Lecture 2: The NLSE method

1. **The NLSE and quantised vortices**
2. **Nucleation and reconnections**
3. **Sound generation**
4. **Finite–temperature effects**

Weakly interacting Bose gas

Imperfect Bose–Einstein condensate, Hartree approx. $\Psi(\mathbf{x}, t)$ = single particle wavefunction for N bosons of mass m

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + \Psi \int |\Psi(\mathbf{x}', t)|^2 V(|\mathbf{x} - \mathbf{x}'|) d\mathbf{x}'$$

called Non Linear Schroedinger Equation (NLSE) or Gross–Pitaevskii (GP) equation)

Normalisation:

$$\int |\Psi|^2 d\mathbf{x}' = N$$

Weakly interacting system: replace potential V with a repulsive delta function of strength V_0:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V_0\Psi|\Psi|^2$$

Mass, momentum and energy:

$$M = \int |\Psi|^2 d\mathbf{x}'$$

$$\mathbf{P} = \frac{\hbar}{2i}\int \left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) d\mathbf{x}'$$

$$\mathcal{E} = \frac{\hbar^2}{2m}\int \left(|\nabla\Psi|^2 + \frac{V_0}{2}|\Psi|^4\right) d\mathbf{x}'$$

Simple solutions of the NLSE

Uniform condensate at rest in the laboratory frame:

$$\Psi = e^{iE_0/\hbar}$$

E_0 = increase of energy when one boson is added (chemical potential).

Look for deviations from that state: $\Psi = \psi e^{iE_0/\hbar}$

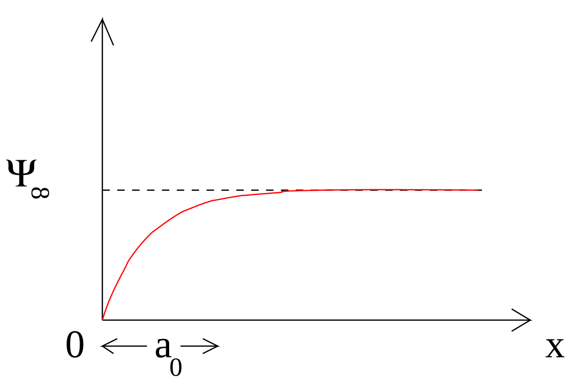
$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V_0|\psi|^2\psi - E_0\psi$$

- Uniform solution:

$$\psi = \psi_\infty = \sqrt{\frac{E_0}{V_0}}$$

- 1–dim solution near wall:

$$\psi = \psi_\infty = \tanh(x/a_0) \qquad a_0 = \sqrt{\frac{\hbar^2}{mE_0}} \text{ healing length}$$



Sound waves

Perturb the uniform state: $\psi = \psi_\infty + \phi$
Substitute into NLSE, assume ϕ small and linearise

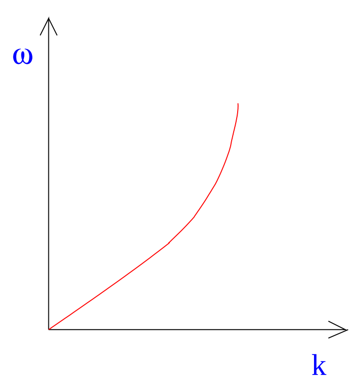
$$i\hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\phi + E_0(\phi + \phi^*)$$

Let $\phi = a + ib$, assume 1–dim for simplicity

$$\begin{aligned} -\hbar b_t &= -\tfrac{\hbar^2}{2m} a_{xx} + 2aE_0, \\ \hbar a_t &= -\tfrac{\hbar^2}{2m} b_{xx} \end{aligned}$$

Let $a = \hat{a} e^{ikx - i\omega t}$, $b = \hat{b} e^{ikx - i\omega t}$. Non-trivial solutions $\hat{a}, \hat{b}$ exist only if

$$\omega = k\sqrt{\frac{E_0}{m}\left(1 + \frac{\hbar^2 k^2}{4mE_0}\right)}$$



$k \ll 1$: $\omega \approx ck$ where $c = \sqrt{E_0/m}$ = sound speed
$k \gg 1$: $\omega \approx \hbar^2 k^2/(2m)$ free particles

Fluid dynamics interpretation of NLSE

Apply Madelung transformation

$$\psi = Re^{iS}$$

Define

$$\rho_s = mR^2$$

$$\mathbf{v}_s = \frac{\hbar}{m}\nabla S \text{ (hence } \nabla \times \mathbf{v}_{\mathrm{s}} = 0)$$

then the NLSE is equivalent to:

- The continuity equation:

$$\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s \mathbf{v}_s) = 0$$

- The (quasi) Euler equation:

$$\rho_s \left(\frac{\partial v_{sj}}{\partial t} + v_{sk} \frac{\partial v_{sj}}{\partial x_k} \right) = -\frac{\partial p}{\partial x_j} + \frac{\partial \Sigma_{jk}}{\partial x_k}$$

where the pressure p and the quantum stress Σ_{jk} are

$$p = \frac{V_0}{2m^2}\rho_s^2, \qquad \Sigma_{jk} = \left(\frac{\hbar}{2m} \right)^2 \rho_s \frac{\partial^2 \ln \rho_s}{\partial x_j \partial x_k}$$

The quantum stress Σ_{jk} makes the NLSE different from the Euler equation.

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The quantum stress Σ_{jk} makes the NLSE different from the Euler equation.

Let L be the typilcal lengthscale of the problem. The ratio of the pressure term and the quantum stress term scales as

$$\frac{\text{pressure}}{\text{quantum stress}} \sim \frac{\hbar^2}{mE_0L^2}$$

which is unity for $L \sim a_0$. So the quantum stress term is important only at scales smaller then the healing length, $L \ll a_0$, and is responsible for

- vortex nucleation
- vortex reconnections

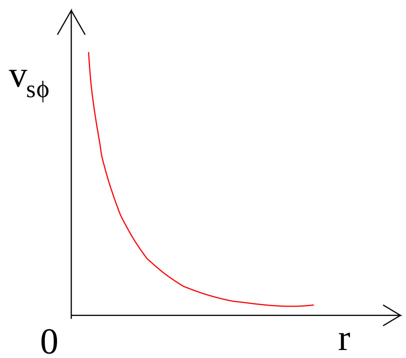
Away from vortices, where ρ_s varies little, the NLSE is essentially the Euler equation.

Vortex line solution of NLSE

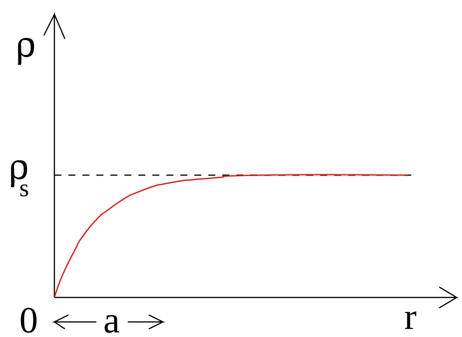
Cylindrical coordinates (r, ϕ, z). Let $S = \phi$, then

$$\mathbf{v}_s = \frac{\hbar}{m}\nabla S = \frac{\hbar}{mr}\hat{\boldsymbol{\phi}} = \frac{\kappa}{2\pi r}\hat{\boldsymbol{\phi}}$$

which is a vortex line aligned along z



and



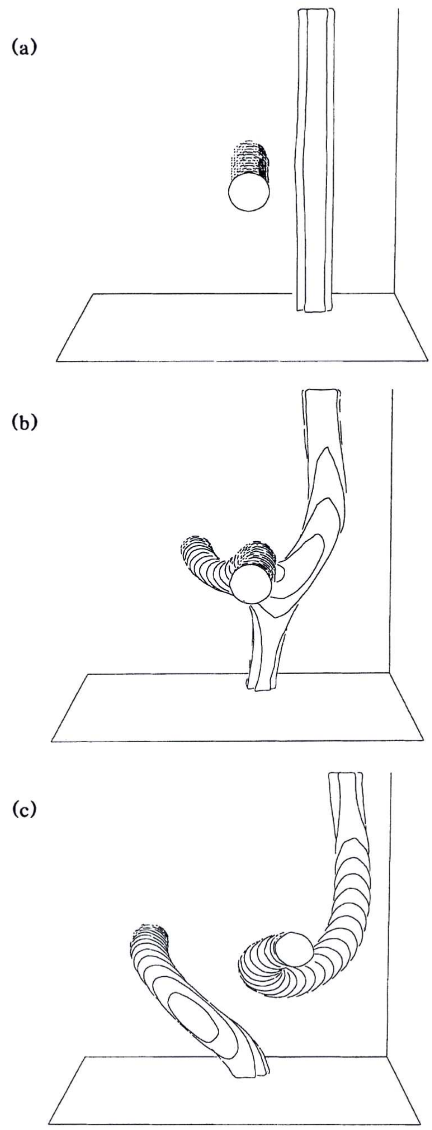
Note the hollow core.

Circulation

$$\oint_C \mathbf{v}_s \cdot \mathbf{d}\boldsymbol{\ell} = \int_0^{2\pi} v_{s\phi} r d\phi = \kappa$$

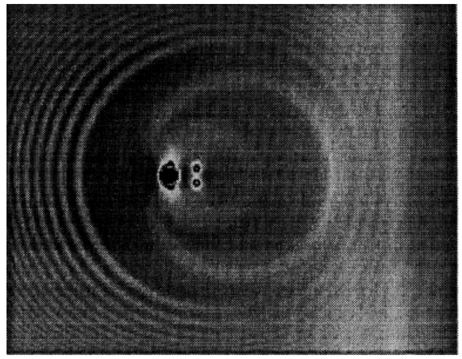
Vortex reconnections

J Koplik and H Levine, PRL 71, 1375, 1993:
First evidence of vortex reconnections

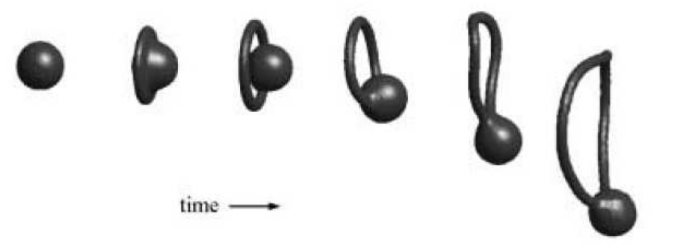


Vortex nucleation

Frish, Pomeau and Rica, PRL, 69, 1644, 1992



Winiecki and Adams, Europhysics Lett 52, 257, 2000



Energy dissipation near absolute zero

Experiments show that at temperatures of few mK, so low that the normal fluid is virtually absent, vorticity decays (Davis, Hendry and McClintock, Physica B 280, 43, 2000).

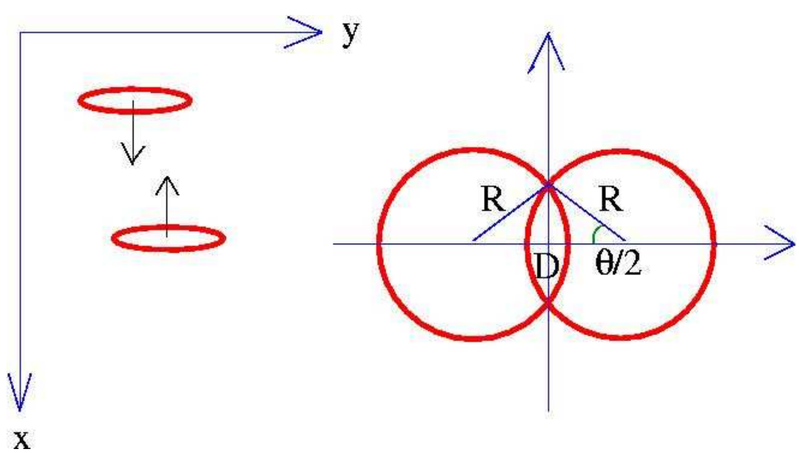
Question: What is the energy sink in the absence of viscosity ?

Lecture 1: sound emission is responsible for decay of kinetic energy. Kelvin wave cascade to wavenumbers k large enough that sound is radiated.

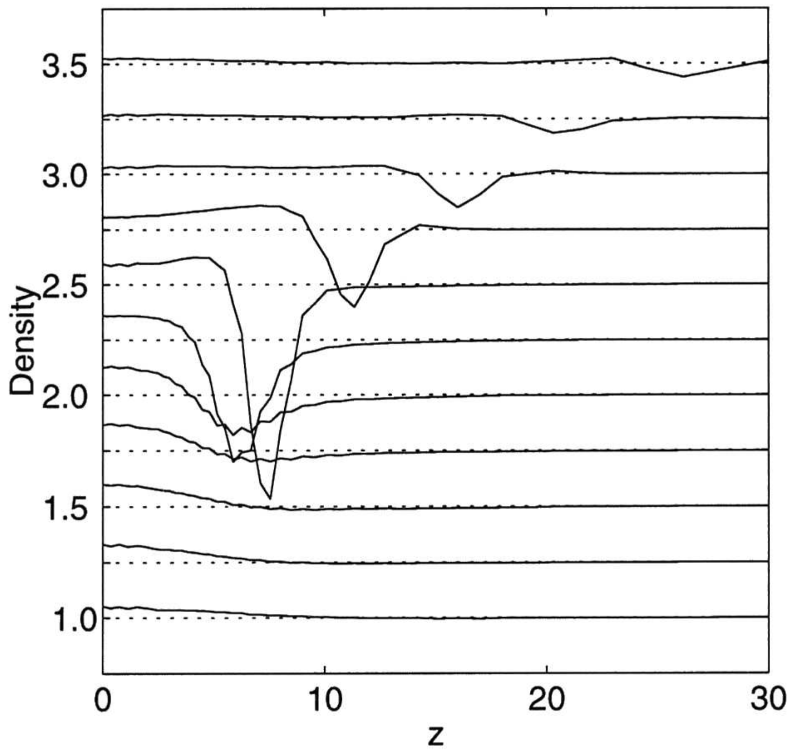
The NLSE model confirms that reconnections trigger Kelvin waves and that vortices radiate sound. It also shows that reconnection events generate a sound pulse which decreases the kinetic energy.

Sound emission at vortex reconnections

Leadbeater, Winiecki, Samuels, Barenghi and Adams (PRL 86, 1410, 2001) studied collisions of vortex rings and found that a short, intense sound pulse is emitted at a reconnection event.

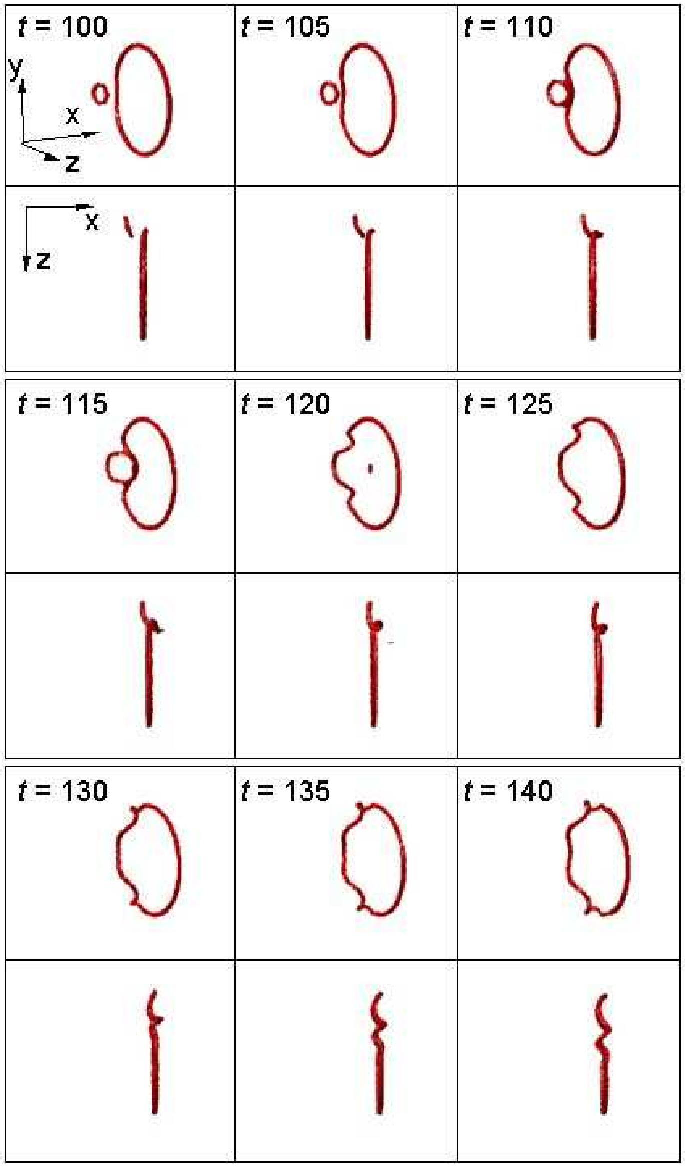


Density profiles at different times



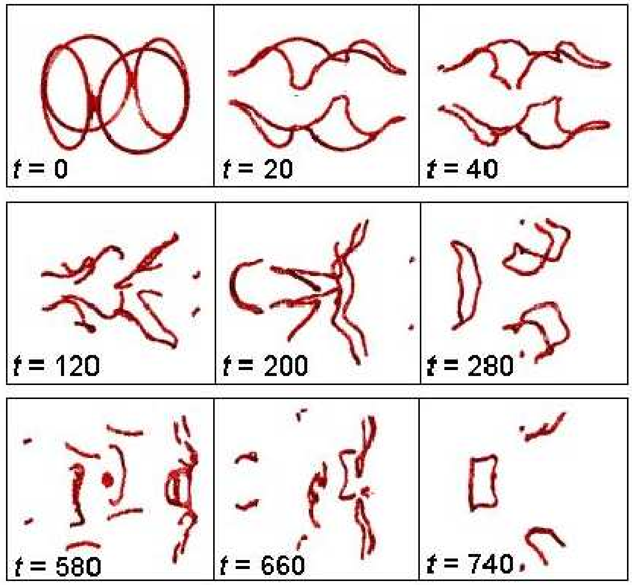
$$E_{lost} \sim \tan^2(\theta/2)$$

In a vortex tangle, sound pulses and Kelvin wave cascade are present together - both arise from vortex reconnections (Leadbeater, Samuels, Barenghi and Adams, PRA 67, 015601, 2002)

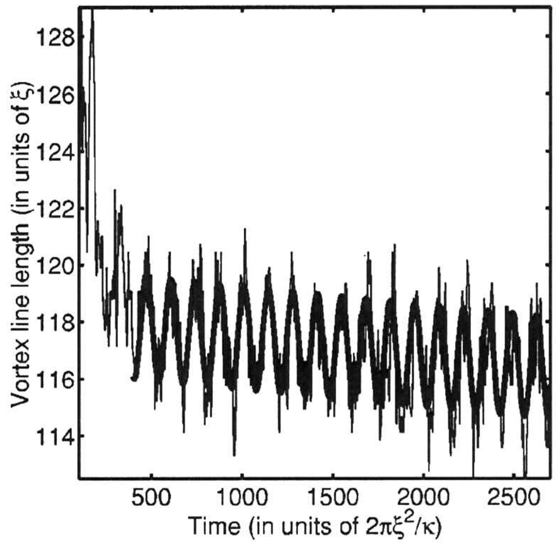


Note the sound pulse (the dot at $t = 120$) and the Kelvin wave created by the reconnection.

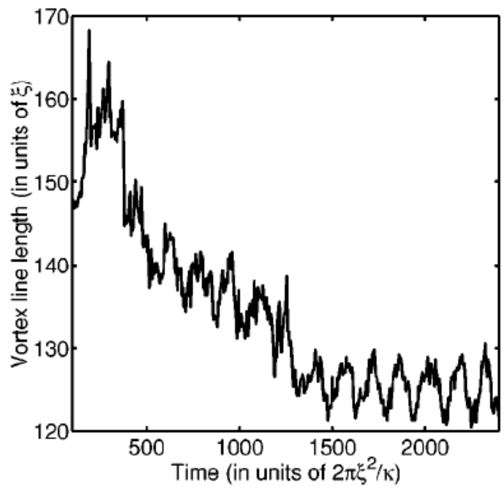
Collision of four rings:



Energy loss



Vortex line length (in units of ξ)
Time (in units of 2πξ²/κ)



Vortex line length (in units of ξ)
Time (in units of 2πξ²/κ)

Trapped condensate

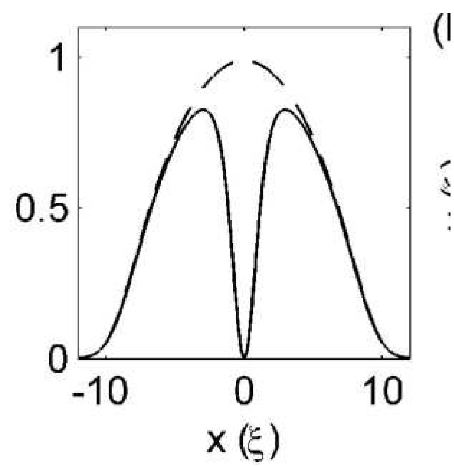
$$i\hbar\frac{\partial\psi}{\partial t}=\left(-\frac{\hbar^2}{2m}\nabla^2+V+gn_c\right)\psi$$

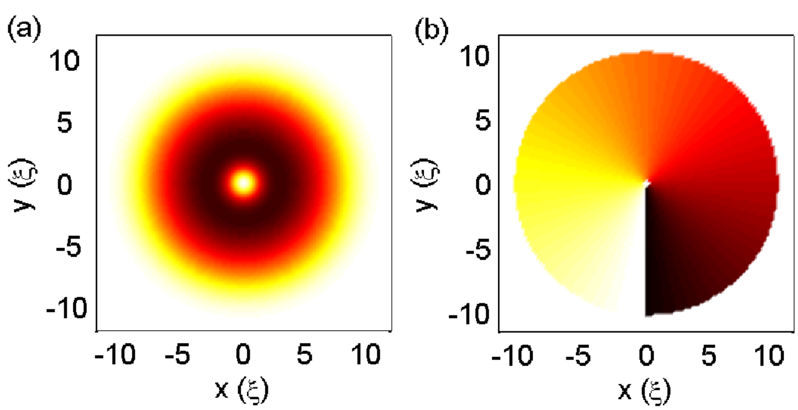
Condensate density

$$n_c(\mathbf{r},t)=|\psi(\mathbf{r},t)|^2,$$

Interaction parameter and trap potential:

$$g=4\pi\hbar^2N_c a/m,\qquad V(\mathbf{r})=\frac{m}{2}(\omega_\perp^2 r^2+\omega_z^2 z^2)$$

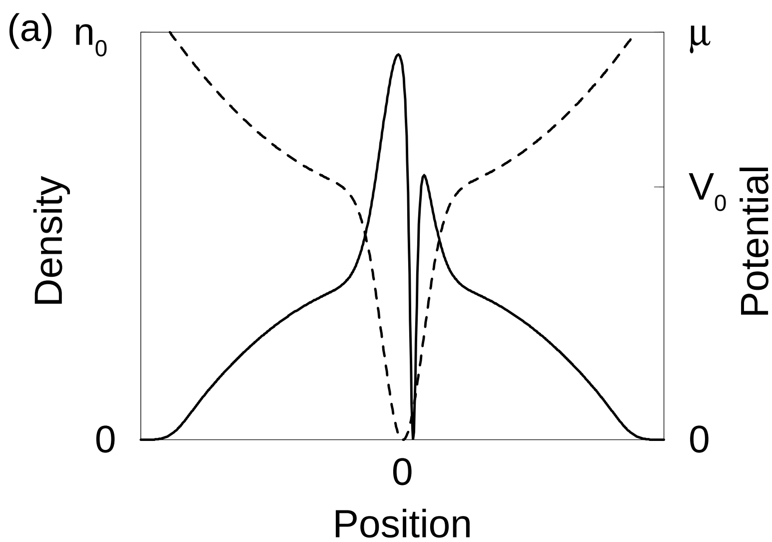




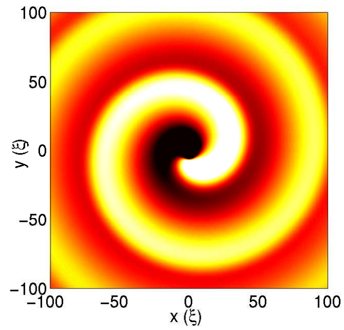
Sound emission

Barenghi, Parker, Proukakis and Adams studied the sound emission by accelerating vortices (J. Low Temp. Physics 138, 629, 2005).

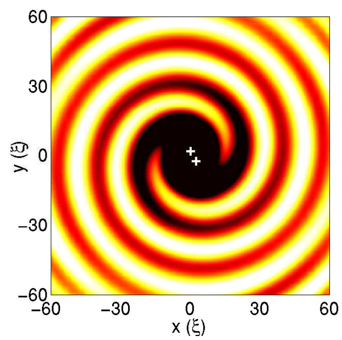
Dimple trap



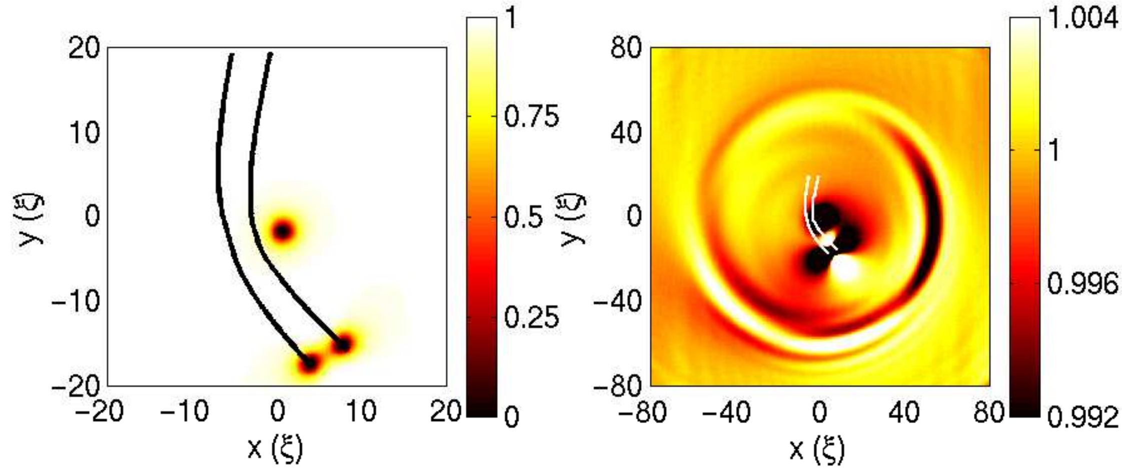
Dipolar radiation pattern emitted by a vortex precessing in a trapped atomic Bose–Einstein condensate (NLSE model):



Quadrupolar sound emission of a co–rotating vortex–vortex pair in a homogeneous condensate (NLSE model):



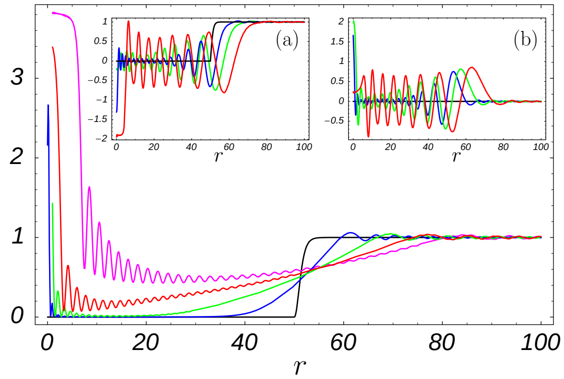
A vortex–antivortex pair in a homogeneous condensate interacts with an isolated vortex. Note the sound which is radiated away. Because of this loss of energy, the size of the vortex–antivortex pair after the interaction is less.



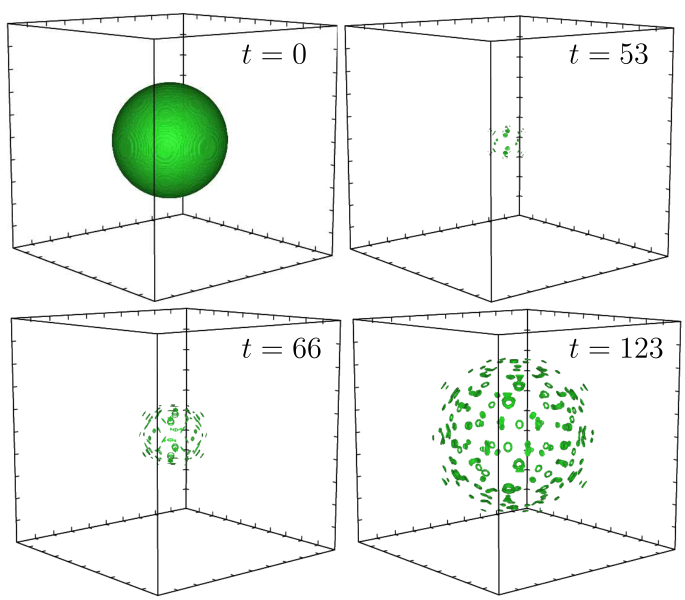
Collapsing ultrasound bubble

Berloff and Barenghi, PRL 93, 090401, 2004

Condensate density vs distance from centre of cavity at different times. Insets show real and imaginary parts of ψ.



If bubble radius exceeds 28 healing lengths, vortex rings are generated.



Finite T: ZNG formalism

Zaremba, Nikuni, Griffin, JLTP, 116, 277, 1999

Generalised GP equation for trapped condensate

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V + gn_c + 2g\tilde{n} - iR\right)\psi$$

Boltzmann equation:

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m}\cdot\nabla f - \nabla U \cdot \nabla_{\mathbf{p}} f = C_{22} + C_{12}$$

Condensate density and thermal cloud density:

$$n_c(\mathbf{r},t) = |\psi(\mathbf{r},t)|^2, \qquad \tilde{n}(\mathbf{r},t) = \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} f(\mathbf{p},\mathbf{r},t)$$

Interaction parameter and trap potential:

$$g = 4\pi\hbar^2 N_c a/m, \qquad V(\mathbf{r}) = \frac{m}{2}(\omega_\perp^2 r^2 + \omega_z^2 z^2)$$

Effective potential:

$$U(\mathbf{r},t) = V(\mathbf{r}) + 2g\left(n_c(\mathbf{r},t) + \tilde{n}(\mathbf{r},t)\right)$$

C_{22} = collisions between atoms in the thermal cloud
C_{21} = collisions between condensate and thermal atoms

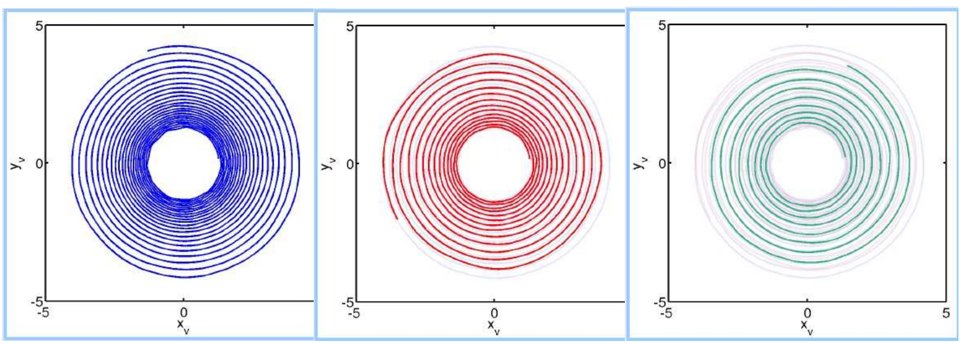
$$R(\mathbf{r},t) = \frac{\hbar}{2n_c}\int \frac{d\mathbf{p}}{(2\pi\hbar)^3} C_{12}$$

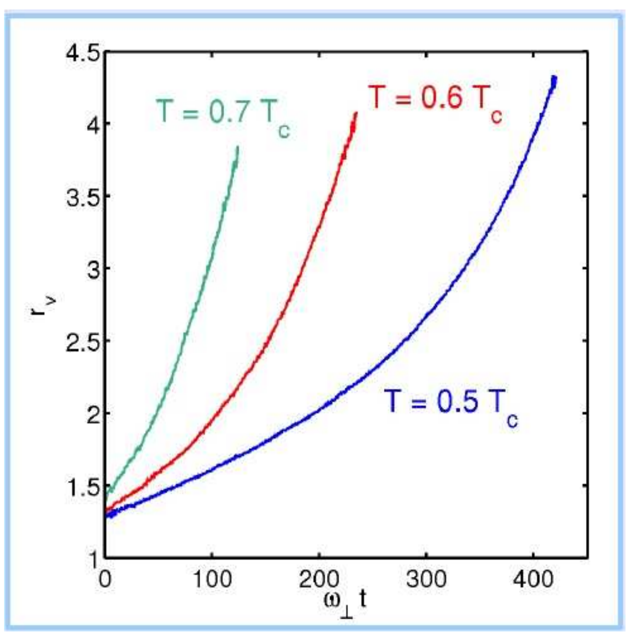
- Solve GP equation for ψ
- Evolve Kinetic equation using N-body simulations

Decay of a vortex

Vortex set initially off–centre

Vortex trajectories at $T = 0.5$, 0.6 and 0.7 T_c





Connection to vortex dynamics

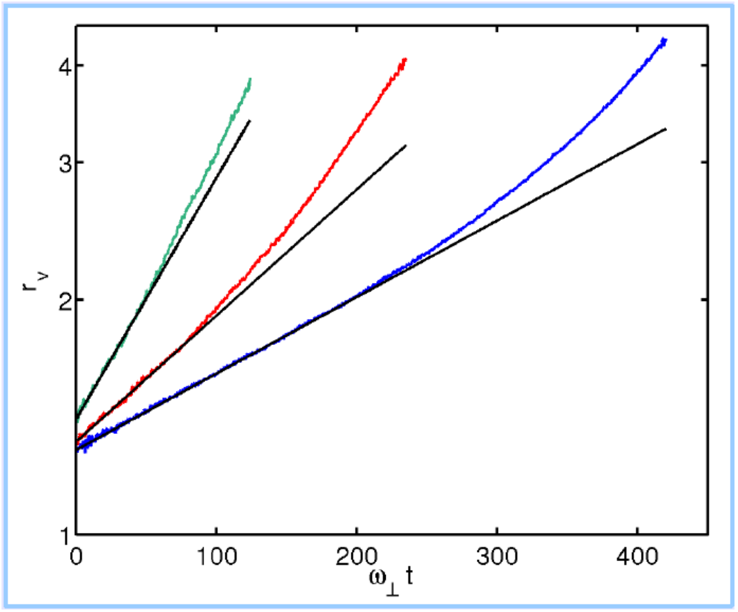
$$\frac{d\mathbf{s}}{dt} = \mathbf{v}_{self} - \alpha \mathbf{s}' \times \mathbf{v}_{self} - \alpha' \mathbf{v}_{self}$$

$$\frac{dr_v}{dt}\hat{\mathbf{r}} + r_v \frac{d\phi_v}{dt}\hat{\boldsymbol{\phi}} = \alpha v_{self}\hat{\mathbf{r}} + (1-\alpha')v_{self}\hat{\boldsymbol{\phi}}$$

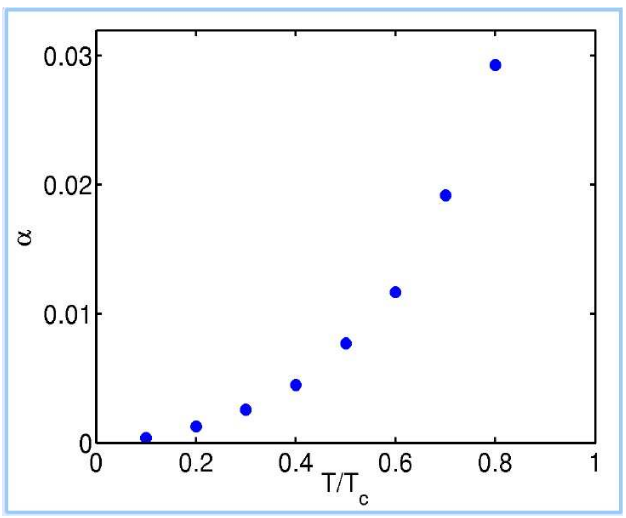
$$\frac{d\phi_v}{dt} = (1-\alpha')\frac{v_{self}}{r_v}, \qquad \frac{dr_v}{dt} = \alpha \omega_v r_v$$

$$\omega_v = (1-\alpha')\omega_v^0, \qquad r_v = r_v^0 e^{\alpha \omega_v t}$$

α': changes precession speed, but we find $\alpha' \approx 0$
α: exponential radial decay



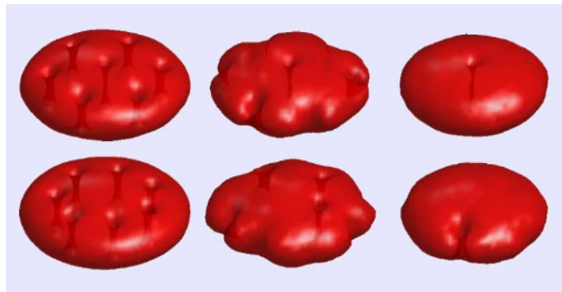
4
3
2
1
r_v
0
100
200
300
400
$\omega_\perp t$

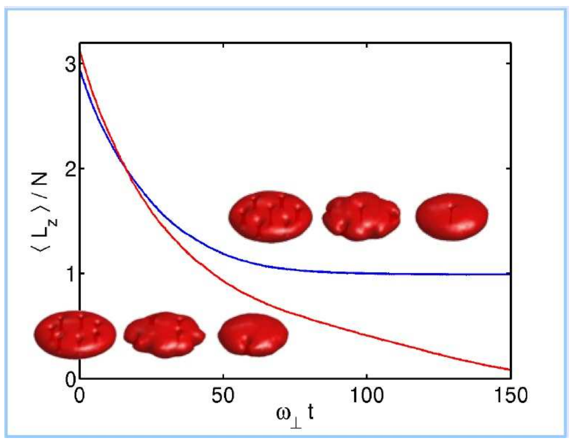


0.03
0.02
0.01
0
α
0
0.2
0.4
0.6
0.8
1
T/T_c

Decay of vortex lattice

$t = 0.7\ T_c$





Central vortex decays slowly