Lecture 3: Motion of small particles in He II

1. The problem of flow visualisation
2. Particle Image Velocimetry (PIV) in liquid helium
3. One–way model
4. Two–way model
5. Interaction of particle and quantised vortex

Flow visualisation in classical fluids

- Ink
- Smoke
- Kalliroscope flakes
- Hydrogen bubbles
- Baker's pH method
- Hot wire anemometry
- Laser Doppler velocimetry
- Particle Image Velocimetry (PIV)

Flow visualisation in Helium II

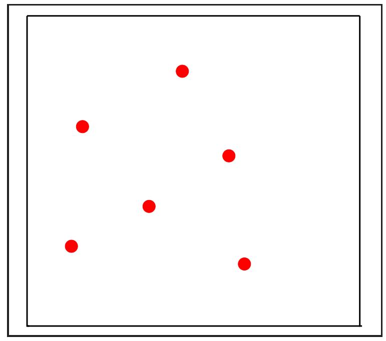
- Second sound
- Ion trapping
- Temperature, pressure and chemical potential gradients

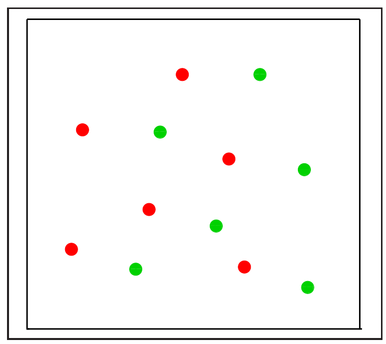
Difficulty: poor resolution in space and time (for turbulent fluctuations), no flow patterns.

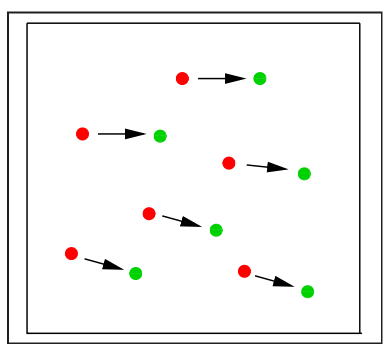
Current work:

- Improve resolution (Roche, Ihas)
- New methods:
  - shadography (Lucas)
  - laser induced fluorescence (McKinsey, Vinen)
  - PIV (the subject of this lecture)

PIV method







PIV in liquid helium

Ri. J. Donnelly, A. N. Karpetis, J.J. Niemela, K.R. Sreenivasan, and W.F. Vinen, J. Low Temp. Physics **126**, 327 (2002)

D. Celik and S. W. Van Sciver, Exp. Therm. Fluid Sci. i**26**, 971 (2002) T Zhang, D Celik, and S W. VanSciver, J. Low Temp. Phys.**134**, 985 (2004)

T. Zhang and S. W. Van Sciver, Nature Physics **1**, 36 (2005)

G. P. Bewley, D. P. Lathrop, and K. R. Sreenivasan, Nature **441**, 588 (2006).

T. Zhang and S. W. Van Sciver, J. Low Temp. Phys. **138**, 865 (2005).

Properties of the tracer particles

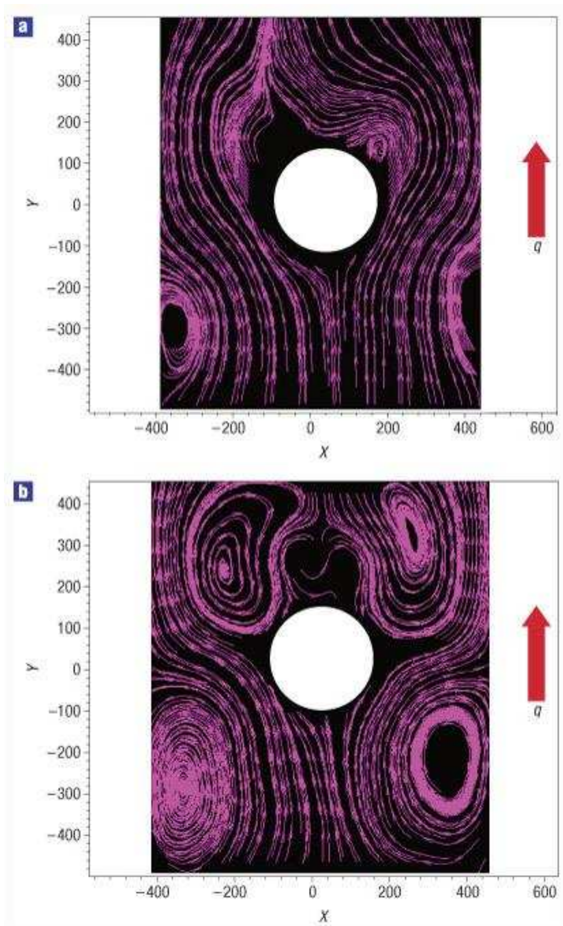
Material: Hollow glass or polymer spheres, solid hydrogen

Size: typically $a_p \approx 10^{-4}$ cm

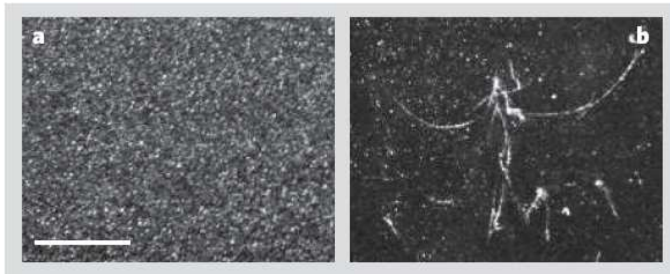
Density: from $\rho_p \approx \rho$ (neutrally buoyant) to 10ρ

PIV results

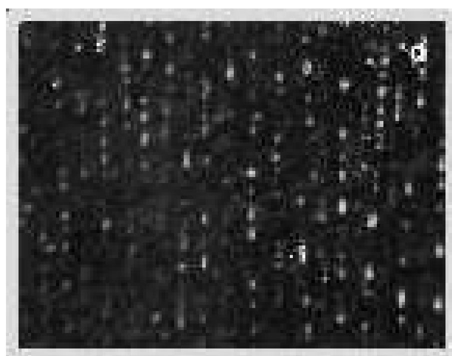
Van Sciver et al: counterflow blocked by an object



Sreenivasan et al: Left: He I. Right: remnant vortex lines in He II

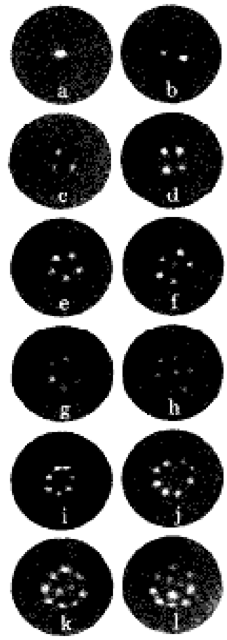


Vortex lines in rotating helium (side view)

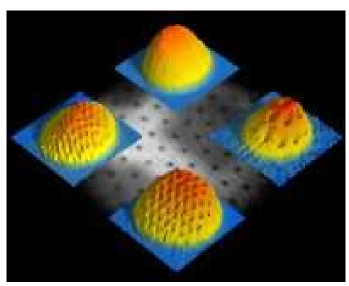


Other iconic images of quantised vorticity:

He II: Packard et al using electrons



Atomic BEC: Ketterlee et al using laser



What do tracer particles actually trace ?

The normal fluid ? The superfluid ? Neither ? Do they get trapped into the quantised vortex lines, hence trace the superfluid vorticity ?

One–way interaction

Generalise classical approach to helium two–fluid hydrodynamics:
normal fluid = classical viscous Navier–Stokes fluid
superfluid = classical ideal Euler fluid

Assume that the particles are small enough that:

- Presence of particles does not affect flow and do not get trapped in quantised vortices or disturb them
- Flow velocity varies little in distance of order of particle size

  $a_p \ll \delta$ typical intervortex spacing
  $a_p \ll \eta$ Kolmogorov length

$$Re = \frac{a_p|\mathbf{v}_p - \mathbf{v}_n|}{\nu} \ll 1,$$

- Linear Stokes drag
- Neglect
  - Faxen correction to Stokes drag
  - Basset history force
  - Saffman lift force on particle in a shear
  - Magnus lift force on rotating particle in uniform flow

Lagrangian equations of particle motion

$$\frac{d\mathbf{r}_p}{dt} = \mathbf{u}_p$$

$$\frac{d\mathbf{u}_p}{dt} = \frac{1}{\tau}(\mathbf{v}_n - \mathbf{u}_p) + \frac{(\rho_p - \rho)}{\rho_o}\mathbf{g} + \frac{3\rho_n}{2\rho_o}\frac{D\mathbf{v}_n}{Dt} + \frac{3\rho_s}{2\rho_o}\frac{D\mathbf{v}_s}{Dt}$$

where

$$\tau = \frac{2\rho_o a_p^2}{9\mu_n}, \qquad \rho_o = \rho_p + \frac{1}{2}\rho$$

and

$$\frac{D\mathbf{v}_n}{Dt} = \frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n \cdot \nabla)\mathbf{v}_n, \qquad \frac{D\mathbf{v}_s}{Dt} = \frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s,$$

For neutrally buoyant particles ($\rho_p = \rho$)

$$\rho_o = \frac{3}{2}\rho, \qquad \tau = \frac{a_p^2 \rho}{3\mu_n}$$

$$\frac{d\mathbf{u}_p}{dt} = \frac{1}{\tau}(\mathbf{v}_n - \mathbf{u}_p) + \frac{\rho_n}{\rho}\frac{D\mathbf{v}_n}{Dt} + \frac{\rho_s}{\rho}\frac{D\mathbf{v}_s}{Dt}$$

Note the viscous effect (arising from the normal fluid) and the inertial effects (arising from the normal fluid and the superfluid)

Sedimentation

Let $\mathbf{v}_n = \mathbf{v}_s = 0$. Then, in the presence of gravity, particles fall with terminal velocity

$$u_\infty = \frac{2a_p^2 g(\rho_p - \rho)}{9\mu}, \tag{1}$$

which can be used to determine the particle size.

Classical limit

Single fluid of density ρ_f and velocity $\mathbf{v}_f$. We have

$$\frac{d\mathbf{r}_p}{dt} = \mathbf{u}_p,$$

$$\frac{d\mathbf{u}_p}{dt} = \frac{1}{\tau}(\mathbf{v}_f - \mathbf{u}_p) + \frac{(\rho_p - \rho_o)}{\rho_o}\mathbf{g} + \frac{3\rho}{2\rho_o}\frac{D\mathbf{v}_f}{Dt}.$$

Comparison with ideal Lagrangian tracer

Ideal Lagrangian tracer (fluid parcel):

$$\frac{d\mathbf{r}_p}{dt} = \mathbf{v}_f$$

where $\mathbf{v}_f = \mathbf{v}_f(\mathbf{r}_p)$.

If the particle is neutrally buoyant we have

$$\frac{d\mathbf{u}_p}{dt} = \frac{1}{\tau}(\mathbf{v}_f - \mathbf{u}_p) + \frac{D\mathbf{v}_f}{Dt}.$$

If we approximated

$$\frac{D\mathbf{v}_f}{Dt} = \frac{\partial \mathbf{v}_f}{\partial t} + (\mathbf{v}_f \cdot \nabla)\mathbf{v}_f \approx \frac{\partial \mathbf{v}_f}{\partial t} + (\mathbf{u}_p \cdot \nabla)\mathbf{v}_f = \frac{d\mathbf{v}_f}{dt},$$

then we would have

$$\frac{d(\mathbf{u}_p - \mathbf{v}_f)}{dt} = -\frac{1}{\tau}(\mathbf{u}_p - \mathbf{v}_f),$$

of solution

$$\mathbf{u}_p - \mathbf{v}_f = [\mathbf{u}_p(0) - \mathbf{v}_f(0)]e^{-t/\tau},$$

that is $\mathbf{u}_p \to \mathbf{v}_f$ for $t \gg \tau$. However this is not correct.

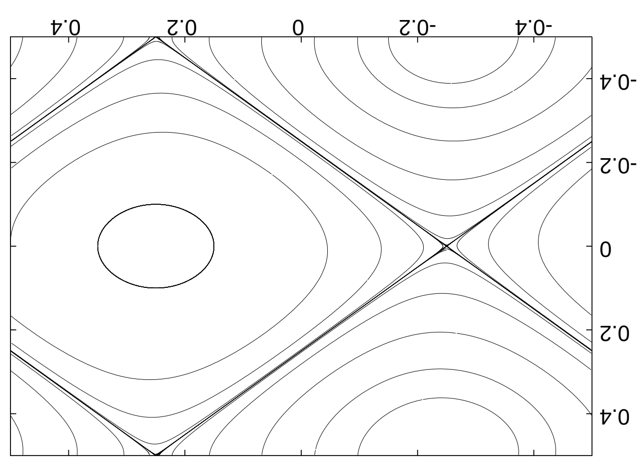
Neutrally buoyant particles (Babiano 2000) move away from regions of high vorticity and preferentially segregate in regions of high strain.

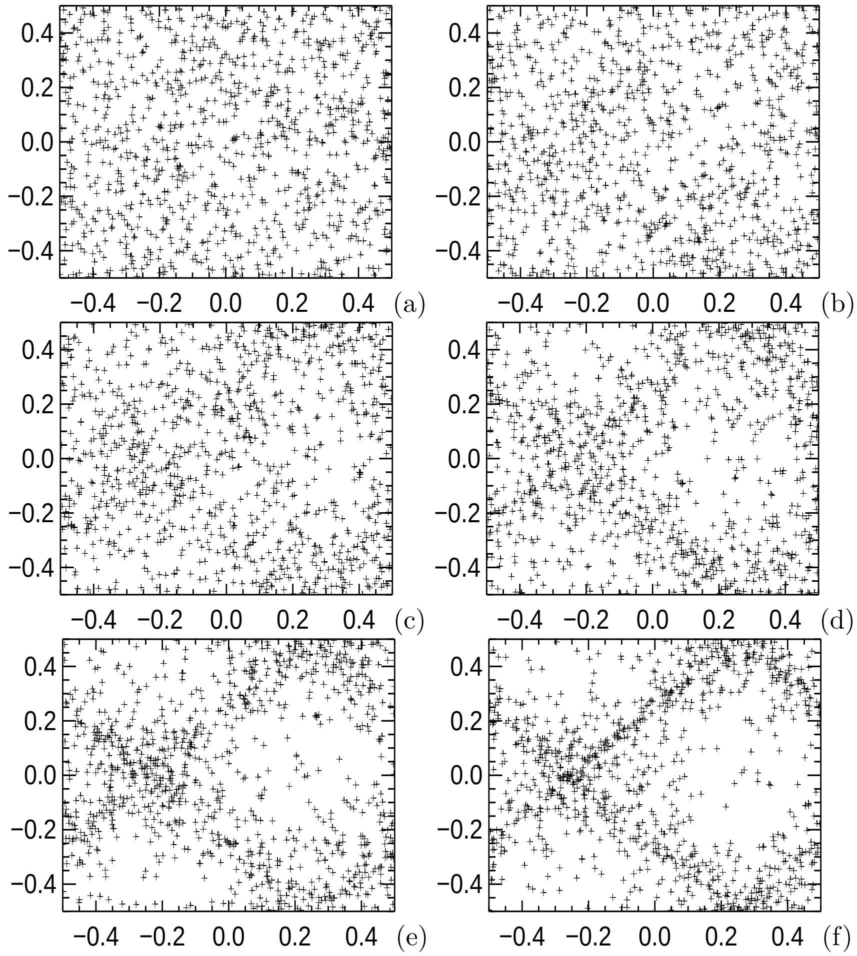
Example: ABC flow:

$$v_x = A \sin(2\pi z) + C \cos(2\pi y),$$

$$v_y = B \sin(2\pi x) + A \cos(2\pi z),$$

$$v_z = C \sin(2\pi y) + B \cos(2\pi x),$$





Note the segregation in regions of high rate of strain.

Fortunately, the time taken to segregate is of the order of few turnover times, and the typical lifetime of turbulent eddies is of the order of the turnover time too.

This example shows that the classical PIV technique has its limitations. Still, PIV has proved itself very useful in classical fluid dynamics.

Tracer particles at very low temperatures

At $T < 1$ K $\rho_n \approx \rho \to 0$ and helium II is effectively a pure superfluid. The equation of motion of a neutrally buoyant particle becomes

$$\frac{d\mathbf{u}_p}{dt} = \frac{D\mathbf{v}_s}{Dt}.$$

Use Euler's equation:

$$\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s = -\frac{1}{\rho_s}\nabla p,$$

hence the particle obeys

$$\frac{d\mathbf{u}_p}{dt} = -\frac{1}{\rho_s}\nabla p.$$

Right hand side = force per unit mass acting on a parcel of superfluid = force on the particle that replaces that superfluid. Therefore

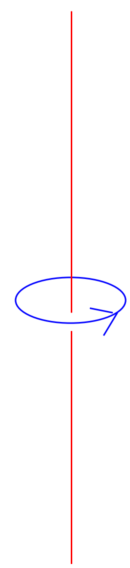
$$\mathbf{u}_p(t) = \mathbf{v}_s(\mathbf{r}_p(t), t), \qquad \mathbf{r}_p(t) = \mathbf{r}_s(t)$$

is a formal solution of the equation of motion, where $\mathbf{r}_s(t)$ is the Lagrangian trajectory of a superfluid parcel.

It would seem that small particles are ideal Lagrangian tracers suitable for studying low temperature turbulence, but it is not the case: in the absence of damping forces, the issue of stability of particle trajectories becomes crucial.

Motion of particle around vortex at T=0

Vortex line: $\mathbf{v}_s = (v_{sr}, v_{s\theta}, v_{sz}) = (0, \kappa/(2\pi r), 0)$.



Particle's equation becomes

$$\frac{d\mathbf{u}_p}{dt} = \frac{\kappa^2}{8\pi^2}\nabla\left(\frac{1}{r^2}\right),$$

like unit mass in the central potential $-\kappa^2/(8\pi^2 r^2)$.

Let $\mathbf{r}_p = (r, \theta)$ and $\mathbf{u}_p = (u_r, u_\theta) = (dr/dt, r d\theta/dt)$.
Angular momentum:

$$l = r u_\theta$$

Energy:

$$u_\theta^2 + u_r^2 - \frac{\kappa^2}{4\pi^2 r^2}$$

Let $u_r = 0$ when $r = r_0$. Express radial position as $x = r/r_0$, so that $x = 1$ at $t = 0$.

- If $l > \kappa/2\pi$ then

$$x^2 = 1 + \left(l^2 - \frac{\kappa^2}{4\pi^2}\right)\frac{t^2}{r_0^4}.$$

$$\theta = \left(1 - \frac{\kappa^2}{4\pi^2 l^2}\right)^{-1/2} \tan^{-1}\left[\left(l^2 - \frac{\kappa^2}{4\pi^2}\right)^{1/2} \frac{t}{r_0^2}\right].$$

  Particle spirals outwards towards $r \to \infty$.

- If $l < \kappa/2\pi$ then

$$x^2 = 1 - \left(\frac{\kappa^2}{4\pi^2} - l^2\right)\frac{t^2}{r_0^4},$$

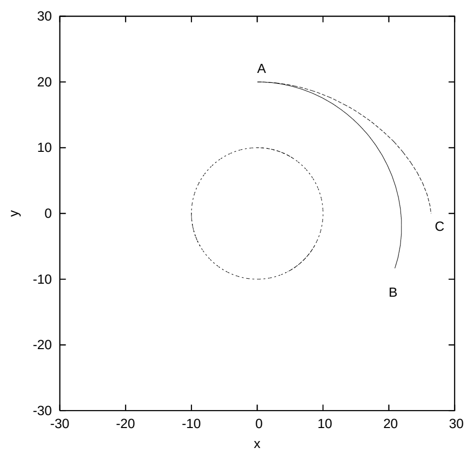
$$\theta = \left(\frac{\kappa^2}{4\pi^2 l^2} - 1\right)^{-1/2} \tanh^{-1}\left[\left(\frac{\kappa^2}{4\pi^2} - l^2\right)^{1/2} \frac{t}{r_0^2}\right].$$

  Particle spirals inwards towards the vortex.

- If $l = \kappa/2\pi$ (initially the velocity of the particle is exactly equal to the velocity of the superfluid) the particle follows the superfluid but it can be shown (Sergeev et al 2006) that the orbit is unstable.

We conclude that the tracer particle cannot even trace the orbit around a single vortex line.

Similar instability for the more complicated case of the motion of a particle in the presence of three vortices:



Dashed line A to C: solid particle
Solid line A to B: superfluid particle
Dotted line: three vortices

Why does a vortex tend to trap particles ?

- Energy
- Pressure
- Forces

- **Energy**

When a particle of radius a_p is trapped on a vortex the energy of the helium is reduced by an amount approximately equal to the kinetic energy ΔE of the displaced superfluid.

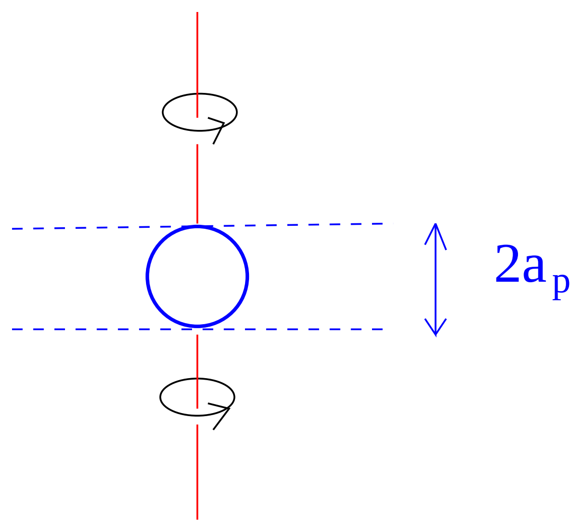
Consider a straight vortex line along the z direction. The kinetic energy per unit length is

$$E = \int_0^{2\pi} d\phi \int_0^{\infty} dr\ r \frac{1}{2} \rho_s v_{s\phi}^2 \approx \frac{\rho_s \kappa^2}{4\pi} \ln\left(b/a_0\right), \quad (2)$$

Then

$$\Delta E \approx \frac{\rho_s \kappa^2 a_p}{4\pi} \log\left(a_p/a_0\right)$$

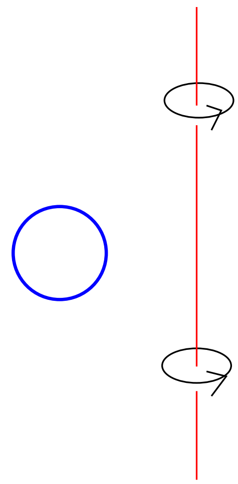
Since $\Delta E \gg k_B T$ for typical a_p, thermal effects do not inhibit trapping.



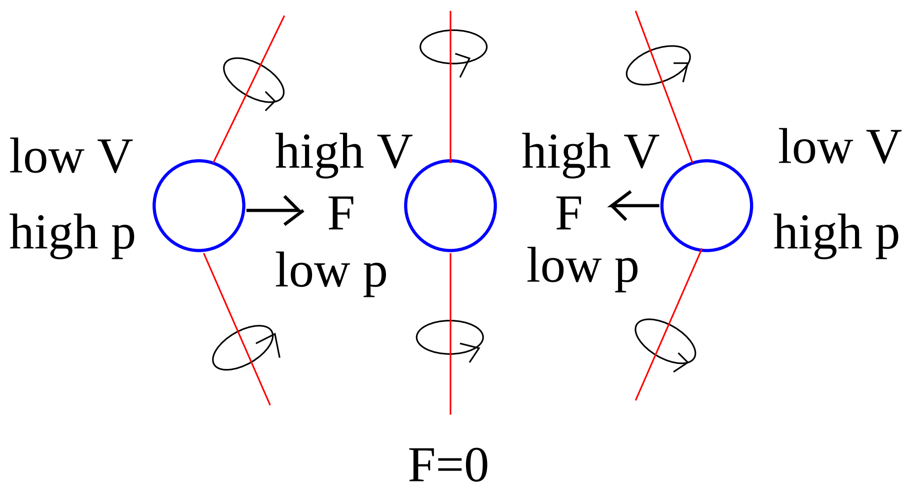
- **Pressure**

Flow field around vortex creates a pressure gradient

$$\nabla p = \frac{\rho_s \kappa^2}{8\pi^2} \nabla \left(\frac{1}{r^2} \right)$$



- **Forces**



Approach of particle to vortex

Assume that the particle is neutrally buoyant and at rest at distance r_0 from an isolated stationary rectilinear vortex which does not respond to the particle (one-way model). Let $\mathbf{v}_n = 0$. The particle obeys

$$\frac{d\mathbf{u}_p}{dt} = -\frac{1}{\tau}\mathbf{u}_p + \frac{3\rho_s}{2\rho_o}(\mathbf{v}_s \cdot \nabla)\mathbf{v}_s.$$

hence the radial motion is driven by $(\mathbf{v}_s \cdot \nabla)\mathbf{v}_s$ which has the form of a radial pressure gradient $\nabla(1/r^2)$. Let u_p denote the radial component of $\mathbf{u}_p$. Then

$$\frac{du_p}{dt} = -\frac{u_p}{\tau} - \frac{2\beta}{r^3}$$

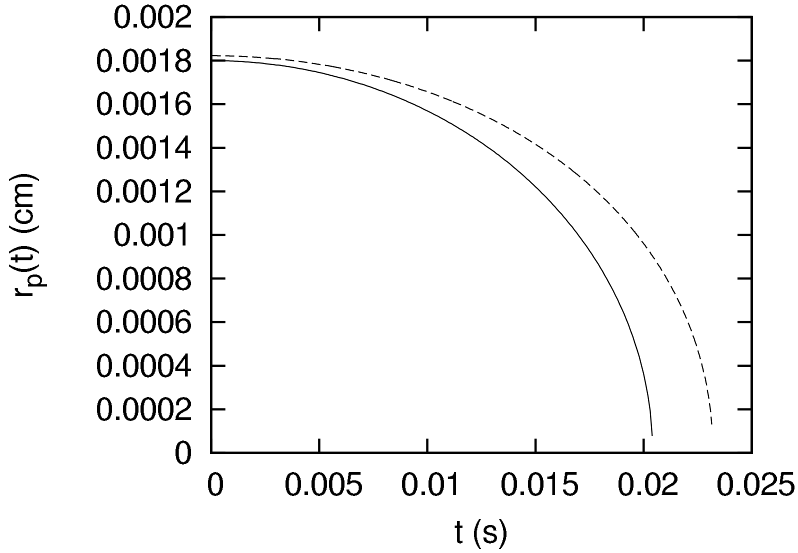
where

$$\beta = \frac{\rho_s \kappa^2}{8\pi^2 \rho},$$

We solve this equation and conclude that the particle, which started at distance r_0 from the vortex, arrives at radial distance $r = 2a_p$ from the vortex at time t_a given by

$$t_a = \frac{r_0^4}{8\beta\tau}\left(1 - \frac{(2a_p)^4}{r_0^4}\right).$$

The approximation (solid line) is not too bad, compared to the dynamically self–consistent calculation (dashed line):



Self–consistent interaction

Allow the (neutrally buoyant) tracer particle which approaches the vortex to affect it in a dynamically self–consistent way (Kivotides et al 2006, Kivotides et al 2007).

Particle's equation of motion:

$$\frac{d\mathbf{r}_p}{dt} = \mathbf{u}_p,$$

and

$$m_{eff}\frac{d\mathbf{u}_p}{dt} = \mathbf{f} = \mathbf{f}_d + \mathbf{f}_t + \mathbf{f}_b,$$

where

$$m_{eff} = m + 2\pi\rho a_p^3/3$$

$$\mathbf{f}_d = 6\pi a_p \mu(\mathbf{v}_n - \mathbf{u}_p),$$

$$\mathbf{f}_t = 2\pi\rho_s a_p^3 \frac{\partial \mathbf{v}_{si}}{\partial t},$$

$$\mathbf{f}_b = \frac{\rho_s}{2}\int_S (\mathbf{v}_{si} + \mathbf{v}_b)^2 \widehat{\mathbf{n}} dS,$$

with $\mathbf{v}_n = 0$ and $\widehat{\mathbf{n}}$ = unit radial vector pointing out of the surface S of the sphere.

Represent the vortex as a space curve $\mathbf{X} = \mathbf{X}(\xi, t)$ where ξ is arclength which obeys Schwarz's equation

$$\frac{d\mathbf{X}}{dt} = \mathbf{v}_{si} + \mathbf{v}_b + \mathbf{v}_\phi + \mathbf{v}_{mf},$$

• **First term:** It is the velocity which the vortex induces upon itself due to its own curvature:

$$\mathbf{v}_{si}(\mathbf{X}) = -\frac{\kappa}{4\pi} \int d\xi \frac{\mathbf{X}' \times (\mathbf{X} - \mathbf{x})}{|\mathbf{X} - \mathbf{x}|^3},$$

The integral extends on the entire vortex configuration, thus describing the advection of a vortex line by another vortex line (multiple vortex loops can be generated by the particle–vortex interaction).

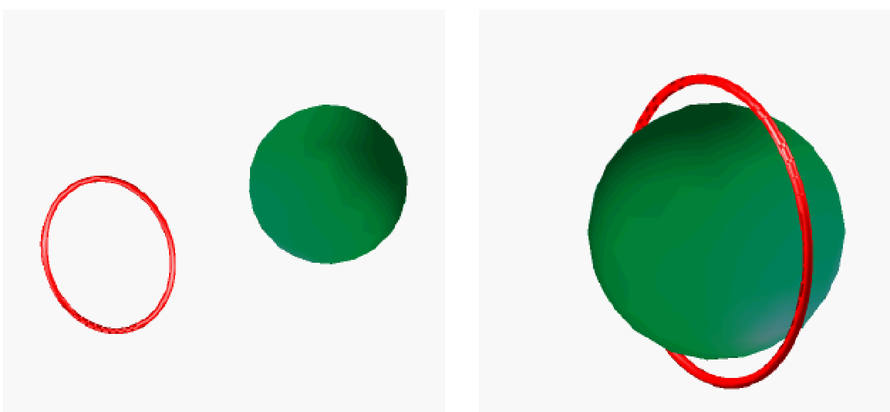
$\mathbf{X}' = d\mathbf{X}/d\xi$ is the unit tangent vector at the point $\mathbf{X}$ in the direction of the circulation.

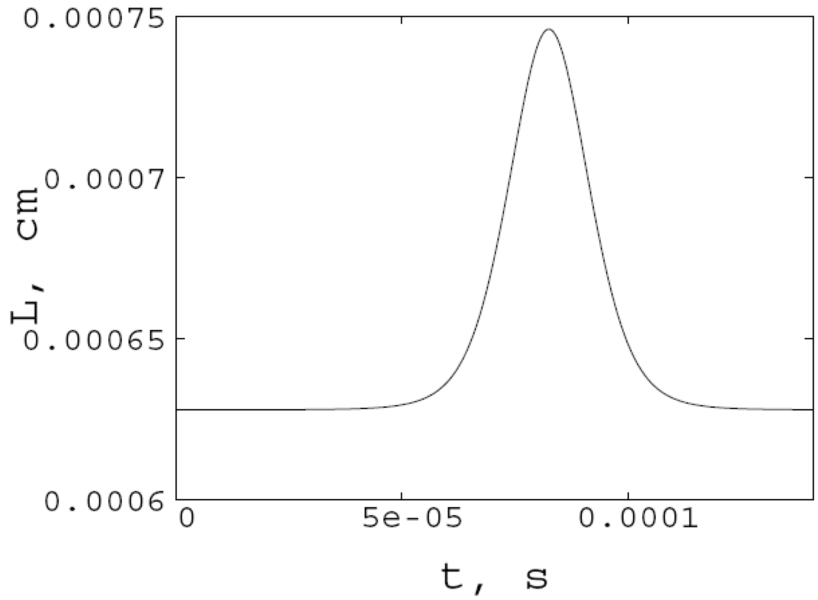
• **Second term:** It makes sure that the combined flow induced by the vortex and by the particle has zero radial component at the particle's surface. $\mathbf{v}_b = \nabla\Phi_b$ is obtained by solving

$$\nabla^2\Phi_b = 0$$

in terms of an expansion of N associated Legendre functions. The number N is variable to keep the same level of approximation throughout the time evolution (more terms are needed when the the particle is close to the vortex).

Example: vortex ring flying around a sphere:




0.00075
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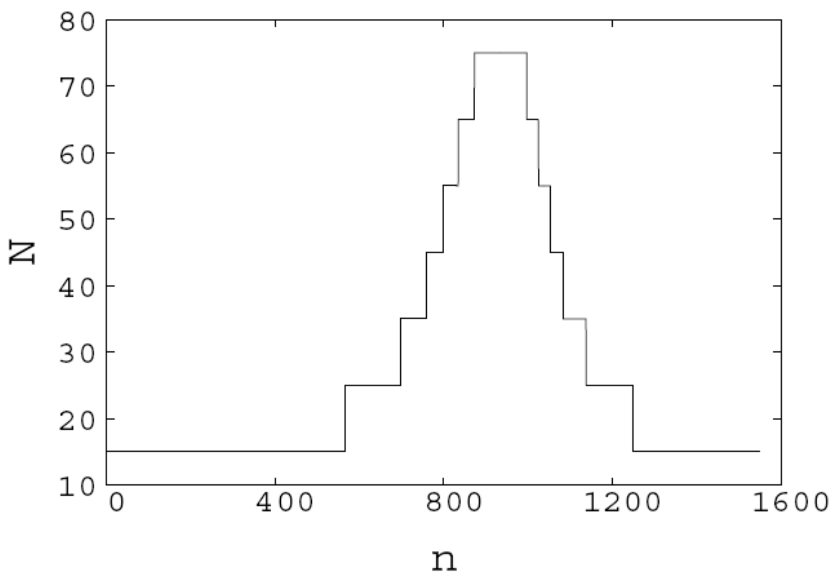

80
70
60
50
40
30
20
10
N
0
400
800
1200
1600
n

• **Third term:** It is the potential flow field induced by the motion of a spherical particle with velocity $\mathbf{u}_p$:

$$\mathbf{v}_\phi = -\frac{1}{2}\left(\frac{a_p}{r}\right)^3 \mathbf{u}_p \cdot \left(\mathbf{I} - 3\frac{(\mathbf{x}-\mathbf{z})(\mathbf{x}-\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^2}\right)$$

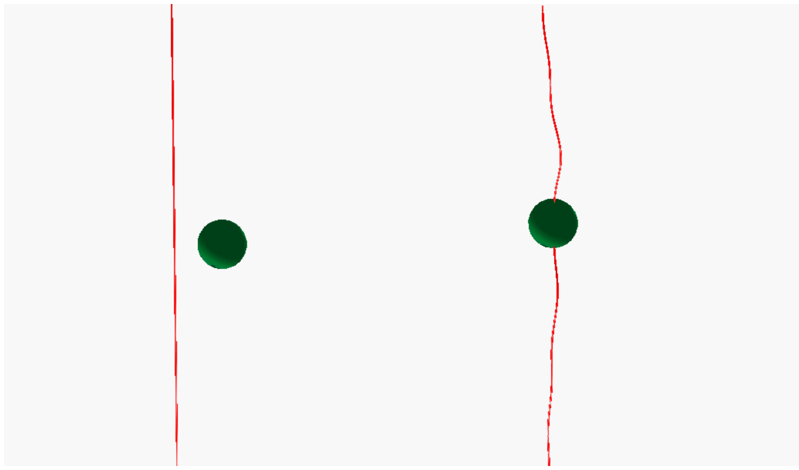
• **Fourth term:** It describes the friction on the vortex arising from the stationary normal fluid;

$$\begin{aligned}\mathbf{v}_{mf} = h_{\star\star}(\mathbf{v}_{si} + \mathbf{v}_b + \mathbf{v}_\phi)\\ +h_\star \mathbf{X}' \times (\mathbf{v}_n - (\mathbf{v}_{si} + \mathbf{v}_b + \mathbf{v}_\phi))\\ +h_{\star\star}\mathbf{X}' \times (\mathbf{X}' \times \mathbf{v}_n)\end{aligned}$$

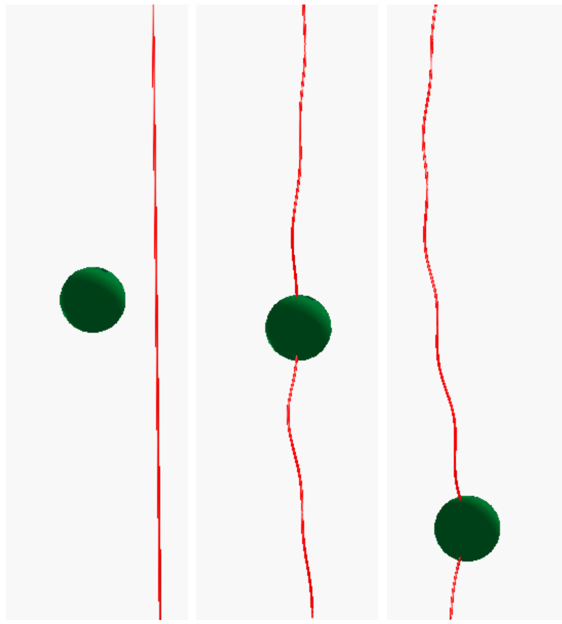
where $\mathbf{v}_n = 0$.

$T = 1.3$ K, particle initially at rest at distance $2a_p$ from the vortex.

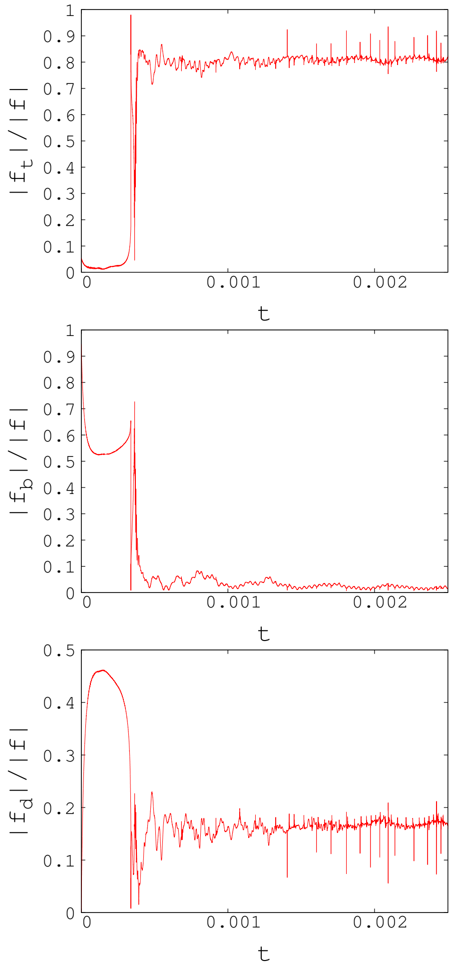
As the particle approaches the vortex, the superflow generated by the particle tends to push away the vortex. Then the vortex effectively sees an image vortex with the opposite polarity behind the surface of the particle and moves to the side, rotating under the velocity self–induced by the curvature, and reconnects with the particle.



Trapping



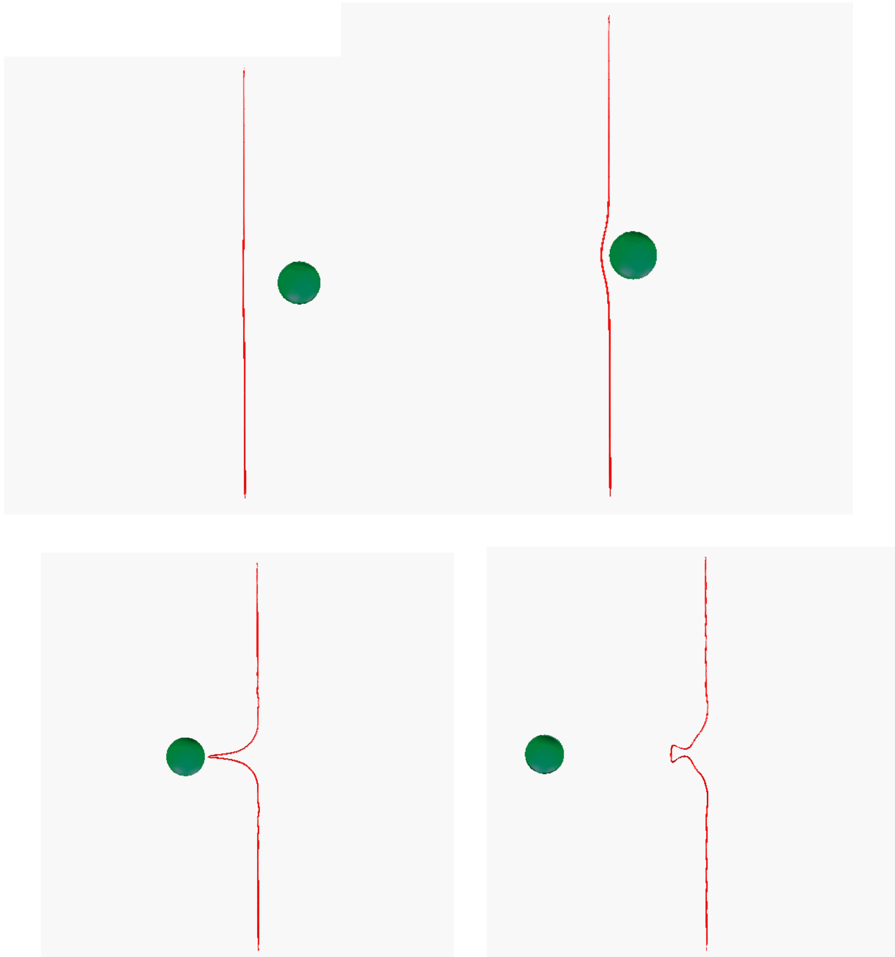
Drifting



Force balance

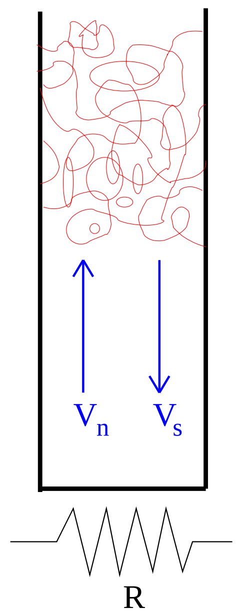
Relative importance of local force, f_t (top), boundary force, f_b (middle), and drag force, f_d (bottom) compared to total force $f = |\mathbf{f}_t + \mathbf{f}_b + \mathbf{f}_d|$.

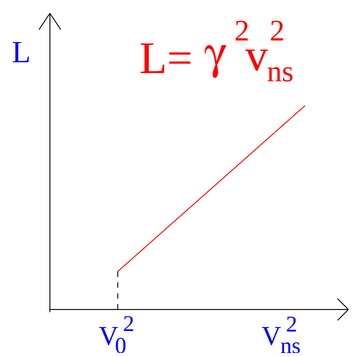
Initially, when the particle is at rest, the boundary force is the most important force and attracts the particle towards the vortex. As the particle accelerates, the main balance is between boundary and drag forces. The local (time–dependent) force is initially negligible because the vortex barely moves, but becomes important when the particle and the vortex are close to reconnection. After the reconnection it is the most important force, as the particle is shaken by Kelvin waves.



Escape

Vertical counterflow



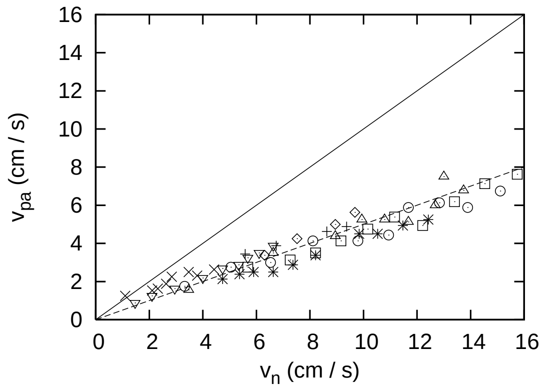


Zhang and Van Sciver expected the particles ($\rho_p \gg \rho$) to fall with velocity v_{slip} given by the balance of viscous drag (up) and gravity (down), so that the adjusted particle velocity, defined as

$$v_{pa} = v_p + v_{slip}$$

was $v_{pa} = v_n$.

Instead they found $v_{pa} = v_n - v_{add}$, where $v_{pa}/v_n \approx 1/2$.



As a particle moves through the tangle, it traps vortex lines. Although vortex lines may later disconnect, on the average the particle is likely to have one or more loops attached. We expect that vortex reconnections with the particle are not very frequent, because $a_p \ll \delta$.

Because the vortices, two extra forces act on a particle:

First extra force

$$\mathbf{F_1} = \int_S p\hat{\mathbf{n}}dS = \frac{\rho_s}{2}\int_S (\mathbf{v}_\ell + \mathbf{v}_b)^2 \hat{\mathbf{n}} dS,$$

where S is the surface of the particle, $\hat{\mathbf{n}}$ is the radial unit vector point out of S into the fluid, $\mathbf{v}_b$ comes from the boundary condition that the normal component of the total superfluid velocity at S vanishes, and $\mathbf{v}_\ell$ is the velocity field around the vortex line. If the radius of curvature of the vortex is larger than a_p, then $\mathbf{v}_b$ is negligible because the velocity field $\mathbf{v}_\ell$ at S is approximately tangential to S, and

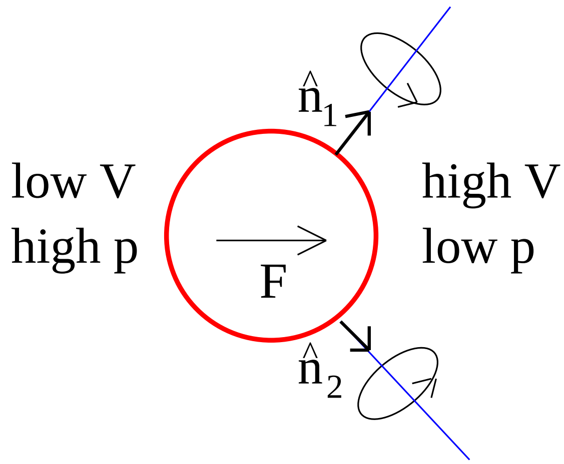
$$\mathbf{F_1} \approx (\frac{\rho_s}{2}) \int_S (\mathbf{v}_\ell)^2 \hat{\mathbf{n}} dS \approx (\frac{\rho_s}{2}) 2\pi (\int_{a_0}^{a} (\frac{\kappa}{2\pi r})^2 r dr) \hat{\mathbf{n}}_0$$

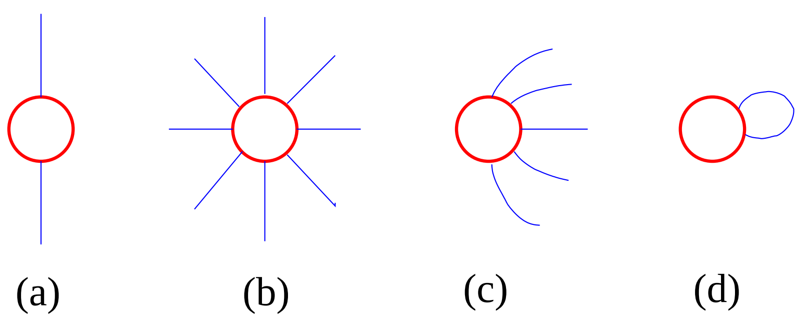
$$\approx \frac{\rho_s \kappa^2}{4\pi} \ln{(a/a_0)} \hat{\mathbf{n}}_0,$$

where $\hat{\mathbf{n}}_0$ is the normal unit vector along one vortex strand pointing out of the plane which represents S.

Generalisation to N vortices attached to the sphere:

$$\mathbf{F_1} \approx \frac{\rho_s \kappa^2}{4\pi} \ln(a/a_0) \sum_{i=1}^{N} \hat{\mathbf{n}}_i.$$





It is likely that the particle, after connecting to a vortex line, keeps moving, dragging a vortex loop (two attachments) along for a fraction of the relative distance to the next vortex with respect to its own size. We expect that the particle suffers a body force of magnitude

$$F_1 \approx \frac{\rho_s \kappa^2}{4\pi} \ln{(a/a_0)} \, (\frac{2\beta a}{\delta}) \, ,$$

where β is a geometrical factor of the order unity which depends on the number of vortex pairs attached to the sphere and the relative distance of travel where they remain attached. For a single vortex, $2\beta a$ is the length of this vortex.

Second extra force

The second force on the sphere arises from the drag of the attached vortex with the normal fluid. We expect $F_2 = \gamma_0 \ell (v_n - v_\ell)$ where γ_0 is a known temperature–dependence friction coefficient; setting $v_\ell = v_p$ (as vortex and particle move together) and interpreting $\ell = 2\beta a$, the friction is thus $2\beta a \gamma_0 (v_n - v_p)$.

Taking into account F_1 and F_2, we have

$$v_n - v_p = v'_{slip} + v_{add},$$

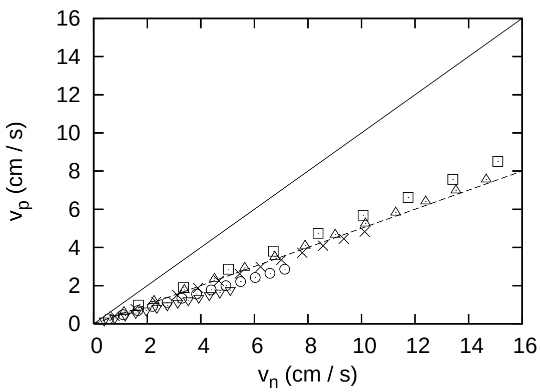
where $v'_{slip} = v_{slip}/f$, $f = 1 + \beta\gamma_0/(3\pi\mu)$ and

$$v_{add} = \frac{\beta\kappa^2\rho_s}{12\pi^2 f\mu\delta}\ln(a/a_0) = \\ = \frac{\beta\kappa^2\gamma\rho}{12\pi^2 f\mu}\ln(a/a_0)v_n$$

because $1/\delta \approx L^{1/2} = \gamma v_{ns} = \gamma\rho v_n/\rho_s$. We obtain

$$v_{pa} = (\frac{f-1}{f})v_{slip} + (1 - \frac{\beta\kappa^2\gamma\rho\ln(a/a_0)}{12\pi^2\mu f})v_n\,.$$

But $v_{slip}(f-1)/f$ is negligible and we conclude that v_{pa} is essentially proportional to v_n:



for $\beta = 4.5$ over the independent ranges of T and q used in the experiment.

Note the same linear dependence of v_{pa} on v_n and the same temperature independence of the slope v_{pa}/v_n which was observed in the experiment. The value of β which best fit the observed slope ($\beta = 4.5$) suggests that the loops which remain attached to the particles are not big.