van der Wals interfacial energies

- shear bands in rubber-like materials: Knowles & Sternberg observed that loss of strong ellipticity of Hessian matrix

$\Downarrow$

spatial hyperbolicity
gives location of bands

in real materials, bands have a width $\Rightarrow$ higher gradient terms will impart width.

$\parallel$

suggestion of Triantafyllidis in the 90's

$\Downarrow$

Introduce higher ∇'s at μ-level & homogenize.
Investigate macro-stability.

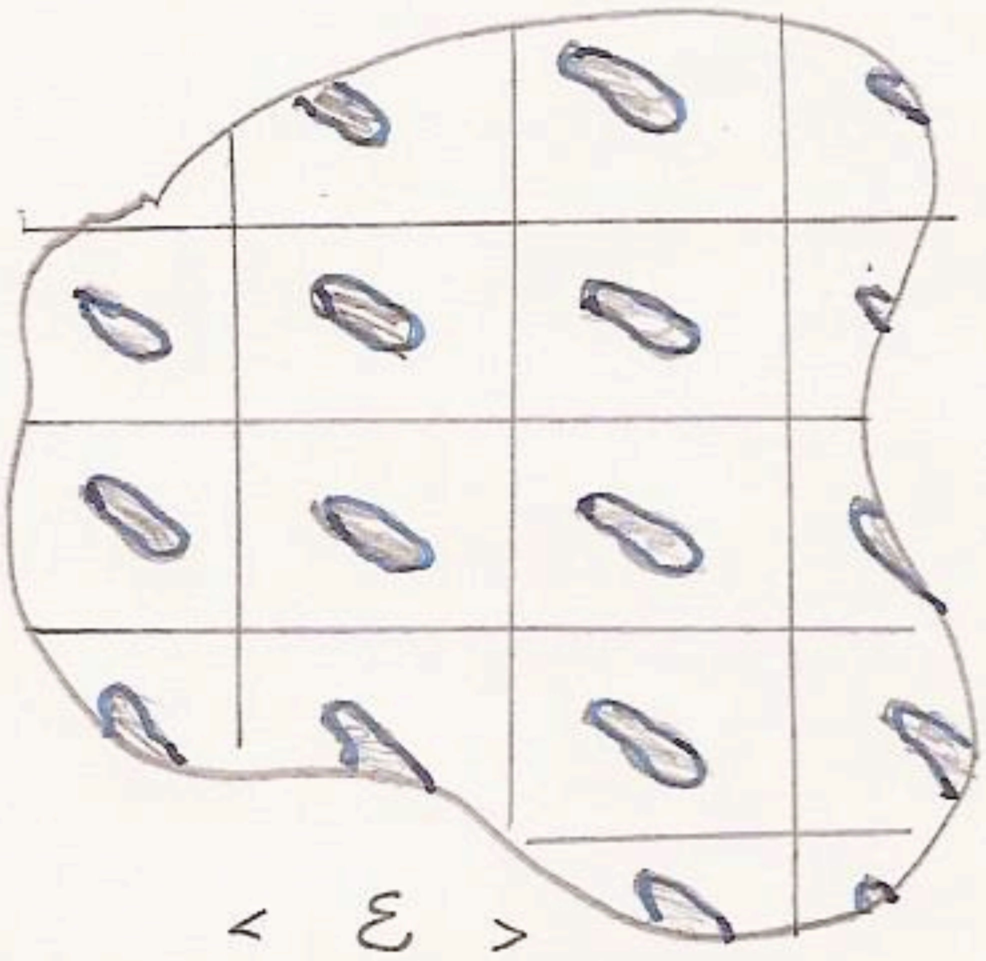
- polycrystalline thin films sensitive to interfacial effects: Bhattacharya & James observed exact attainment for films originating in 3d-models with fixed interfacial energy

$\Downarrow$

Investigate correlation between strength of interfacial energy and thickness of film.

Model

- $\gamma > 0$



$$E^{\gamma}_{\varepsilon}(u;\Omega) := \varepsilon^{\gamma}\int_{\Omega}|D^2u|^2\,dx + \int_{\Omega} W\left(\tfrac{x}{\varepsilon}, Du\right)dx$$

- $\Omega_{\varepsilon} = \omega \times \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$

$$\varepsilon^{\gamma-1}\int_{\Omega^{\varepsilon}}|D^2u|^2\,dx + \frac{1}{\varepsilon}\int_{\Omega^{\varepsilon}} W(Du)\,dx$$

$\downarrow$ $1/\varepsilon$ dilation

$\Omega := \omega \times (-1,1)$

$$\varepsilon^{\gamma}\int_{\Omega}\left(|D_{\alpha\beta}u|^2 + \frac{1}{\varepsilon^2}|D_{\alpha 3}u|^2 + \frac{1}{\varepsilon^4}|D_{33}u|^2\right)dx + \int_{\Omega} W\left(D_{\alpha}u \,\Big|\, \frac{1}{\varepsilon}D_3u\right)dx$$

$$=: E^{\gamma}_{\varepsilon}(u;\omega)$$

energy density

$$\begin{cases} 1 < p < \infty \\ W(F) \sim |F|^p \end{cases}$$

Goal

Find $\Gamma - \liminf E^{\gamma}_{-}$ of $E^{\gamma}_{\varepsilon}(u; \Omega \text{ or } \omega)$

knowing : $u^{\varepsilon} \xrightarrow[L^2]{} u$

+ case of thin films :

$\frac{1}{\varepsilon} D_{\varepsilon} u^{\varepsilon} \longrightarrow b$ (Cosserat vector)

thin film case
$\left\{ \begin{array}{l} u : \text{describes mid-plane behavior} \quad \text{independent of } x_3 \\ \quad\quad \text{while} \\ b : \text{cross-sectional behavior} \quad \text{independent of } x_3 \text{ iff } \gamma < 2 \end{array} \right.$

- homogenization case (old) with S. Müller

- thin film case (new) with I. Fonseca, G. Leoni.

Local character of limit energy

- $E^{\gamma}_{-}(u \text{ (or } (u,b)); \bullet)$ is a finite non-negative Radon measure μ a.c. w.r.to $\mathcal{L}^N$.

(variant of De Giorgi's slicing method).

Homog. case

$\Downarrow$

$$E^{\gamma}_{-}(u; A) = \int_A W^{h}(Du)\,dx$$

it remains to characterize

$W^{h}(F)$.

thin film case

$\Downarrow$

cannot apply Buttazzo's result

must characterize

$$\frac{d\mu}{d\mathcal{L}^2}.$$

Sub - critical case $\gamma < 2$

expectation : the strength of the singular perturbation induces "strong" convergence

- thin film case : b ind^{t} of x_3
  - $E_-^{\gamma}(u, b; \omega) = \int_\omega Q_2 \times C_2 [W](D_\alpha u | b)\, dx_\alpha$

$$\inf \underline{\lim} \int_{A\times(-\frac{1}{2},\frac{1}{2})} W\left(D_\alpha u_\varepsilon \Big| \frac{1}{\varepsilon} D_3 u^\varepsilon\right) dx \leq E_-^{\gamma}(u, b; A) \leq \int_A W(D_\alpha u | b)\, dx_\alpha$$

$u + \varepsilon x_3 b$ as test fct.

$$\inf \underline{\lim} \int_{A\times(-\frac{1}{2},\frac{1}{2})} Q_3 \times C_3 [W](\text{idem})$$

Cross quasiconvex - convex

$$\geq \int_A Q_3 \times C_3 [W] (D_\alpha u | b)\, dx$$

$$\Downarrow$$

$$\int_A Q_3 \times C_3 [W](D_\alpha u | b)\, dx_\alpha \leq E_-^{\gamma}(u, b; A) \leq \int_A Q_2 \times C_2 [W](D_\alpha u | b)\, dx_\alpha$$

But

$$Q_3 \times C_3 [W](\bar{F} | b) = Q_2 \times C_2 [W](\bar{F} | b).$$

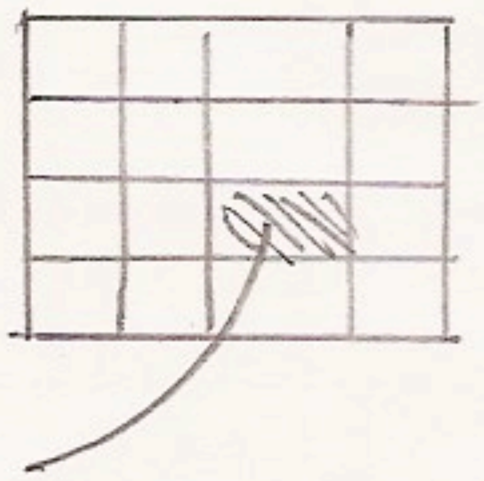
- homogenization case:

$$E_-^{\gamma}(u,\Omega) = \int_\Omega Q[\overline{W}](Du)\,dx$$

$$\lim_\varepsilon \int_\Omega W\left(\tfrac{x}{\varepsilon}, D\varphi_\varepsilon\right)dx + \varepsilon^{\gamma}\int_Q |D^2\varphi_\varepsilon|^2 = E_-^{\gamma}(Fx; \underbrace{Q}_{\text{unit cube}}) = W^{k}(F) \leq \overline{W}(F)$$

φ_ε p-equi-int. / $\varphi_\varepsilon \xrightarrow[W^{1,p}]{} Fx$

Q



Q_k^ε

Set: $F_\varepsilon^{k} := ⨍_{Q_k^\varepsilon} D\varphi_\varepsilon\,dx$

$$\|D\varphi_\varepsilon - F_\varepsilon^{k}\|^2_{L^2(Q_k^\varepsilon)} \overset{\text{P.W.}}{\leq} \varepsilon^2 \|D^2\varphi_\varepsilon\|_{L^2(Q_k^\varepsilon)}$$

$$\Downarrow$$

$$\|D\varphi_\varepsilon - \underset{\text{piecewise affine}}{F_\varepsilon}\|^2_{L^2(Q)} \leq \varepsilon^{2-\gamma}\,\underbrace{\varepsilon^{\gamma}\|D^2\varphi_\varepsilon\|^2_{L^2}}_{\text{bounded}} \Rightarrow D\varphi_\varepsilon - F_\varepsilon \xrightarrow{a.e.} 0$$

Egorov

$$\lim_\varepsilon \geq \liminf_\varepsilon \int_{Q^\eta} W\left(\tfrac{x}{\varepsilon}, F_\varepsilon\right)dx \geq \liminf_\varepsilon \int_Q W\left(\tfrac{x}{\varepsilon}, F_\varepsilon\right)dx - O(\eta)$$

F_ε bd in L^p by $\|D\varphi_\varepsilon\|_{L^p}$ + p-equi-int

$$= \liminf_\varepsilon \int_Q \overline{W}(F_\varepsilon)dx - O(\eta) \geq \liminf_\varepsilon \int_Q \overline{W}(D\varphi_\varepsilon)dx - O(\eta) \geq Q[\overline{W}](F) - O(\eta)$$

p-equiint. of $D\varphi_\varepsilon$

Critical case $\gamma = 2$

expectation : singular perturbation is felt.

- Homogenization case : "easy"

$$E_-^2(u;\Omega) = \int_\Omega \hat{W}^h(Du)\,dx$$

where

$$\hat{W}^h(F) := \inf_{k} \inf_{\varphi \in W_0^{1,p}(kQ) \cap W^{2,2}(kQ)} \fint_{kQ} \left\{ |D^2\varphi|^2 + W(y, F + D\varphi) \right\} dy$$

formula à la Braides Müller

- thin film case :

additional assumption : W p- Lip.

$$|W(F) - W(G)| \leq C\left(1 + |F|^{p-1} + |G|^{p-1}\right)|F - G|$$

Note: $D_3 b \in L^2(\Omega; \mathbb{R}^3)$

Thin film - $\gamma = 2$

- $$E^2_-(u,b;\omega) = \int_\omega \overline{W}_2\left(D_\alpha u \,|\, b(x_\alpha,\cdot)\right) dx_\alpha$$

where

$$\overline{W}_2(\overline{F} \,|\, \underbrace{b(\cdot)}_{\in W^{1,2}(-1/2,\,1/2)}) := \inf_{L>0} \ \inf_{\begin{cases}\varphi \\ \varphi(\cdot, x_3) \ Q'\text{-per.} \\ \int_{Q'} D_3\varphi(x_\alpha, x_3)\,dx_\alpha = 0\end{cases} \text{for a.e. } x_3} \Big\{$$

$$\int_Q \Big\{ W\left(\overline{F} + D_\alpha\varphi \,|\, b(x_3) + L D_3\varphi\right) + \frac{1}{L}|D^2_{\alpha\beta}\varphi|^2 + |D_{\alpha 3}\varphi|^2 + |b'(x_3) + L D_{33}\varphi|^2 \Big\} dx$$

- Proof: typical blow up argument. A bit messy....

- $$\int_{-1/2}^{1/2} (Q_3 \times C_3)[W]\left(\overline{F} \,|\, b(x_3)\right) dx_3 + \int_{-1/2}^{1/2} |D_3 b(x_3)|^2 dx_3 \leq \overline{W}_2(\overline{F}, b) \leq \int_{-1/2}^{1/2} W\left(\overline{F} \,|\, b(x_3)\right) dx_3 + \int_{-1/2}^{1/2} |D_3 b(x_3)|^2 dx_3$$

$$\Downarrow$$

$$\overline{W}_2(\overline{F} \,|\, b) = \int_{-1/2}^{1/2} W\left(\overline{F} \,|\, b(x_3)\right) dx_3$$

if W is cross quasiconvex - convex.

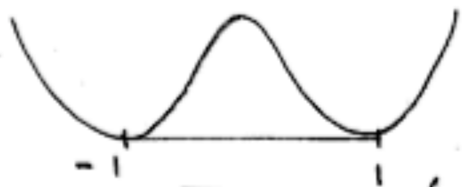
Thin film - $\gamma = 2$

Dal Maso - Fonseca - Leoni

- Questions of locality: $\Omega = (0,1)^2$

$$\overline{F}(b;\Omega) := \inf \ \underline{\lim}_m \left\{ \int_\Omega \{ W(b_m) + |D_2 b_m|^2 \} dx \ : \ b_m \xrightarrow{L^2} b \right\}$$

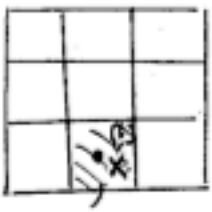
is **not** local



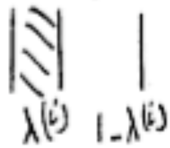
Indeed: locality $\Longrightarrow$ $\overline{F}(b;A) = \int_A \{ \underbrace{W^{**}}_{\text{convexified}}(b) + |D_3 b|^2 \} dx$

$b = x_2$

$$b_k^{(i)} := \chi_k^{(i)}(x_1)\left(-1 + x_2 - x_2^{(i)}\right) + \left(1 - \chi_k^{(i)}(x_1)\right)\left(1 + x_2 - x_2^{(i)}\right)$$



$\chi_k^{(i)}(x_1) \simeq$



$\xrightarrow[k]{L^2} x_2$

$\left\{ \begin{array}{l} A_i \text{ with } \operatorname{diam} A_i \text{ small} \\ \text{so that } W(\pm 1 + x_2 - x_2^{(i)}) < \varepsilon \\ \text{on } A_i \end{array} \right.$

$x_2^{(i)} = \lambda^{(i)}(-1) + (1 - \lambda^{(i)})(+1)$

$F(b_k^{(i)}; A_i) \leq \varepsilon |A_i| + \underset{D_2 b_k^{(i)} = 1}{\underbrace{|A_i|}}$

$\Downarrow$

$F(b_k; \Omega) \leq \varepsilon(1 + \delta) + 1$

- diagonalization $\Longrightarrow$ $\exists\, b_m \to x_2$ with $F(b_m, \Omega) \to 1$
- l.s.c $\Longrightarrow$ $W(b_m) \xrightarrow{L^1} 0$, $D_2 b_m \xrightarrow{L^2} 1$ $\Longrightarrow$

$\exists\, b_{m'} \xrightarrow{a.e.} \begin{cases} -1 \\ 1 \end{cases}$ contradiction

Super - critical case $\gamma > 2$

expectation : Same as if no singular perturbation

Yes !

- homogenization case : easy

$$E^{\gamma}_{-}(u;\Omega) = \int_{\Omega} W^{h}(Du)\,dx$$

where

W^{h} is "classical" homogenized energy.

$$\int_{\Omega} W^{h}(Du)\,dx \leq E^{\gamma}_{-}(u;\Omega) \leq \inf_{\mu>0} \varliminf_{\varepsilon} \mu \left\{ \varepsilon^{2}\int_{\Omega} |D^{2}u^{\varepsilon}|^{2} + \frac{1}{\mu}\int_{\Omega} W\left(\frac{x}{\varepsilon} \middle| Du^{\varepsilon}\right)dx \right\}$$

|| critical case

$$\mu \int_{\Omega} \hat{W}^{h}_{\mu}(Du)\,dx$$

But $\mu\, \hat{W}^{h}_{\mu}(F) \Big\downarrow^{\mu \downarrow 0} W^{h}(F)$.

- thin film case : same should apply, but ?

Thin film case - $\gamma > 2$

- result of Bouchitté - Fonseca - Mascarenhas:

~~sing part.~~

$$\Gamma\text{-}\underline{\lim} \int_\Omega W\left(D_\alpha u \,\middle|\, \tfrac{1}{\varepsilon} D_3 u\right) dx = \int_\omega Q_\infty[W]\left(D_\alpha u \,\middle|\, b(x_\alpha,\cdot)\right) dx_\alpha$$

where

$$Q_\infty[W](\bar F | b(\cdot)) = \sup_m Q_m[W](\bar F | b(\cdot))$$

and

$$Q_m[W](\bar F | b(\cdot)) = \inf_L \inf_\varphi \Big\{ \int_Q W(\bar F + D_\alpha \varphi \,|\, b(x_3) + L D_3 \varphi) :$$

$$\varphi(\cdot, x_3)\ Q'\text{- per. for a.e. } x_3$$

$$\Big| \int_Q L D_3 \varphi\, \theta_i(x_3)\, dx \Big| \leq 1/m \quad i = 1, \dots, m \Big\}$$

$\{\theta_i\}$ dense in $L^{p'}(I; \mathbb{R}^3)$.

Clearly $\quad E_-^\gamma(u, b; A) \geq \int_\omega Q_\infty[W](D_\alpha u | b(x_\alpha, \cdot))\, dx_\alpha$

Other inequality through an explicit construction starting from $Q_\infty[W](\bar F | b(\cdot))$.