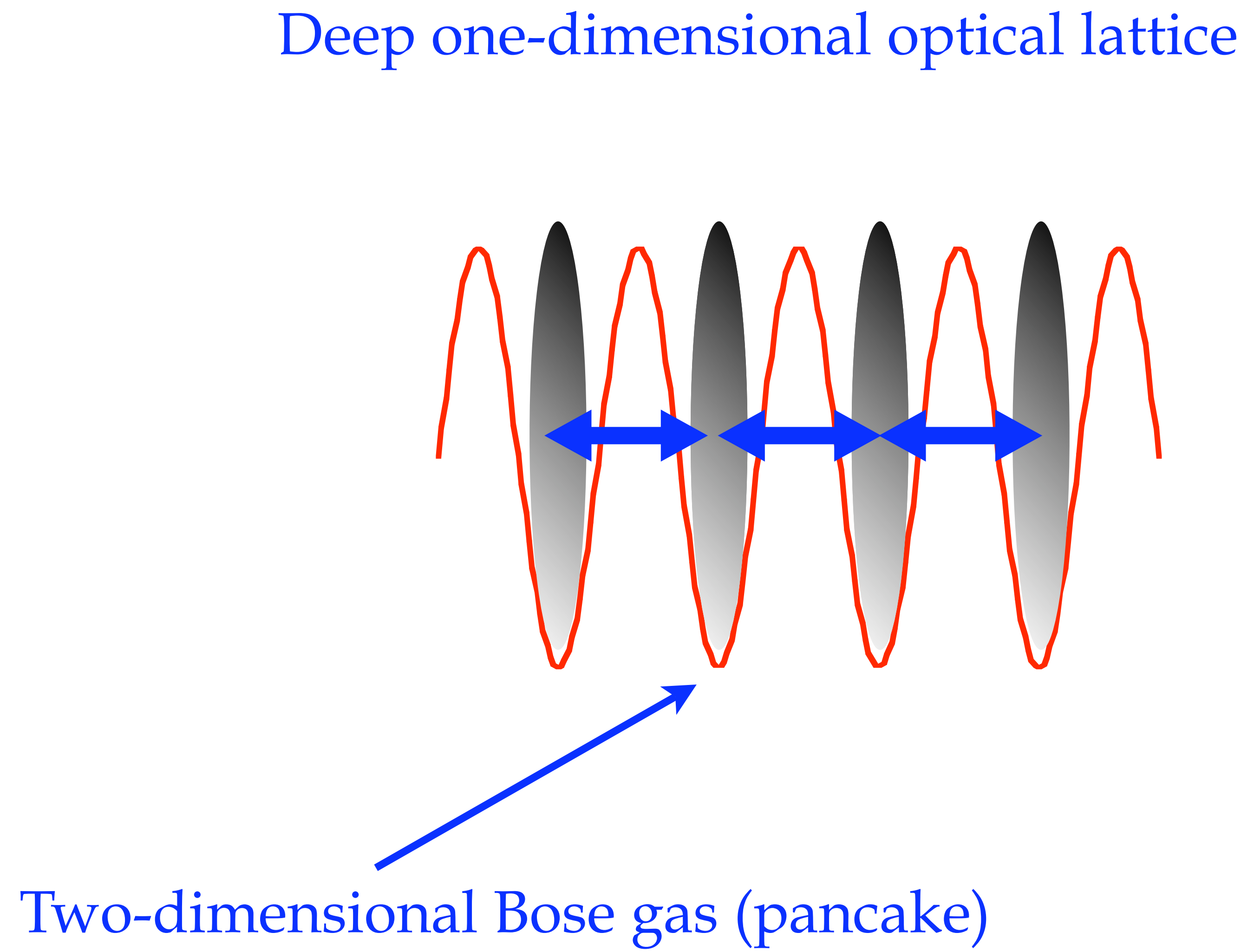
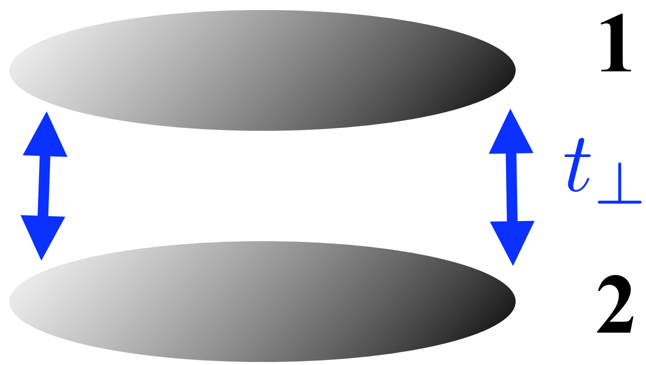


Josephson coupled classical 2D Bose gases



Fixing the phase: calling Mr. Josephson



$t_\perp \ll \mu$

Tunneling tends order the relative phase

$$\Psi(\mathbf{r}) \sim e^{i\Theta(\mathbf{r})}$$

$$H_J = -t_\perp \int d\mathbf{r}\ \left[\Psi_1^\dagger(\mathbf{r})\Psi_2(\mathbf{r}) + h.c.\right] \simeq -2t_\perp \rho_0(T) \int d\mathbf{r}\ \cos\left[\Theta_1(\mathbf{r}) - \Theta_2(\mathbf{r})\right]$$

Mr Josephson is happy when $\Theta_1(\mathbf{r}) = \Theta_2(\mathbf{r}) \pmod{2\pi}$ <u>for all $\mathbf{r}$</u>

However

Entropy wants to have <u>both</u> $\Theta_1(\mathbf{r}), \Theta_2(\mathbf{r})$ disordered

Phase diagram. Deconfinement

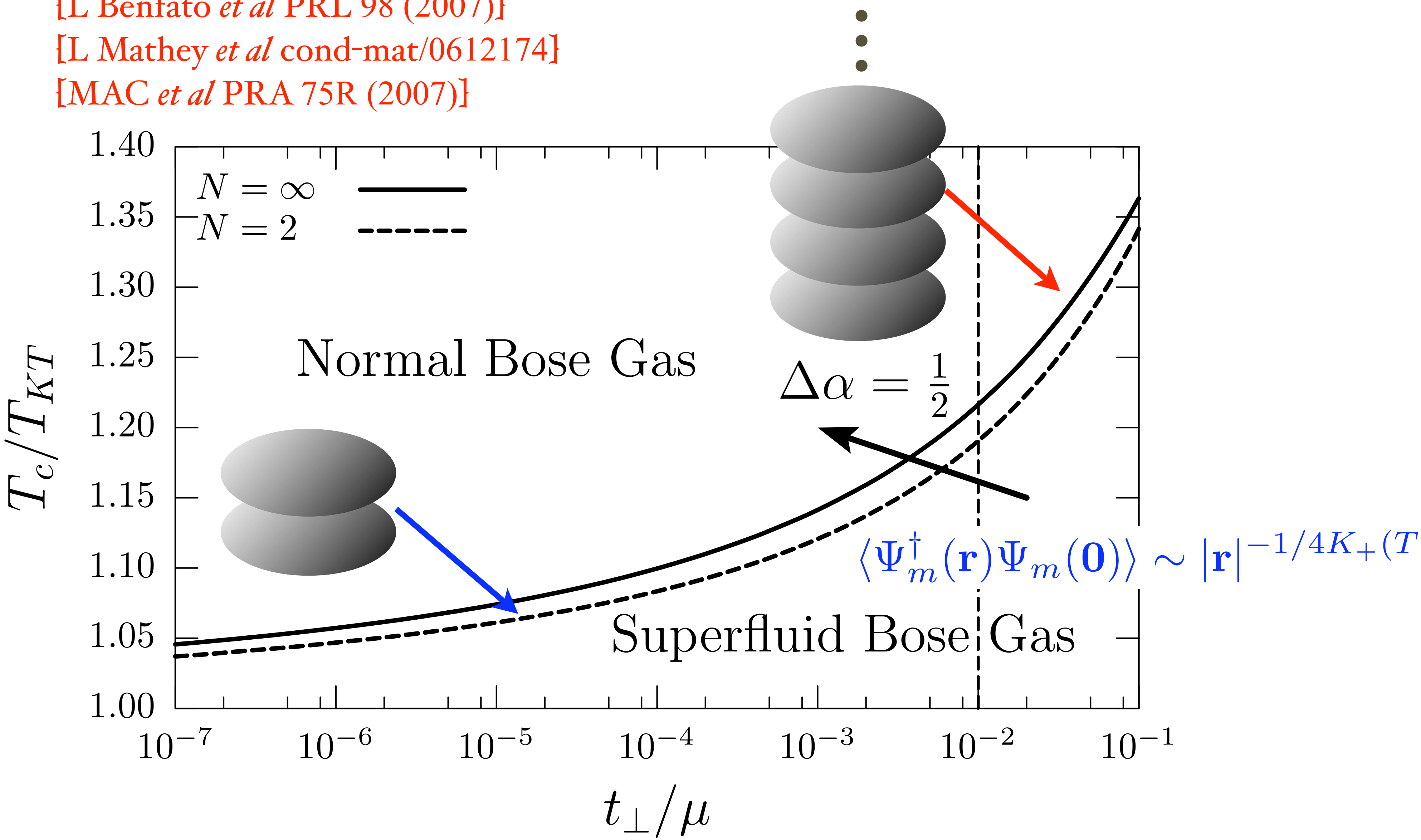
[P Donohoue & T Giamarchi, PRB 63 (2001)]

[AF Ho, MAC & T Giamarchi, PRL 92(2004), MAC *et al* New J of Phys 8 (2006)]

[L Benfato *et al* PRL 98 (2007)]

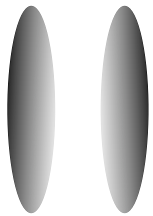
[L Mathey *et al* cond-mat/0612174]

[MAC *et al* PRA 75R (2007)]



Interference pattern

1 2



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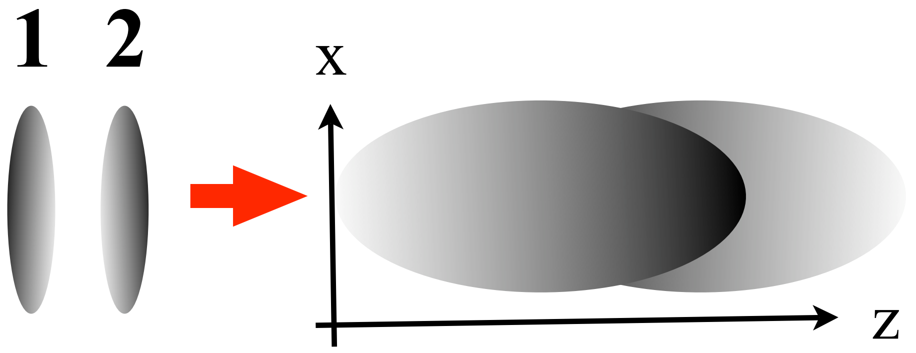

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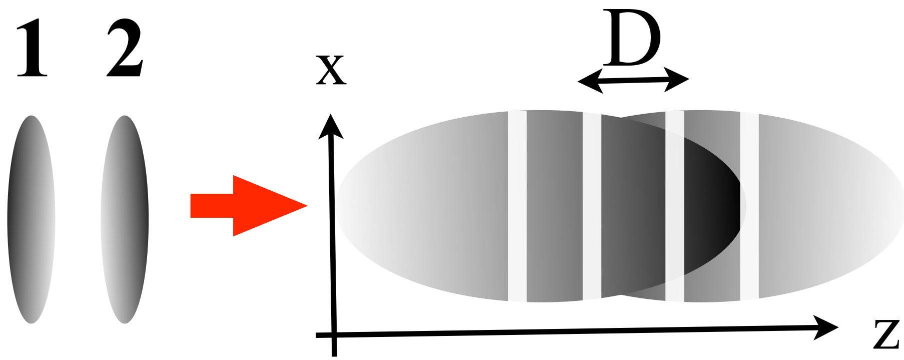

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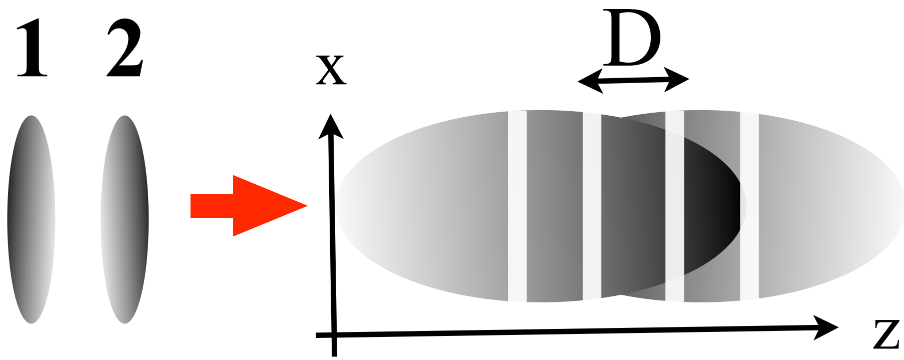

[A Polkovnikov, E Altman & E Demler, PNAS 103 (2006)]

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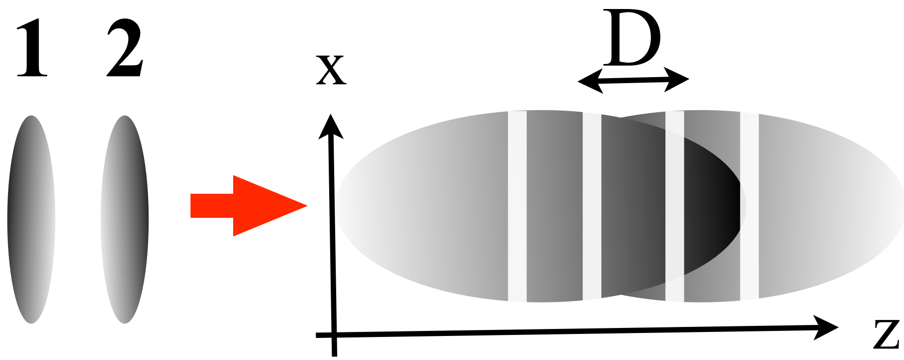
$$\rho(\mathbf{r},t) = \Psi^\dagger(\mathbf{r},t)\Psi(\mathbf{r},t) = \text{non osc.} + \left[\hat{B}_Q(\mathbf{r})\exp\left(\frac{2\pi i z}{D}\right) + \text{h.c.}\right]$$

Interference pattern



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Independent clouds $(t_\perp = 0)$

$$\langle \rho(\mathbf{r},t)\rangle_{\text{osc}} = 0 \quad \left[\langle \hat{B}_Q(\mathbf{r})\rangle = 0\right]$$

$$\langle \rho(\mathbf{r},t)\rho(\mathbf{r}',t)\rangle_{\text{osc}} \propto \langle \hat{B}_Q(\mathbf{r})\hat{B}_Q^\dagger(\mathbf{r}')\rangle$$

$$\langle \Psi_1^\dagger(\mathbf{r})\Psi_1(\mathbf{r}')\rangle\langle \Psi_2^\dagger(\mathbf{r})\Psi_2(\mathbf{r}')\rangle^* = |g_1(\mathbf{r},\mathbf{r}')|^2$$

$$\Theta_- = \frac{1}{\sqrt{2}}(\Theta_1 - \Theta_2)$$

Coupled Clouds $(t_\perp \neq 0)$

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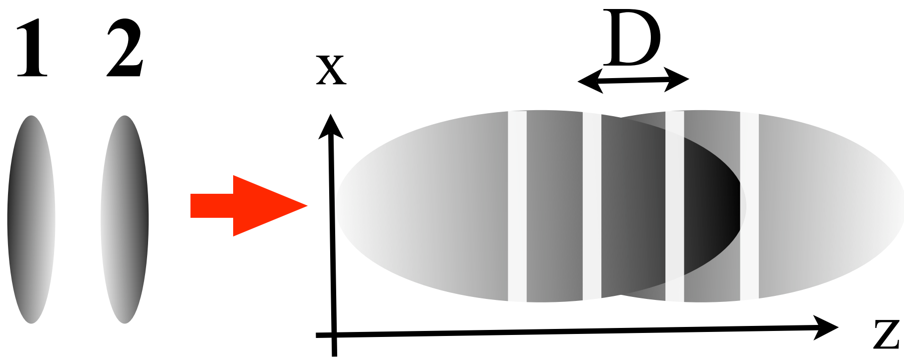
$$\langle \Psi_1^\dagger(\mathbf{r})\Psi_2(\mathbf{r}')\Psi_2^\dagger(\mathbf{r})\Psi_1(\mathbf{r}')\rangle \propto \langle e^{i\sqrt{2}\Theta_-(\mathbf{r})}e^{-i\sqrt{2}\Theta_-(\mathbf{r}')}\rangle$$

Interference pattern



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$$\lim_{|\mathbf{r}|\to\infty} \langle e^{i\sqrt{2}\Theta_-(\mathbf{r})}e^{-i\sqrt{2}\Theta_-(\mathbf{0})}\rangle \to \langle e^{i\sqrt{2}\Theta_-(\mathbf{r})}\rangle\langle e^{-i\sqrt{2}\Theta_-(\mathbf{0})}\rangle = \text{const.}$$

$$C(L_x) = \frac{1}{L_x}\int_0^{L_x} dx\, \langle\hat{B}_Q(x,0)\hat{B}_Q^\dagger(0,0)\rangle = L_x^{-2\alpha} \quad \begin{cases} \alpha = 0 \ \text{(SF)} \\ \alpha = \frac{1}{2} \ \text{(NBG)} \end{cases}$$

Proof of relative coherence but not of superfluidity...

Response to slow rotation

Minimize to find the equilibrium state

$$F(\Omega, T) = -T \log \, \mathrm{Tr} \, e^{-H_{\mathrm{ROT}}(\Omega)/T}$$

$$H_{\mathrm{ROT}}(\Omega) = \sum_{i=1}^{N} \left[\frac{\mathbf{p}_i^2}{2m} - \mathbf{\Omega} \cdot (\mathbf{r}_i \times \mathbf{p}_i) + u(\mathbf{r}_i) \right] + \sum_{i<j}^{N} V(\mathbf{r}_i - \mathbf{r}_j)$$

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$$\Omega L_z = \int d\mathbf{r}\, \mathbf{A}_\Omega(\mathbf{r}) \cdot \mathbf{j}(\mathbf{r})$$

Linear response to a transverse field $(\mathbf{q} \cdot \mathbf{A}_\Omega(\mathbf{q}) = 0)$

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$$\langle L_z \rangle_{\Omega \ll \mu} \simeq \frac{1}{T} \sum_{\alpha,\beta,\gamma} \int d\mathbf{r} d\mathbf{r}' \, \epsilon_{\alpha\beta} r_\alpha \langle j_\beta(\mathbf{r}) j_\gamma(\mathbf{r}') \rangle \, A_\Omega^\gamma(\mathbf{r}')$$

$$\langle j_\alpha(\mathbf{q}) j_\beta(-\mathbf{q}) \rangle = f_{||}(q, T) \frac{q_\alpha q_\beta}{\mathbf{q}^2} + f_\perp(q, T) \left(\delta_{\alpha\beta} - \frac{q_\alpha q_\beta}{\mathbf{q}^2} \right)$$

Response to slow rotation

Minimize to find the equilibrium state

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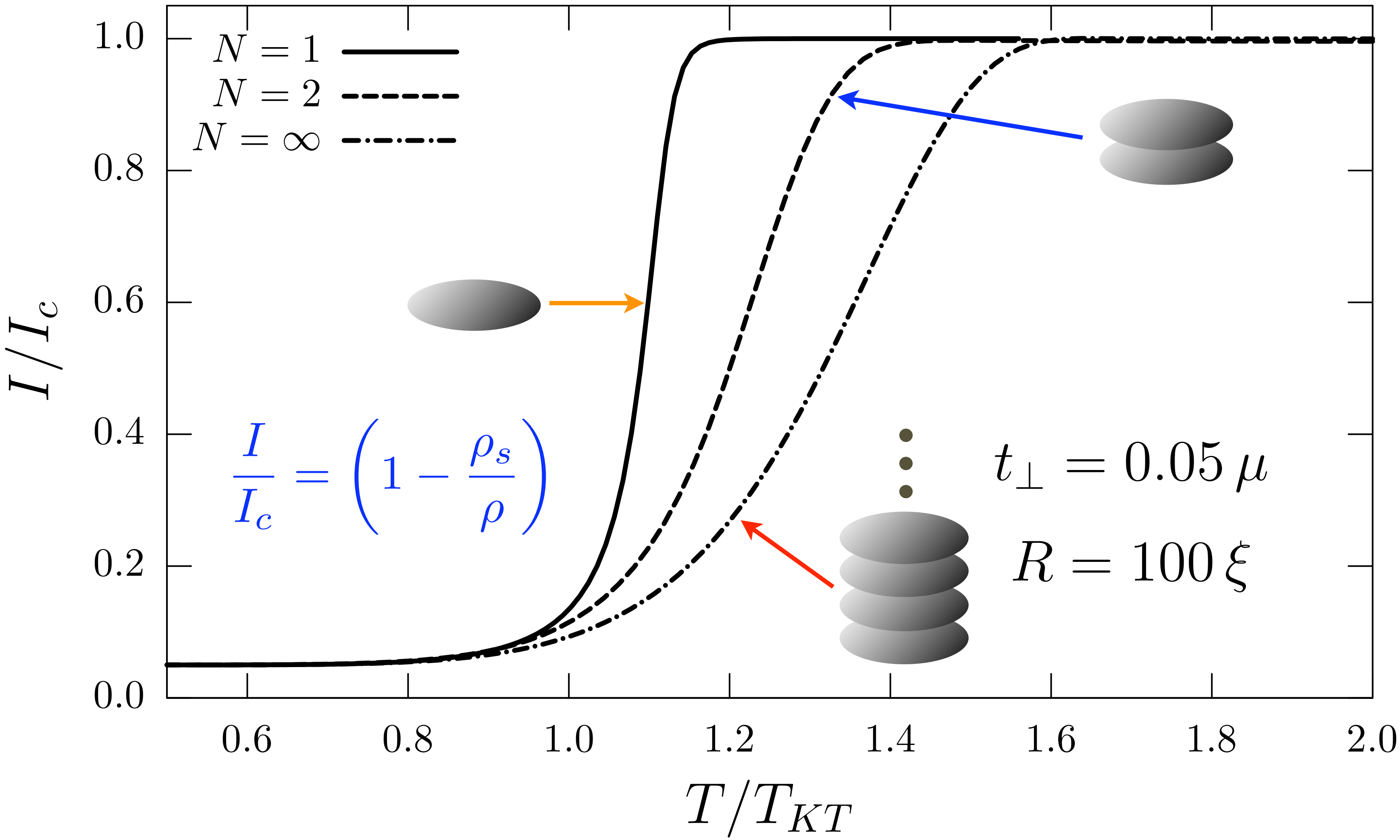
$$\Omega \to 0 \Rightarrow \lim_{q \to 0} f_\perp(q,T) \propto \rho - \rho_s,$$

$\rho_s \propto K_+(T)$ can be obtained from the RG flow

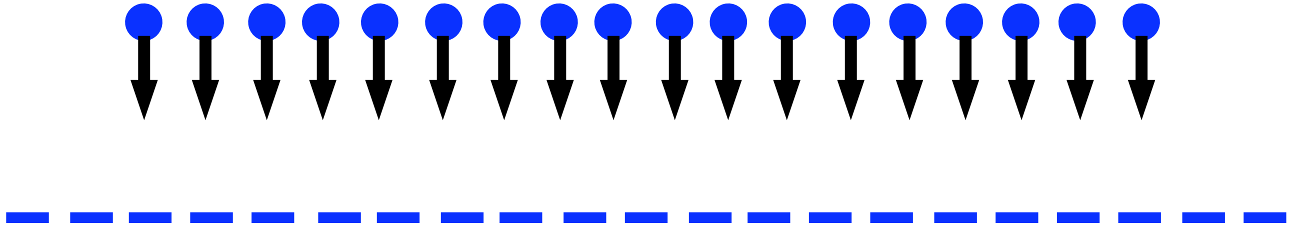
Rotational response

$$L = I\Omega \quad (\Omega \ll \mu)$$

Measure the frequency of the scissors mode
[F Zambelli and S Stringari PRA 63 (2001)]



Fast rotation

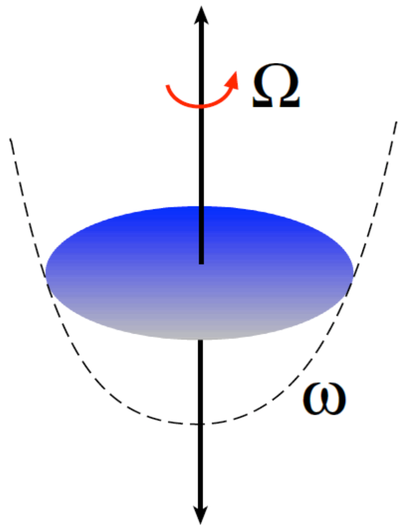


$N \gg 1$

$G \gg N$

Rotation of a superfluid leads to vortices...

$$\Omega \rightarrow \omega^-$$



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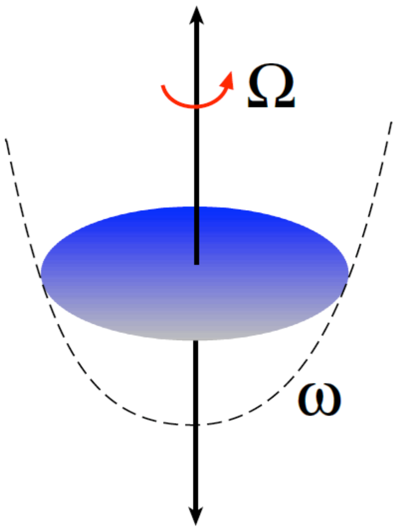

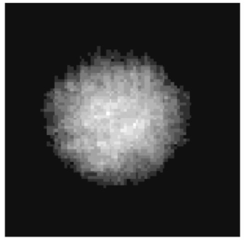

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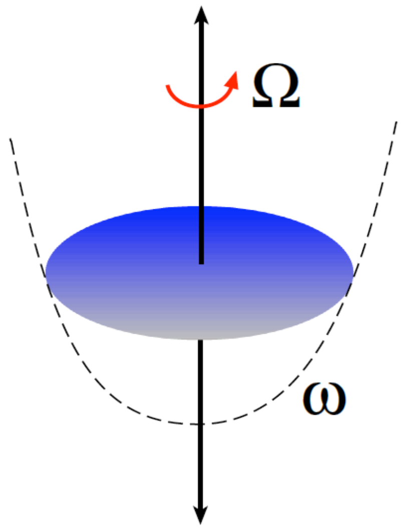

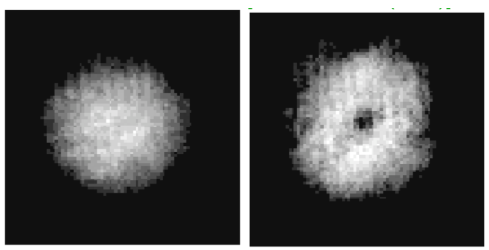

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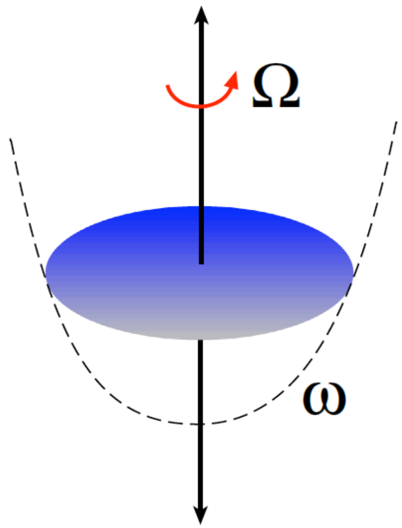

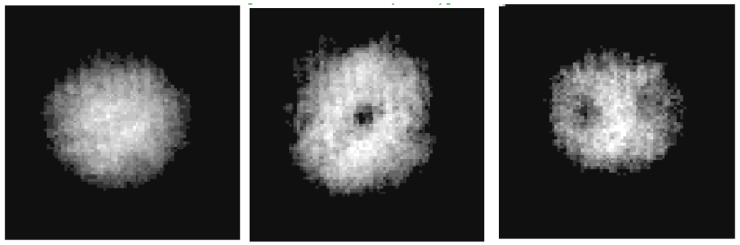

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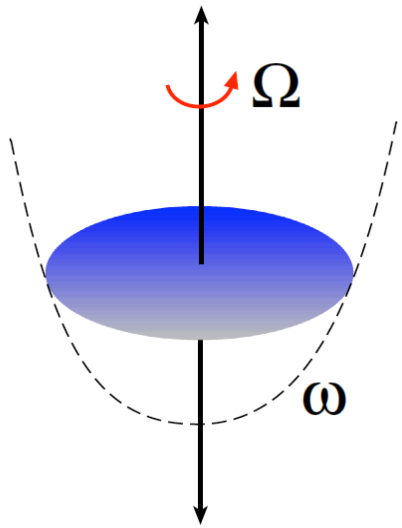

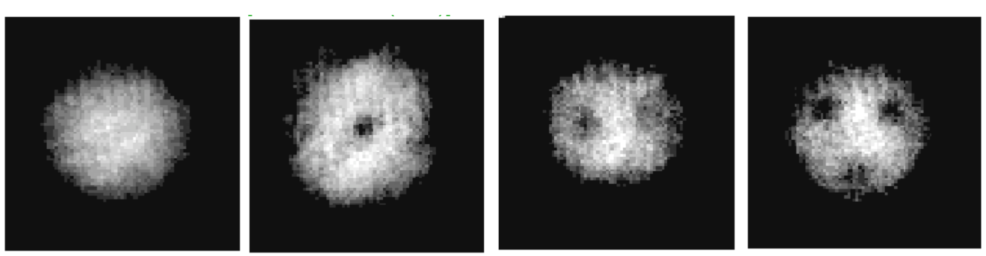

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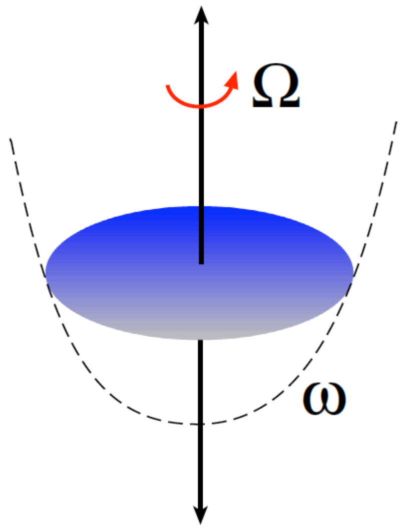

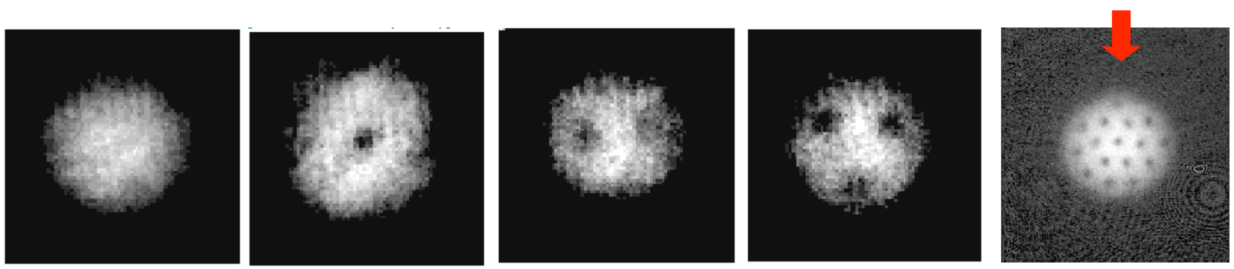

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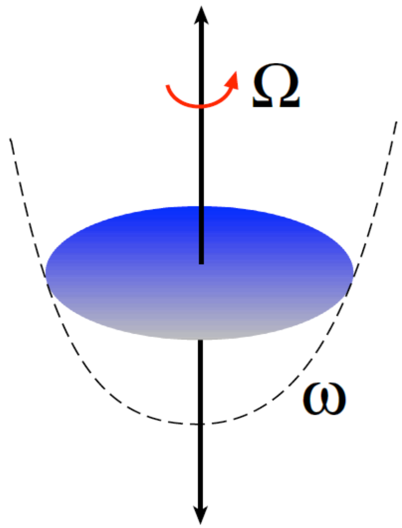

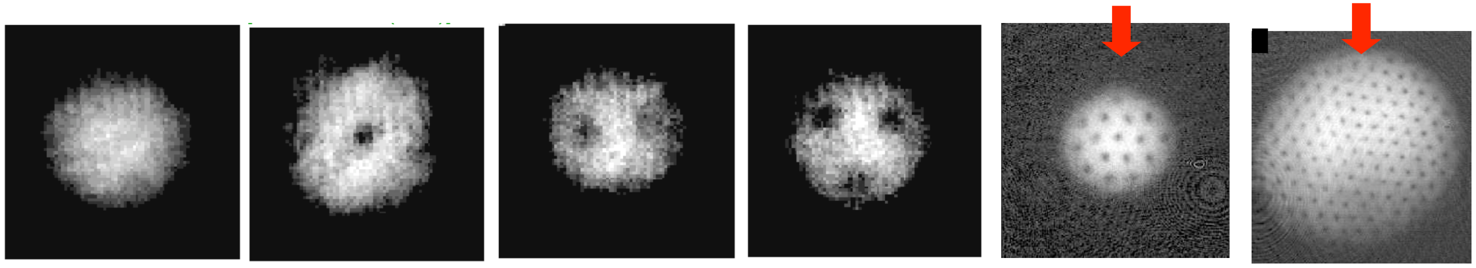

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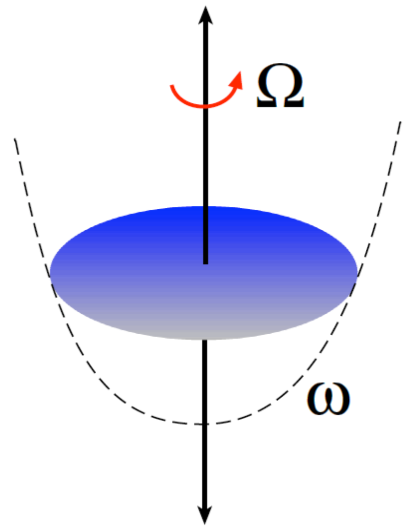
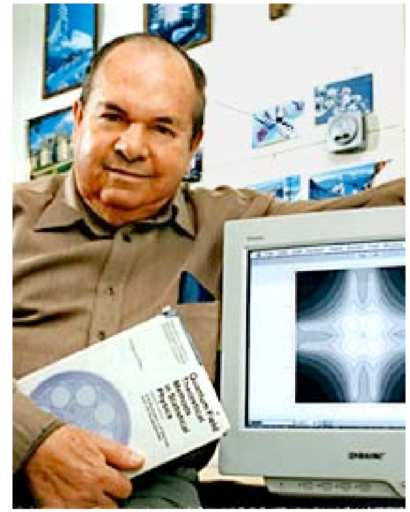
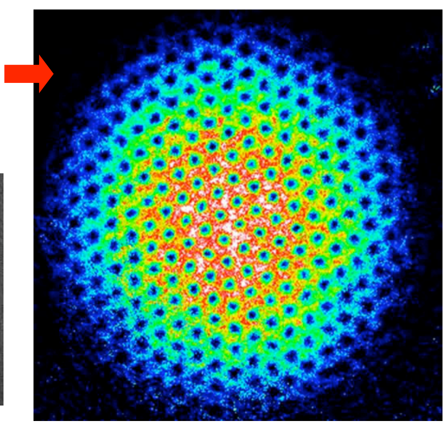
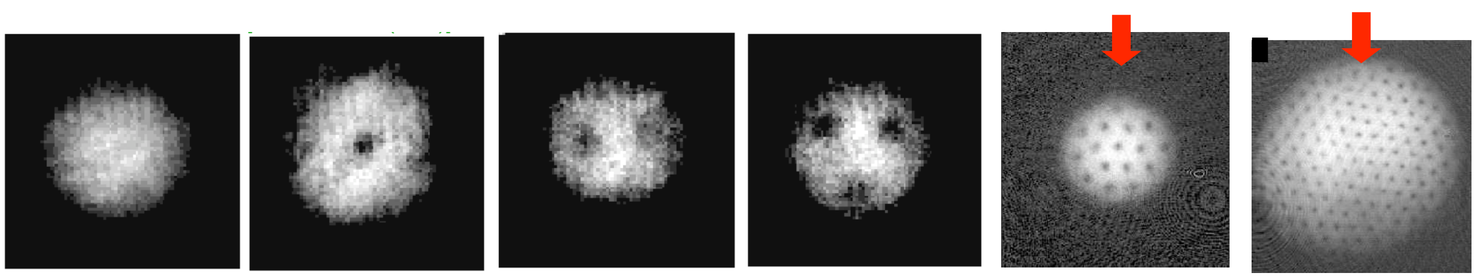




Abrikosov lattice !!





[Madison et al. PRL 84 (2000), Abo-Shaeer et al. Science 476 (2001)]

What happens next?

$$M\left(\frac{d\mathbf{v}}{dt}\right)_{\mathrm{ROT}} = \mathrm{F}_{\mathrm{ext}} + \mathbf{F}_c + \mathbf{F}_c$$

Harmonic trap

$$\mathbf{F}_{\mathrm{ext}} + \mathbf{F}_{\mathrm{C}} = -M(\omega^2 - \Omega^2)\,\mathbf{r}_\perp - M\omega_{||}^2 z\,\hat{\mathbf{z}}$$

Confinement in the rotation plane is greatly reduced: The system becomes quasi-2D

Coriolis force vs. Lorentz force

$$\mathbf{F}_c = \mathbf{v} \times (2M\boldsymbol{\Omega}) \Leftrightarrow F_{\mathrm{L}} = \mathbf{v} \times q\mathbf{B}$$

Coriolis force like an effective magnetic field: Landau levels?

(Rotational) Landau levels (LL's)

$$H_{\mathrm{ROT}} = H_{\mathrm{LAB}} - \Omega L_z$$

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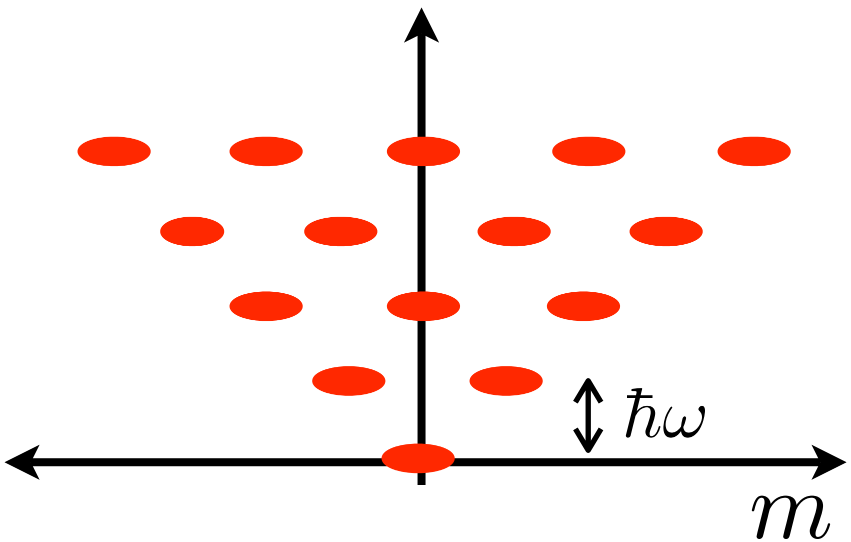
$$H_{\mathrm{ROT}} = H_{\mathrm{LAB}} - \Omega L_z$$

$\Omega = 0$

$$E_n = \hbar\omega(n+1)$$



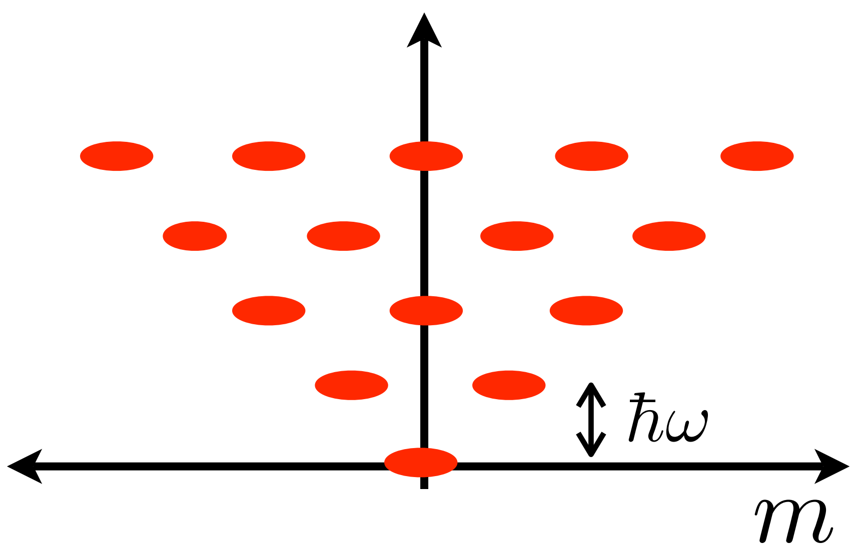
Degeneracy $= n + 1$

(Rotational) Landau levels (LL's)

$$H_{\mathrm{ROT}} = H_{\mathrm{LAB}} - \Omega L_z$$

$\Omega = 0$ $\quad\quad$ $0 < \Omega < \omega$

$$E_n = \hbar\omega(n+1) \qquad E_{n,m} = \hbar\omega(n+1) - \hbar\Omega m$$



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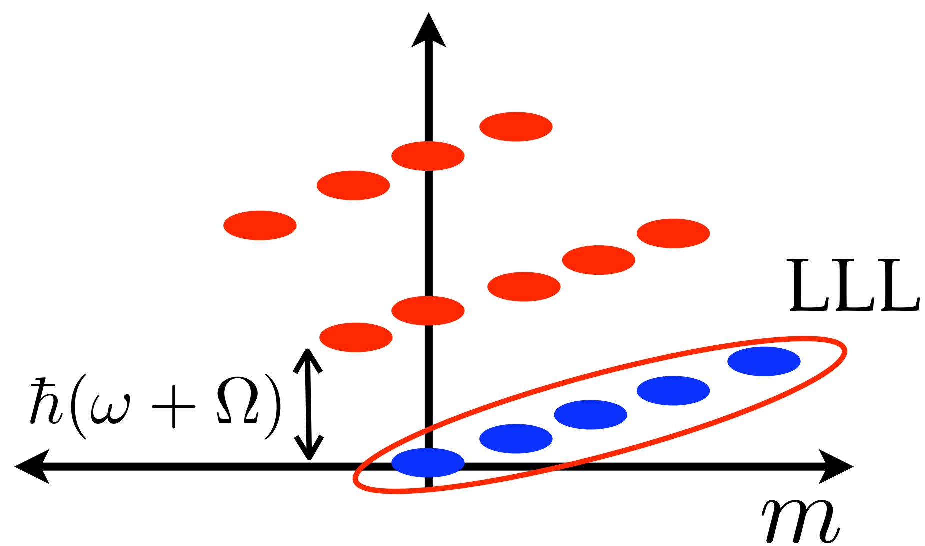


Interactions in the Lowest Landau level (LLL)

$$H_{\mathrm{ROT}} = H_{\mathrm{LAB}} - \Omega L_z$$

$$H_{\mathrm{int}}$$

$$H^{2D}_{\mathrm{ROT}} = \sum_{j=1}^{N} \left[\frac{(\mathbf{p}_j - M\omega \hat{\mathbf{z}} \times \mathbf{r}_j)^2}{2M} + (\omega - \Omega) L_{zj} \right] + g_{2D} \sum_{i<j} \delta(\mathbf{r}_i - \mathbf{r}_j)$$

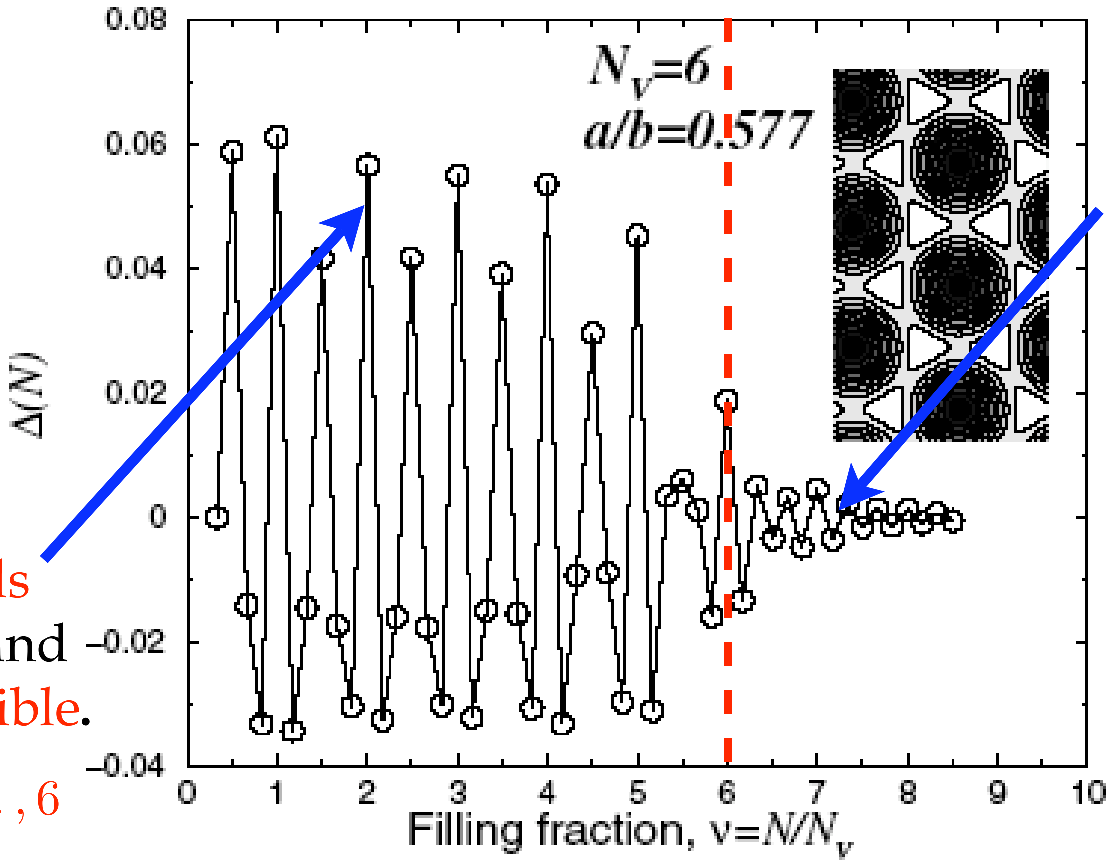
$$\mu, T << \hbar\omega_{||} \qquad \mu, T \ll 2\hbar\omega$$

$$H^{2D}_{\mathrm{LLL}} = N\hbar\omega + (\omega - \Omega) L_z + g_{2D} \sum_{i<j} \delta(\mathbf{r}_i - \mathbf{r}_j)$$

Wave functions in the LLL are analytic functions of $z = x + i\, y$

The fate of Abrikosov lattice: melting and VL's

[N Cooper et al PRL (2001)]



The Abrikosov lattice has shear modes that are gapless. The system is compressible

Vortex liquids exhibit a gap and are incompressible.

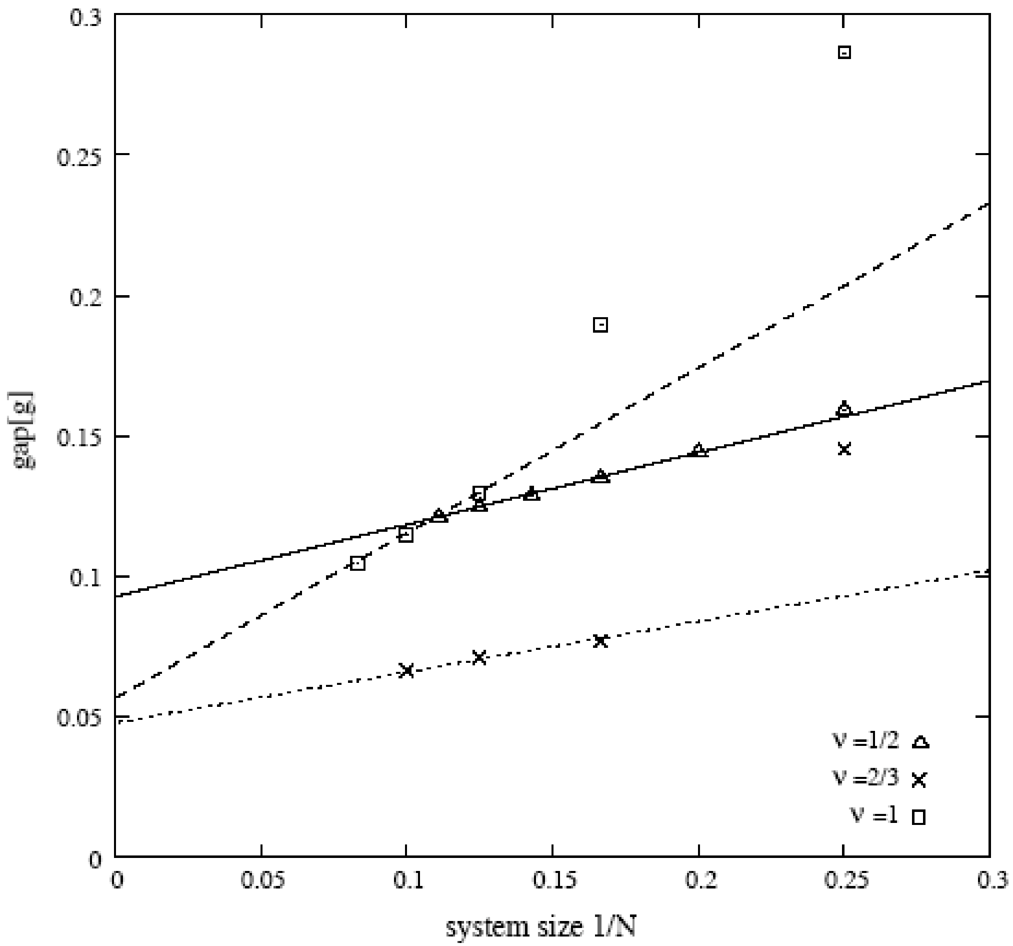
$\nu = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, 6$

$$\nu = \frac{\#\ \text{particles}}{\#\ \text{vortices}} = \frac{N}{N_V}$$

Filling fraction in current experiments ν ~ 100)

Vortex-liquid states on the sphere

[N Reignault and Th Jolicoeur PRL (2003)]



$$2S = \nu^{-1} N - \Delta$$

shift

Some stable fillings

$$\nu = \frac{N_{\text{particles}}}{N_{\text{vortices}}} = \boxed{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}}, 1$$

Principal Jain sequence

$$\Delta = p + 1$$

Pfaffian

$$\Delta = 2$$

‘Simple’ vortex liquids: wavefunctions

Laughlin state $[L_0 = N(N-1)]$ $z = (x+iy)/\ell$ $\left(\ell = \sqrt{\frac{\hbar}{M\omega}}\right)$

$$\Phi_0(\mathbf{r}_1,\ldots,\mathbf{r}_N) = \left\{\prod_{i<j}(z_i - z_j)^2\right\} e^{-\sum_{i=1}^{N}|z_i|^2/2}$$

$$H_{\text{int}}|\Phi_0\rangle = 0$$

Composite Fermion (CF) states

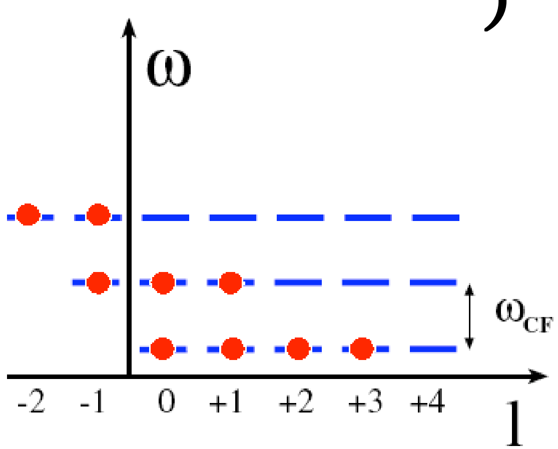
Attaches flux or vorticity

$$\Phi_0(\mathbf{r}_1,\ldots,\mathbf{r}_N) = \mathcal{P}\left\{\prod_{i<j}(z_i - z_j)\det\left[\varphi_{\alpha_i}(\mathbf{r}_1)\cdots\varphi_{\alpha_N}(\mathbf{r}_N)\right]\right\}$$

$$\nu^{-1} = \nu_{CF}^{-1} + 1 = 1 + \frac{1}{p} \quad (N \to +\infty)$$

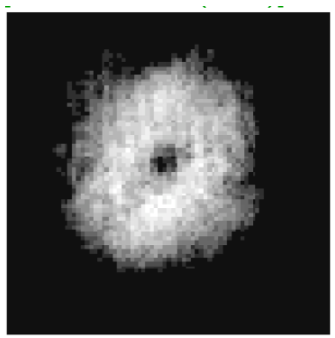
Number of occupied CF LL's

$$H_{\text{int}}|\Phi_0\rangle \neq 0$$



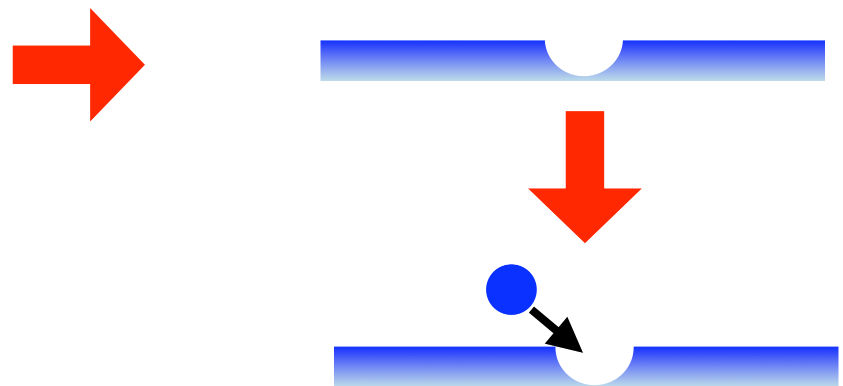
Composite fermion states

[JK Jain in *Perspectives in the Quantum Hall Effects* (Wiley, 1997)]



Vortex

vortex = zero of the wavefunction



bound state of boson + vortex = Composite Fermion (CF)

States of CF's: a CF 'feels' a reduced Coriolis force

Attaches vorticity

$$\Phi_B(\mathbf{r}_1, \cdots \mathbf{r}_N) = \mathcal{P}\left[\prod_{i<j}(z_i - z_j)\, \Phi_{CF}(\mathbf{r}_1, \cdots, \mathbf{r}_N)\right]$$

Filling = 1/2, 2/3, 3/4 ...
Slater determinant of CF

Pfaffian: p-wave
superfluid of CF's

Vortex-liquid states on the sphere

[C-C Chang et al., cond-mat/ 0412253]

ν	N	$\mathcal{O}^2_{\rm gr}$	L	$\mathcal{O}^2_{\rm ex}$	ν	N	$\mathcal{O}^2_{\rm gr}$	L	$\mathcal{O}^2_{\rm ex}$	ν	N	$\mathcal{O}^2_{\rm gr}$	L	$\mathcal{O}^2_{\rm ex}$	ν	N	$\mathcal{O}^2_{\rm gr}$
1/2	4	1.0000	4	0.9972	2/3	4	1.0000	2	1.0000	3/4	9	0.8084(73)	4	0.5613(48)	1	4	1.00000
	5	1.0000	4	0.9965		6	0.9850	4	0.7544(05)		12	0.735(84)	6	0.480(62)		6	0.97279
	6	1.0000	5	0.9959		8	0.9820(10)	5	0.8701(14)							8	0.96687
	7	1.0000	5	0.9954		10	0.9724(89)	6	0.855(12)							10	0.95922
	8	1.0000	6	0.9945												12	0.88435
	9	1.000	6	0.9954 (2)												14	0.88580

Laughlin

$$\nu = \frac{1}{2}$$

Jain sequence

$$\nu = \frac{2}{3}$$

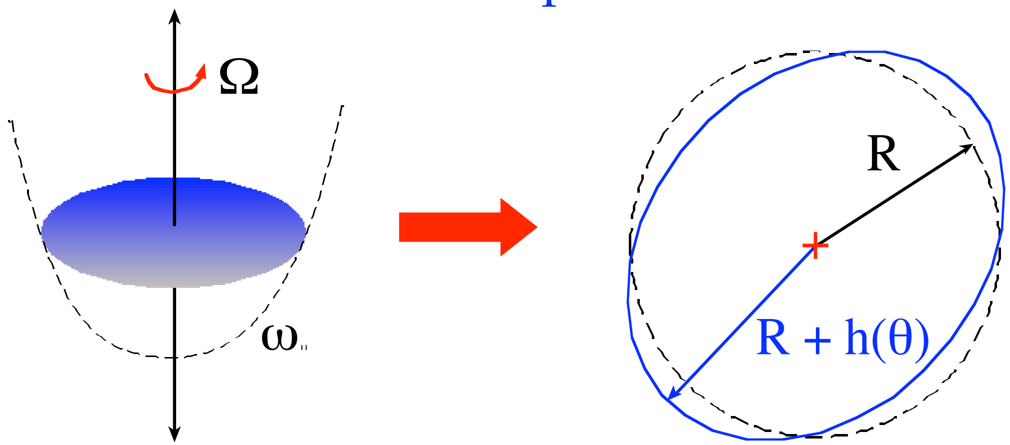
Pfaffian

$$\nu = 1$$

Laughlin state: edge modes

[X-G Wen, PRL 64 (1990) Int J of Mod Phys B (1992), AH MacDonald, PRL 64 (1990)]

Motivation: Detection of the vortex liquids



A chiral Tomonaga-Luttinger liquid [U(1) CFT c = 1]

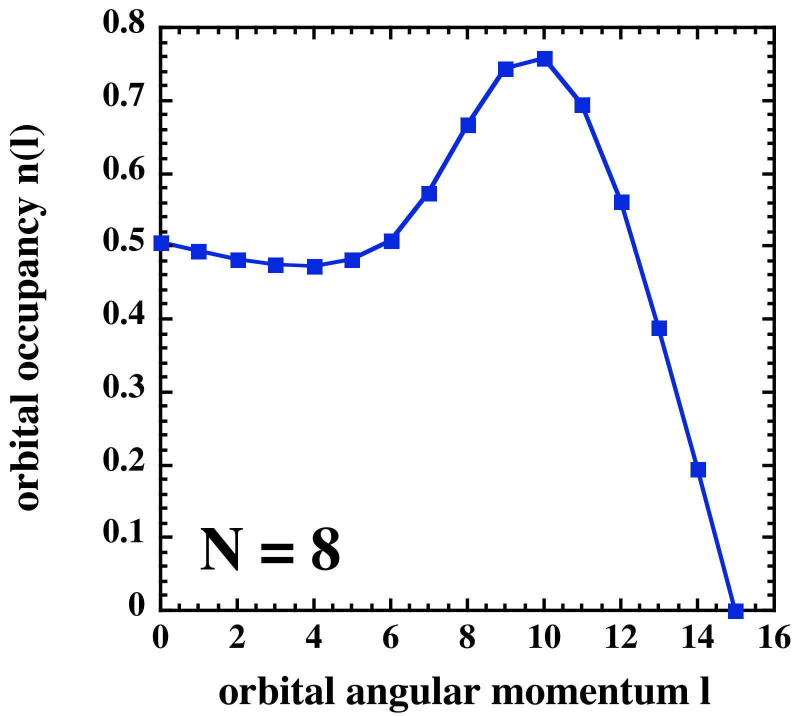
$$\delta\rho(\theta) = \rho_0 R h(\theta) = \frac{\sqrt{\nu}}{2\pi}\partial_\theta\phi(\theta) \quad \left(\rho_0 = \frac{\nu}{\pi\ell^2}\right)$$

$$H_{\text{edge}} = E_0 + \frac{\hbar(\omega-\Omega)}{2}\int_0^{2\pi}\frac{d\theta}{2\pi}\,\left(\partial_\theta\phi(\theta)\right)^2 = \hbar(\omega-\Omega)L$$

$$L = L_0 + \sum_{m>0} m\, b_m^\dagger b_m \quad \left[b_l, b_m^\dagger\right] = \delta_{l,m}$$

Laughlin state: edge theory predictions

Laughlin state (N = 8, L = 56)



Bosonization formula:

$$\Psi(z = Re^{i\theta}) = \mathcal{A}^{1/2}\, e^{il_0\theta}\, e^{i\frac{\phi(\theta)}{\sqrt{\nu}}}$$

$$G_1(\theta) = \langle \Psi^\dagger(Re^{i\theta})\Psi(R)\rangle$$

$$n(l) = \langle a^\dagger(l)a(l)\rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{il\theta} G_1(\theta)$$

$$n(l_{\max}) : n(l_{\max}-1) : n(l_{\max}-2) : n(l_{\max}-3) = 1 : 2 : 3 : 4\,(\text{theory}|)$$

$$n(l_{\max}) : n(l_{\max}-1) : n(l_{\max}-2) : n(l_{\max}-3) = 1.0 : 2.0 : 2.9 : 3.5\,(N=7)$$

$$n(l_{\max}) : n(l_{\max}-1) : n(l_{\max}-2) : n(l_{\max}-3) = 1.0 : 2.0 : 2.9 : 3.6\,(N=8)$$

[MAC, N Barberan, NR Cooper PRB 71R (2004)]
Similar studies for electrons by [JJ Palacios & AH MacDonald, PRL 76 (1996)]

Laughlin state: edge mode wavefunctions

Consider the symmetric polynomials:

$$s_m = \sum_{i_1 < i_2 < \cdots < i_m} z_{i_1} z_{i_2} \cdots z_{i_m} \qquad \left(s_m = \sum_{i=1}^{N} z_i^m \right)$$

$$\Phi_m = s_m \, \Phi_0(z_1, \ldots, z_N) \quad L = L_0 + m \quad E = E_0 + \hbar(\omega - \Omega) m$$

degeneracy: $p(m) (p(1) = 1, \, p(2) = 2, p(3) = 3, \, p(4) = 5, \, p(5) = 7, \ldots)$

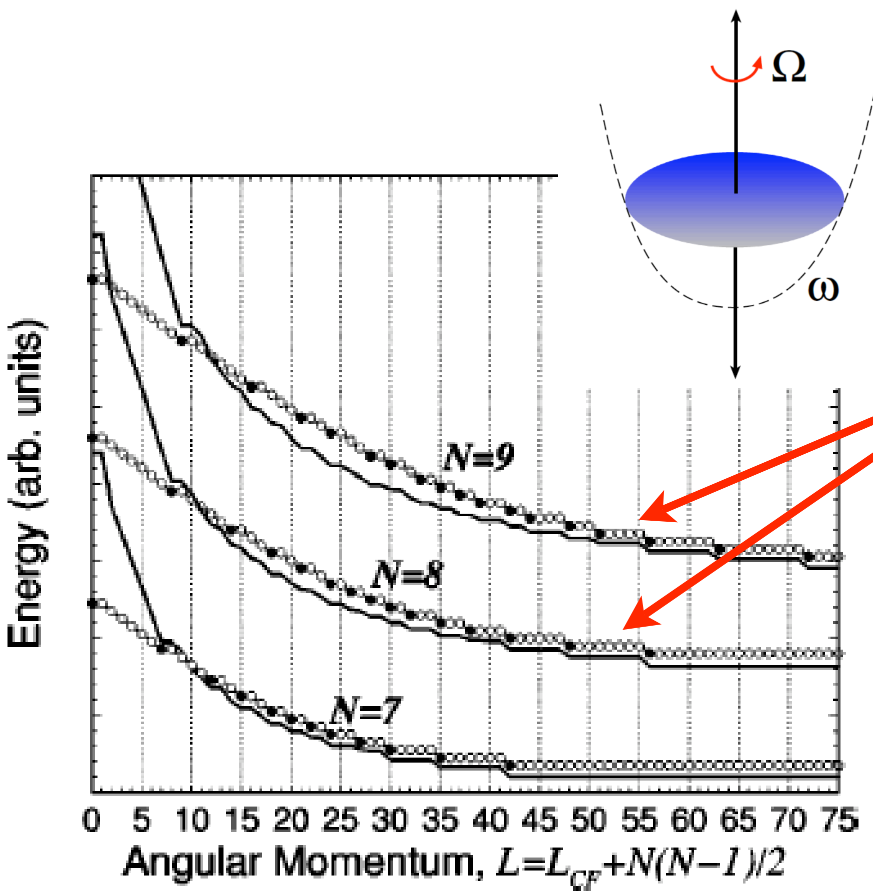
Compare with the quasi-hole excitation

$$\Phi_{\mathrm{qh}} = \left(\prod_{i=1}^{N} z_i \right) \Phi_0(z_1, \ldots, z_N) \quad L = L_0 + N \quad E = E_0 + \hbar(\omega - \Omega) N$$

Provided that $m \ll N$ the s_m are the low-lying excitations

Composite fermion states: edge modes

[N Cooper & N K Wilkin PRB 1999]

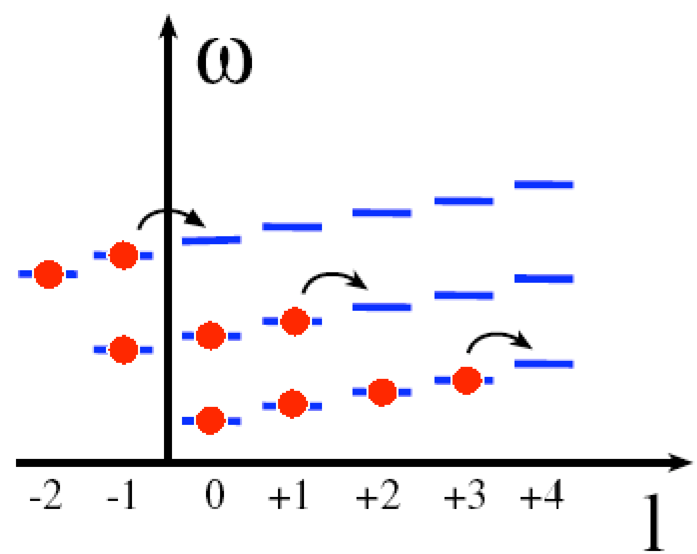


Plateaux correspond to edge structures of compact CF states. These show good overlap with the exact states, and have good variational energy.

p occupied CF LL's → p phonon branches

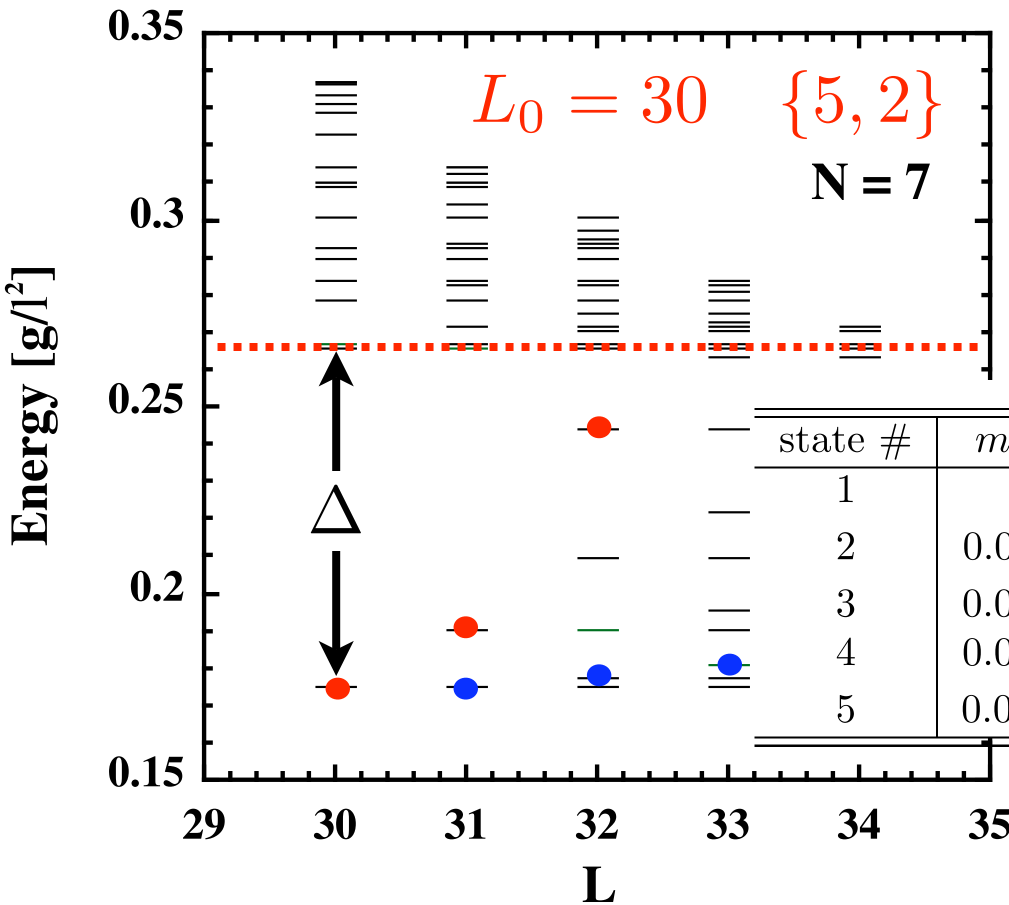
e.g. {4,3,2} → 3 branches

$$\Phi_B(\mathbf{r}_1,\cdots\mathbf{r}_N) = \mathcal{P}\left[\prod_{i<j}(z_i - z_j)\,\Phi_{CF}(\mathbf{r}_1,\cdots,\mathbf{r}_N)\right]$$



Compact CF states: edge spectrum

[MAC, N Barberan, NR Cooper PRB 71R (2005)]



Angular momentum

$$L = L_0 + \sum_{m>0} m \ \left[b_m^\dagger b_m + d_m^\dagger d_m\right]$$

state #	$m = 2$ (th.)	$m = 3$ (th.)	state #	$m = 3$ (th.)
1	0 (0)	0 (0)	6	0.034 (0.030)
2	0.002 $(\hbar\omega_2^{(b)})$	0.002 (0.002)	7	0.047 (0.045)
3	0.015 (0.015)	0.006 $(\hbar\omega_3^{(b)})$	8	0.069 (0.069)
4	0.034 (0.030)	0.015 (0.015)	9	0.088 (0.084)
5	0.069 $(\hbar\omega_2^{(d)})$	0.020 (0.017)	10	$(\hbar\omega_3^{(d)})$

Hamiltonian:

$$H_{\text{edge}} = E_0 + \hbar \sum_{m>0} \left[\omega_m^{(b)} \ b_m^\dagger b_m + \omega_m^{(d)} \ d_m^\dagger d_m\right]$$

Pfaffian or Moore-Read State

[G Moore & N Read Nucl. Phys. B (1991)]

As the number of occupied CF LL's grows... $p \to +\infty \Rightarrow \nu = \frac{p}{p+1} \to 1$

Composite Fermion (Fermi) liquid? → compressible liquid

Composite Fermion pairing? → incompressible liquid again!

$$\Phi_0(z_1, \ldots, z_N) = \prod_{i<j}(z_i - z_j) \left[\frac{1}{2^{N/2}(N/2)!} \sum_{P \in S_N} \mathrm{sgn}(P) \prod_{k=1}^{N/2} \frac{1}{z_{P(2k-1)} - z_{P(2k)}} \right]$$

BCS wavefunction for p-wave paired (spinless) CF's → $\mathrm{Pf}\left[\frac{1}{z_i - z_j}\right]$

$$H_{\text{int}}^{(3)} = g_3 \sum_{i<j<k} \delta(\mathbf{r}_i - \mathbf{r}_j)\delta(\mathbf{r}_j - \mathbf{r}_k) \qquad H_{\text{int}}^{(3)}|\Phi_0\rangle = 0$$

[M Greiter, X-G Wen, and F Wilczek PRL (1991)]

Edge excitations of the Pfaffian

[M Milovanovic & N Read PRB (1996), X-G Wen PRL (1992)]

$$\Phi_0(z_1,\ldots,z_N) = \prod_{i<j}(z_i - z_j)\left[\frac{1}{2^{N/2}(N/2)!}\sum_{P\in S_N}\mathrm{sgn}(P)\prod_{k=1}^{N/2}\frac{1}{z_{P(2k-1)} - z_{P(2k)}}\right]$$

Fermionic-like excitations

$$\Phi_{n_1,n_2,\ldots,n_F} = \prod_{i<j}(z_i - z_j)\frac{2^{-(N-F)/2}}{((N-F)/2)!}\sum_{P\in S_N}\mathrm{sgn}(P)\left[\frac{z_{P(1)}^{n_1} z_{P(2)}^{n_2}\cdots z_{P(F)}^{n_F}}{(z_{P(F+1)} - z_{P(F+2)})\cdots(z_{P(N-1)} - z_{P(N)})}\right]$$

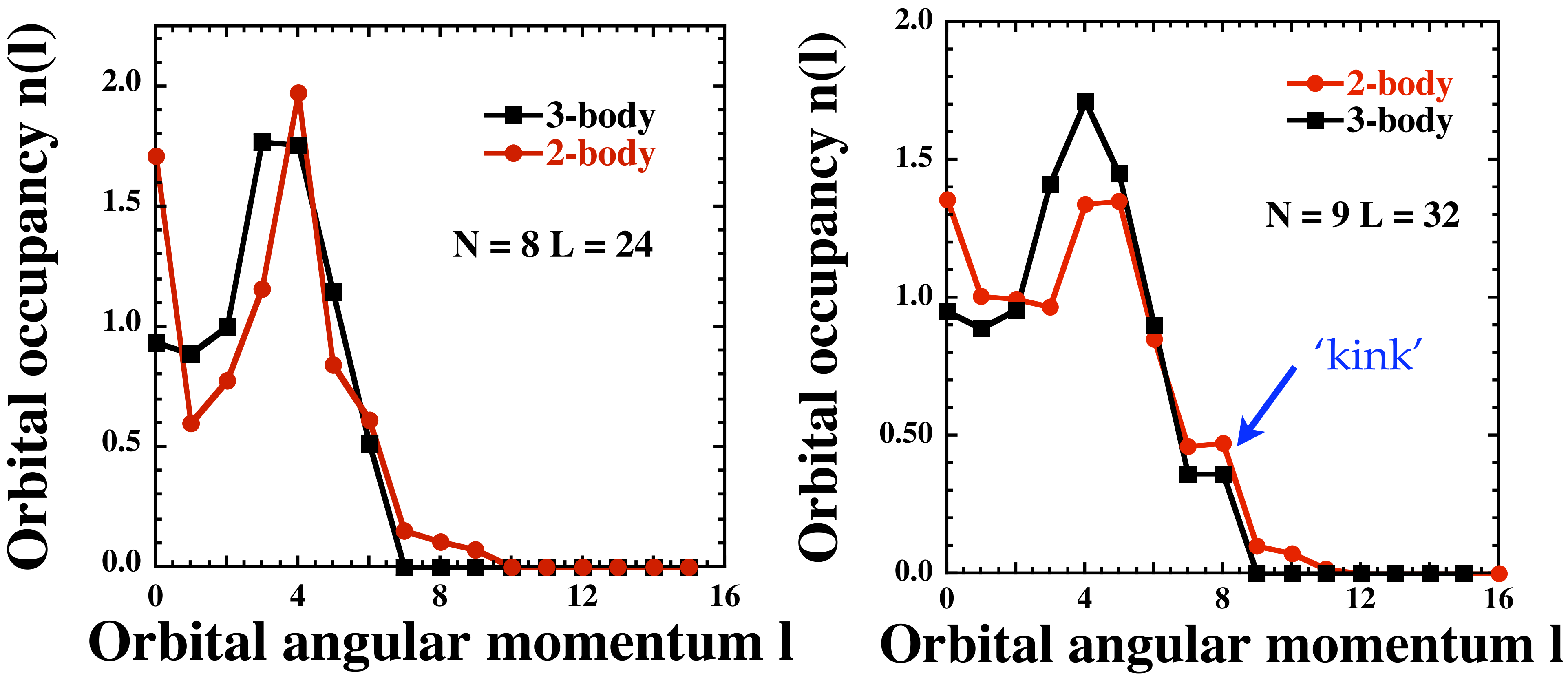
$$H_{\mathrm{edge}} = E_0 + \hbar(\omega - \Omega)L$$

$$L = L_0 + \sum_{m>0}\left[m\, b_m^\dagger b_m + (m - \frac{1}{2})\, c_{m-\frac{1}{2}}^\dagger c_{m-\frac{1}{2}}\right] \qquad \left\{c_{m-\frac{1}{2}}, c_{n-\frac{1}{2}}^\dagger\right\} = \delta_{n,m}$$

$$\textbf{Ising} \times \mathrm{U}(1)\ \textbf{CFT}$$

Even/odd effects in the Pfaffian

The paired nature of the state leads to even/odd effects.
E.g. it can be seen in the orbital occupancy:



Edge excitation degeneracies of the Pfaffian

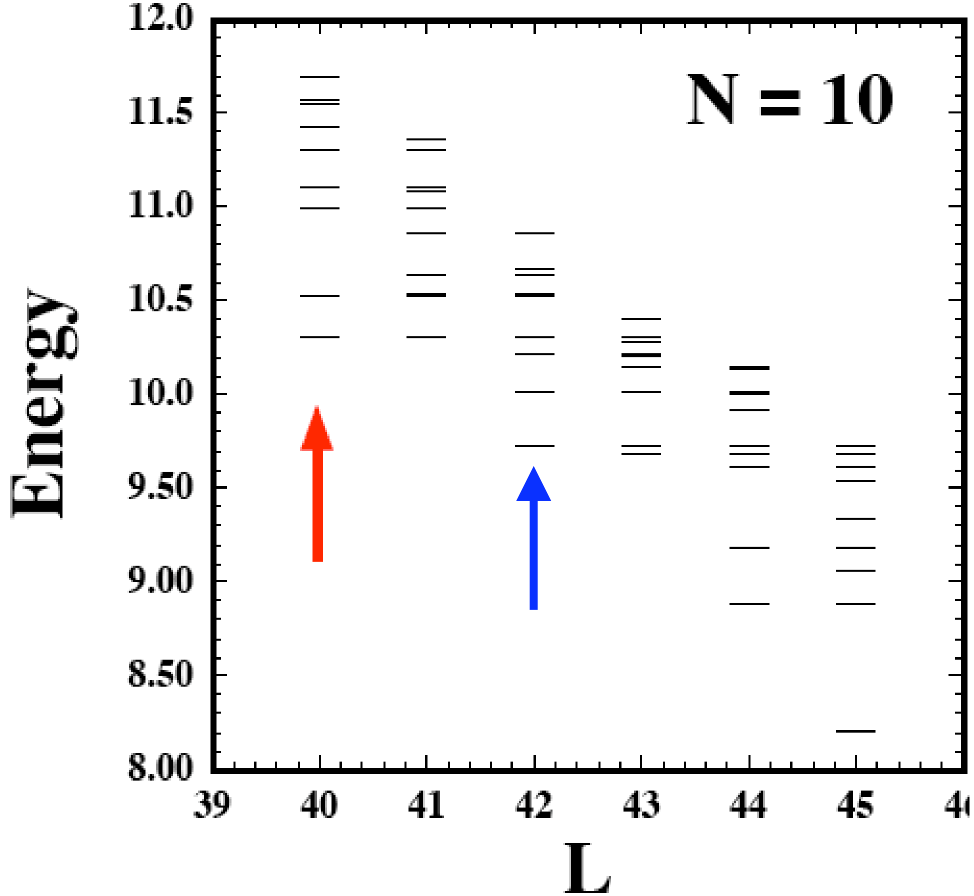
[MAC, N Barberan, NR Cooper PRB 71R (2005)]

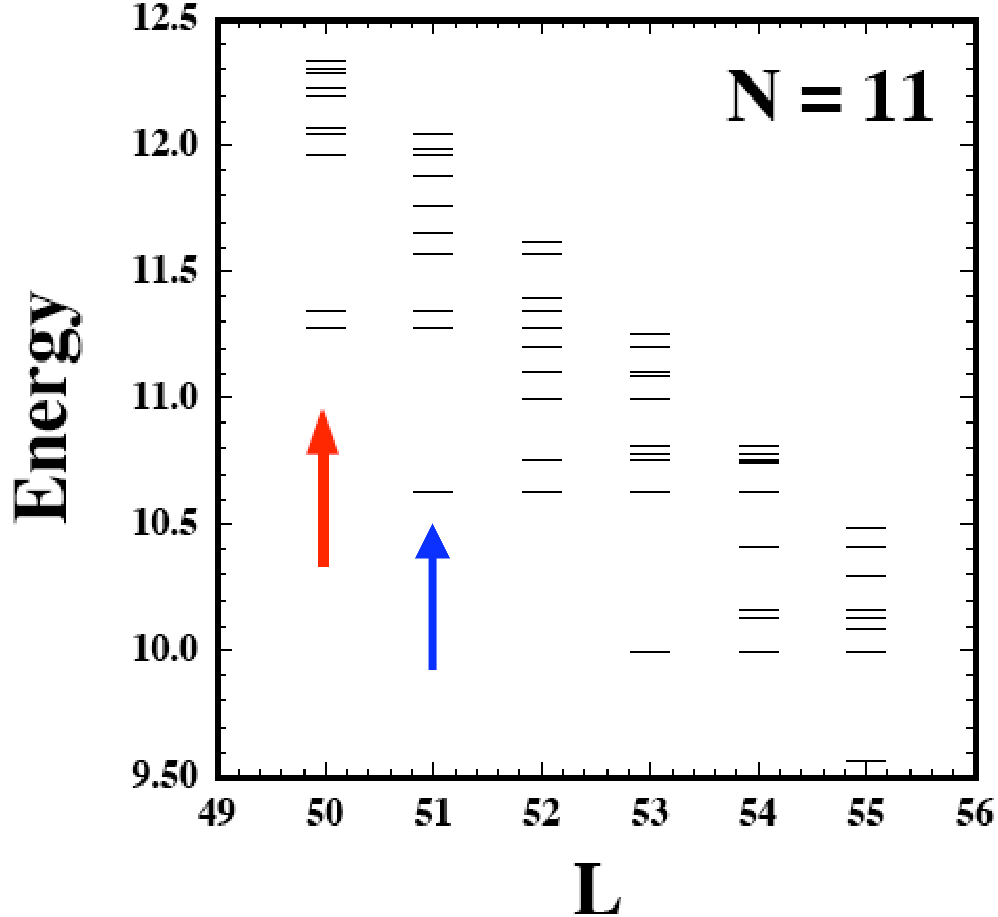
$m = L - L_0$	0	1	2	3	4	5	6
Laughlin ($N = 5,\ L = 20$)	1	1	2	3	5	7	10
Laughlin ($N = 6,\ L = 30$)	1	1	2	3	5	7	11
Laughlin ($N \to \infty$)	1	1	2	3	5	7	11
$\{4, 2\}$ CF ($N = 6,\ L = 20$)	1	2	5	8			
$\{5, 2\}$ CF ($N = 7,\ L = 30$)	1	2	5	9	15		
Jain $\nu = \frac{2}{3}$ ($N \to \infty$)	1	2	5	10	20	36	65
Moore-Read ($\lambda = 0,\ N = 8,\ L = 24$)	1	1	3	5	10	15	
Moore-Read ($\lambda = 0$, even $N \to \infty$)	1	1	3	5	10	16	28
Moore-Read ($\lambda = 1,\ N = 12,\ L = 60$)	1	4	10	21			
Moore-Read ($\lambda = 0,\ N = 7,\ L = 18$)	1	2	4	7	12		
Moore-Read ($\lambda = 0$, odd $N \to \infty$)	1	2	4	7	13	21	35
Moore-Read ($\lambda = 1,\ N = 13,\ L = 72$)	1	6	14	29			

$$H_{\text{int}} = \lambda H_{\text{int}}^{(2)} + (1 - \lambda) H_{\text{int}}^{(3)}$$

Pfaffian: Energy plateaux in $E_0(L)$ vs L

[MAC, N Barberan, NR Cooper PRB 71R (2005)]





No wide plateaux in the interaction energy (for 4 < N < 14)!!

Pfaffian: Overlaps

Torus(PBC) [N Cooper et al. PRL (2001)]

k	ν	$(K_x, K_y) \times$ degeneracy	$\lvert\langle \Psi^{(k)} \mid \Psi \rangle\rvert$	$\lvert\langle \Psi^{GP} \mid \Psi \rangle\rvert$
1	1/2 (Laughlin)	$(0,0) \times 2$	1.000	0.555
2	1 (Moore-Read)	$(3,3) \times 1$	0.989	N/W
2	1 (Moore-Read)	$(3,0) \times 1$	0.982	0.408
2	1 (Moore-Read)	$(0,3) \times 1$	0.981	0.493

Sphere

ν	N	$\mathcal{O}^2_{\mathrm{gr}}$
1	4	1.00000
	6	0.97279
	8	0.96687
	10	0.95922
	12	0.88435
	14	0.88580

[C-C Chang et al., cond-mat/ 0412253]

Harmonic trap (disk)

N	$\lvert\langle \Phi_{\mathrm{MR}} \lvert \Phi(L_0) \rangle\rvert$
5	0.913
6	0.896
7	0.805
8	0.676
9	0.742

[MAC, N Barberan, NR Cooper PRB 71R (2005)]

Experimental consequences

Total oscillator strength at $L = L_0 + m$ (can be measured by absorption spectroscopy)

$$f_m = \sum_\alpha |\langle L_0 + m, \alpha | O_m | L_0 \rangle|^2 = R^{2m} \nu\, m = m N^m \nu^{1-m}$$

$$O_m = \sum_{i=1}^{N} z_i^m \sim R^m \, e^{im\theta} \qquad \frac{f_2}{2 f_1^2} = \nu^{-1}$$

STATE	f_1 (th.)	f_2 (th.)	rel. err. in f_2
Laugnlin $(\nu = \frac{1}{2})$	5 (5)	94.5 (100)	5.5%
Laughlin $(\nu = \frac{1}{2})$	6 (6)	138.2 (144)	4.0%
Pfaffian, $(\nu = 1)$	5 (5)	44.6 (50)	10.8%
Pfaffian $(\nu = 1)$	6 (6)	65.3 (72)	9.3%
$\{4, 2\}$, $(\nu = \frac{2}{3})$	6 (6)	96.9 (108)	10.2%
$\{5, 2\}$, $(\nu = \frac{2}{3})$	7 (7)	141.5 (147)	3.7%

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