Weierstrass Institute for Applied Analysis and Stochastics
in Forschungsverbund Berlin e.V.

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Emerging multiyield behavior of elastoplastic beams and plates

Rate-Independence, Homogenization, and Multiscaling, Pisa, November 15, 2007

Leibniz Gemeinschaft

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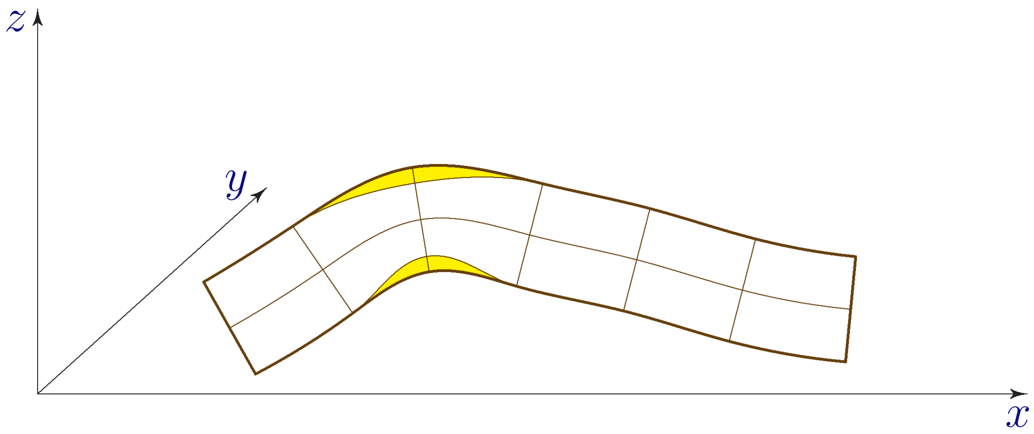
Transversal elastoplastic oscillations (joint work with R. Guenther and J. Sprekels)

Reducing 3D plasticity (e. g. von Mises) to lower dimensional structures (plates, beams, ...) consists in eliminating the transversal variable. We do not observe a sharp transition between the elastic and the plastic deformation regimes any more, but a gradual one.



- The thickness $2h$ is constant
- Existence of plasticized zones
- h is small, and higher order effects in h are neglected

Eccentric layers parallel to the midsurface are subject to larger deformations than the central ones.

Is it possible to let $h \to 0+$, accounting for both the multiyield effect and oscillations?

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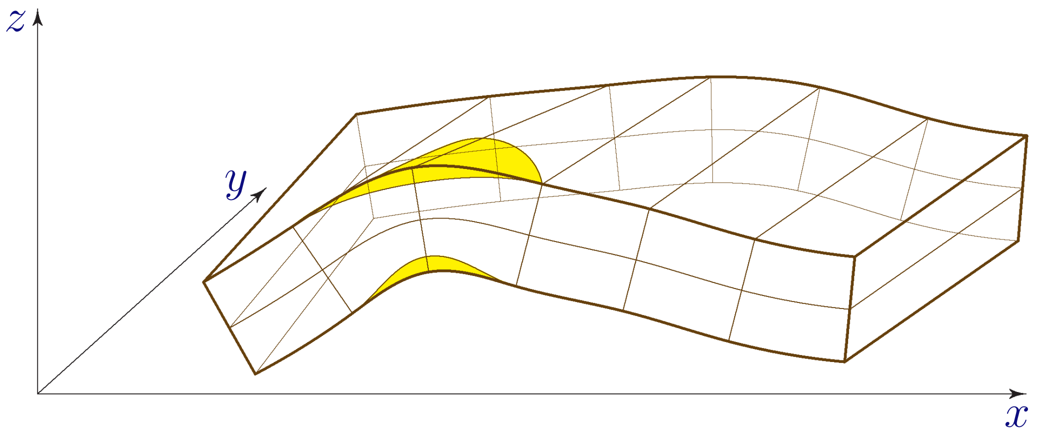

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Displacements and strains in beams

Straight fibers that are perpendicular to the midsurface remain straight and perpendicular after deformation.

$$\mathbf{u}(x,y,0,t) = \begin{pmatrix} 0 \\ 0 \\ w(x,t) \end{pmatrix}$$

Displacement of the midsurface

$$\mathbf{u}(x,y,z,t) = \begin{pmatrix} -zw_x(x,t) \\ 0 \\ w(x,t) \end{pmatrix}$$

Displacement at distance z from the midsurface

$$\boldsymbol{\varepsilon}(x,y,z,t) = \begin{pmatrix} -zw_{xx}(x,t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Strain tensor

Displacements and strains in plates

Straight fibers that are perpendicular to the midsurface remain straight and perpendicular after deformation.

$$\mathbf{u}(x,y,0,t) = \begin{pmatrix} 0 \\ 0 \\ w(x,y,t) \end{pmatrix}$$

Displacement of the midsurface

$$\mathbf{u}(x,y,z,t) = \begin{pmatrix} -zw_x(x,y,t) \\ -zw_y(x,y,t) \\ w(x,y,t) \end{pmatrix}$$

Displacement at distance z from the midsurface

$$\boldsymbol{\varepsilon}(x,y,z,t) = \begin{pmatrix} -zw_{xx}(x,y,t) & -zw_{xy}(x,y,t) & 0 \\ -zw_{xy}(x,y,t) & -zw_{yy}(x,y,t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Strain tensor

Constitutive laws

$\varepsilon = \varepsilon^e + \varepsilon^p$ — Elastic and plastic strain components

$\boldsymbol{\sigma} = 2\mu\varepsilon^e + \lambda(\varepsilon^e : \boldsymbol{\delta})\,\boldsymbol{\delta}$ — Isotropic Hooke's strain-stress law

$\boldsymbol{\delta}$ ... Kronecker tensor

μ, λ ... Lamé's constants

$\mathbf{D}\boldsymbol{\sigma} : \boldsymbol{\sigma} \le \frac{2}{3}R^2$ — von Mises yield criterion

$\mathbf{D}\boldsymbol{\sigma} = \boldsymbol{\sigma} - \frac{1}{3}(\boldsymbol{\sigma} : \boldsymbol{\delta})\,\boldsymbol{\delta}$ ... stress deviator

$K_R = \{\tilde{\boldsymbol{\sigma}}\,;\, \mathbf{D}\tilde{\boldsymbol{\sigma}} : \tilde{\boldsymbol{\sigma}} \le \frac{2}{3}R^2\}$ — admissible stress domain

$$\left.\begin{array}{l} \dot{\varepsilon}^p \in \partial I_{K_R}(\boldsymbol{\sigma}) \\ \boldsymbol{\sigma} \in \partial M_{K_R^*}(\dot{\varepsilon}^p) \end{array}\right\}$$

von Mises flow rule

Special case of beams

$$\boldsymbol{\sigma} = \begin{pmatrix} E\,\varepsilon^e_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hooke's law

$E > 0$ ... Young elasticity modulus

$$\mathbf{D}\boldsymbol{\sigma} = E\,\varepsilon^e_{11} \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}$$

Yield condition $E\,|\varepsilon^e_{11}| \le R$

$$\dot{\boldsymbol{\varepsilon}}^p = \frac{3}{2}\,\dot{\varepsilon}^p_{11} \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}$$

Flow rule

$\dot{\varepsilon}^p_{11}\,(E\,\varepsilon^e_{11} - \tilde{\sigma}) \ge 0$ for all $|\tilde{\sigma}| \le R$

$\varepsilon^p_{11} + \varepsilon^e_{11} \;=\; -z\,w_{xx}(x,t)$

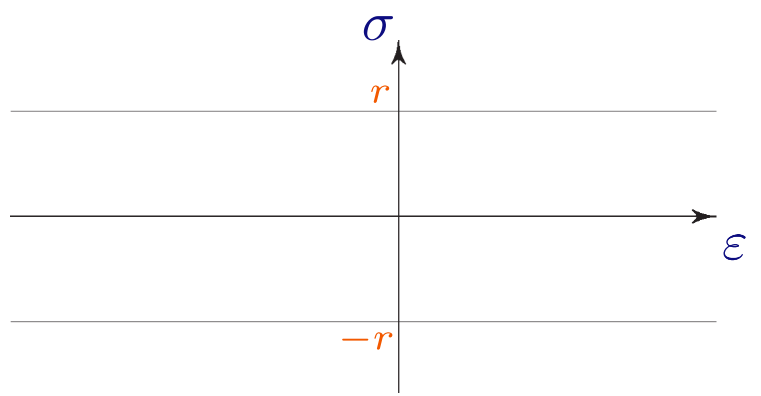
The stop operator

Given a parameter $r > 0$, a function $\varepsilon : [0, T] \to \mathbb{R}$, and an initial condition $\sigma^0 \in [-r, r]$, we look for functions $\sigma, \xi : [0, T] \to \mathbb{R}$ such that $\sigma(0) = \sigma^0$, and

$$\sigma(t) + \xi(t) = \varepsilon(t)$$

$$|\sigma(t)| \leq r$$

$$\dot{\xi}(t)\,(\sigma(t) - \tilde{\sigma}) \geq 0 \quad \forall \tilde{\sigma} \in [-r, r]$$



For every $\varepsilon \in W^{1,1}(0, T)$ and $\sigma^0 \in [-r, r]$, the problem has a unique solution $\sigma \in W^{1,1}(0, T)$. The solution mapping $\mathfrak{s}_r : [-r, r] \times W^{1,1}(0, T) \to W^{1,1}(0, T)$, $\sigma = \mathfrak{s}_r[\sigma^0, \varepsilon]$, is called the stop (or elastoplastic element).

It is Lipschitz continuous and admits Lipschitz continuous extension to $\mathfrak{s}_r : [-r, r] \times C[0, T] \to C[0, T]$.

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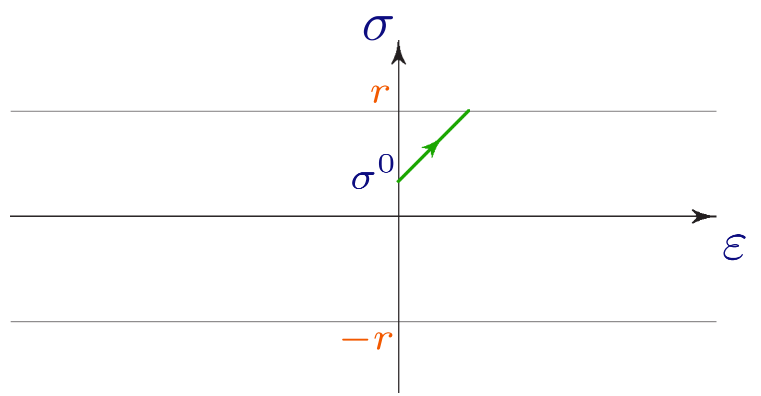

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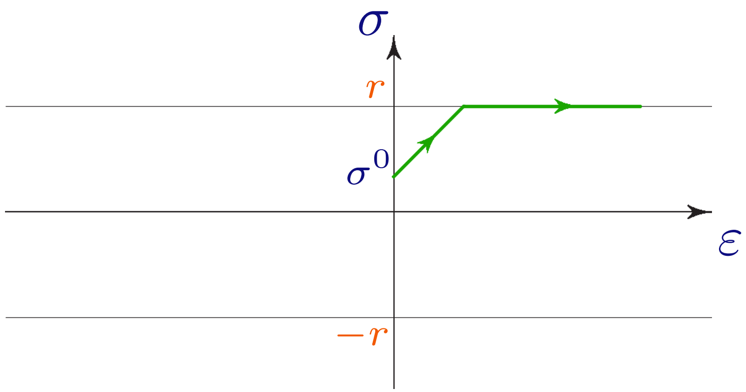

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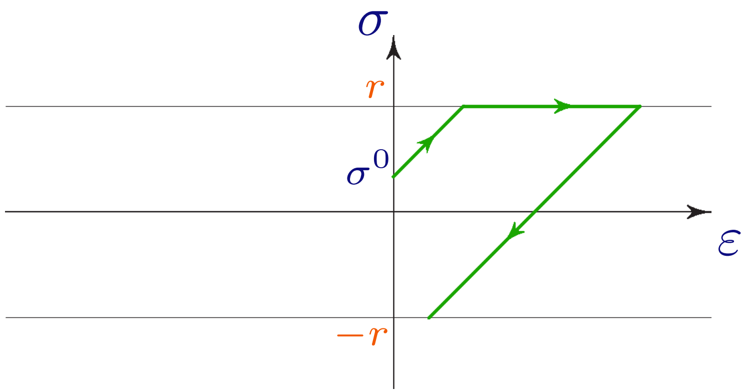

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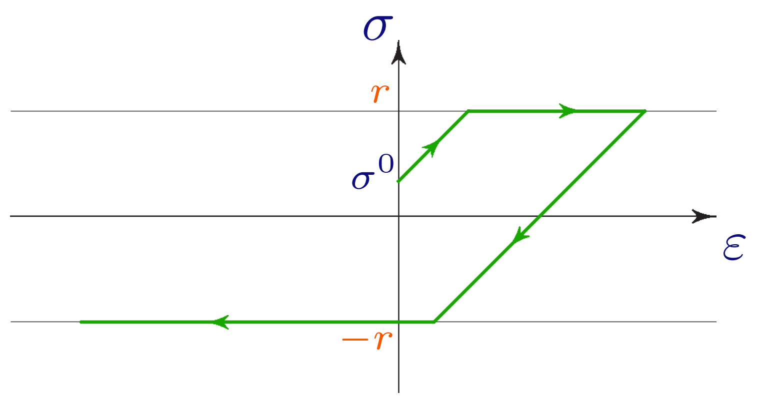

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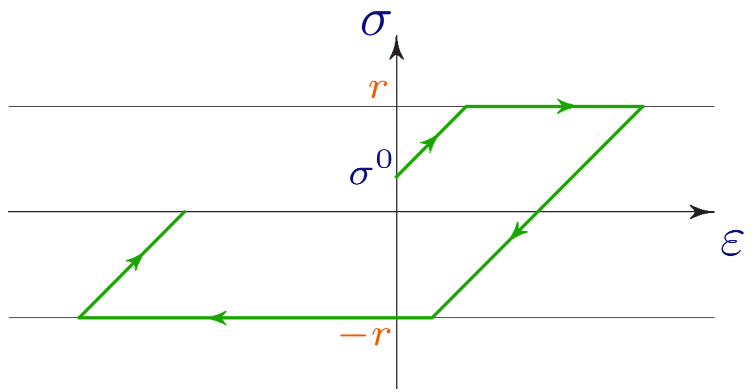

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It is Lipschitz continuous and admits Lipschitz continuous extension to $\mathfrak{s}_r : [-r,r] \times C[0,T] \to C[0,T]$.

The stop operator – continuation

Initial condition

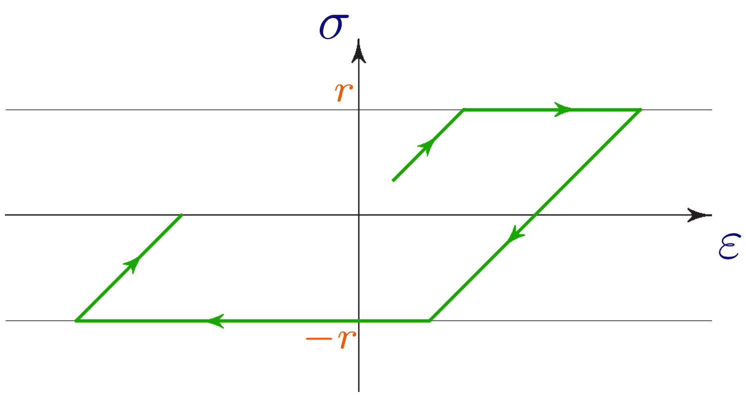
$$\sigma^0 = \max\{-r, \min\{r, \varepsilon(0)\}\}$$

corresponds to the reference initial state with no previous loading history.

$$\sigma(t) + \xi(t) = \varepsilon(t)$$

$$|\sigma(t)| \le r$$

$$\dot\xi(t)\,(\sigma(t) - \tilde\sigma) \ge 0 \;\; \forall \tilde\sigma \in [-r, r]$$



The corresponding stop operator $\sigma = \mathfrak{s}_r[\varepsilon]$

has then for every $c \in \mathbb{R}$ the scaling property $\boxed{\mathfrak{s}_r[c\,\varepsilon] = c\,\mathfrak{s}_{\frac{r}{|c|}}[\varepsilon]}$.

Constitutive law for elastoplastic beams

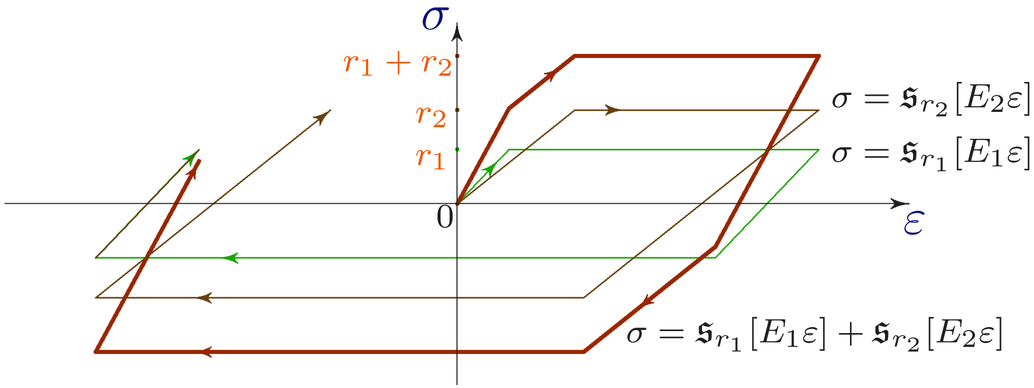
Using the stop operator, and assuming no previous loading history, we may write the elastoplastic constitutive relation for σ_{11} in the form

$$\sigma_{11} = \mathfrak{s}_R[-Ez\, w_{xx}] = -Ez\, \mathfrak{s}_{\frac{R}{E|z|}}[w_{xx}]$$

$\Longrightarrow$ At distance z from the midsurface, the longitudinal fiber behaves like an elastoplastic element with elasticity modulus Ez and yield point $\frac{R}{E|z|}$ with respect to the input w_{xx}.

Eccentric fibers are harder in elasticity and softer in plasticity than the central ones!

Prandtl-Ishlinskii sums of several stops produce gradual transitions from elasticity to plasticity.



Momentum balance for elastoplastic beams

Let $(x, y, z) \in \Omega = (0, L) \times (-b, b) \times (-h, h)$ be a generic point of the beam

(L is the length, $2b$ is the width, $2h$ is the height).

We write the momentum balance in variational form

$$\int_{\Omega} \Big(\rho \mathbf{u}_{tt} \cdot \hat{\mathbf{u}} + \boldsymbol{\sigma} : \hat{\varepsilon} \Big)\, dx\, dy\, dz \;=\; \int_{\partial\Omega} (\boldsymbol{\sigma} \cdot \boldsymbol{\nu}) \cdot \hat{\mathbf{u}}\, ds$$

for every geometrically admissible test functions $\hat{\mathbf{u}}$, where ρ is the mass density and $\boldsymbol{\nu}$ is the unit outward normal.

We prescribe boundary conditions

$$\boldsymbol{\sigma} \cdot \boldsymbol{\nu} = \begin{cases} \begin{pmatrix} 0 \\ 0 \\ f(x,t) \end{pmatrix} & \text{on the upper boundary } z = h \\ 0 & \text{otherwise} \end{cases}$$

Momentum balance for elastoplastic beams - II

We have the identities

$$\int_\Omega \rho \mathbf{u}_{tt} \cdot \hat{\mathbf{u}}\, dx\, dy\, dz = 4hb\,\rho \int_0^L \left(w_{tt}\, \hat{w} + \frac{h^2}{3} w_{xtt}\, \hat{w}_x \right) dx$$

$$\int_\Omega \boldsymbol{\sigma} : \hat{\varepsilon}\, dx\, dy\, dz = 2b \int_0^L \int_{-h}^{h} E\, z^2\, \mathfrak{s}_{\frac{R}{E|z|}}[w_{xx}]\, dz\, \hat{w}_{xx}\, dx$$

$$\int_{\partial\Omega} (\boldsymbol{\sigma} \cdot \boldsymbol{\nu}) \cdot \hat{\mathbf{u}}\, ds = 2b \int_0^L f(x,t)\, \hat{w}\, dx$$

To eliminate vertical rigid body displacements, we prescribe additional boundary conditions

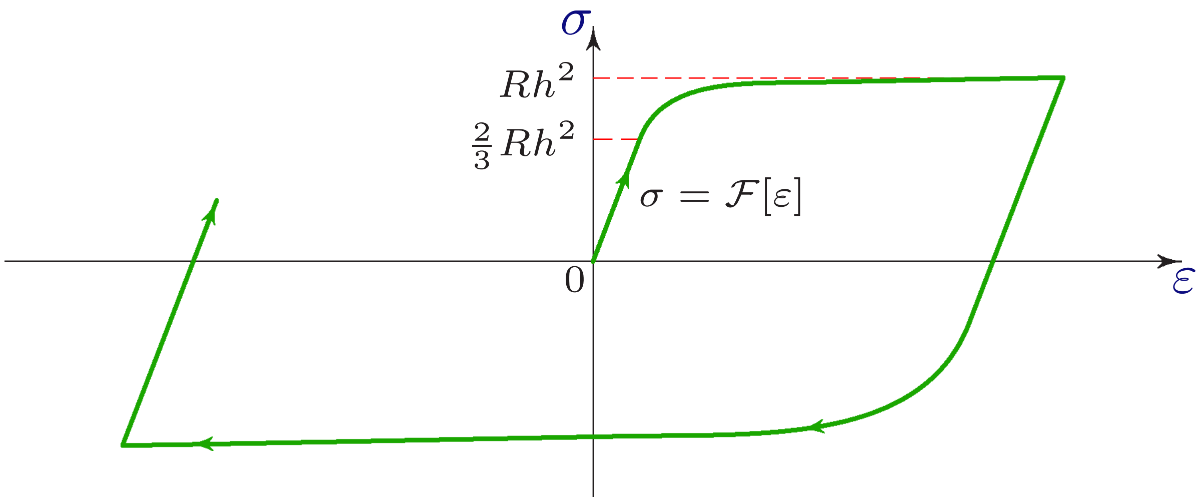
$$w(0,t) = w(L,t) = 0$$

The test functions $\hat{w}$ are chosen in $H^2(0,L) \cap H_0^1(0,L)$.

Prandtl-Ishlinskii operators

The mapping $\mathcal{F}$ which with each function $\varepsilon \in C[0,T]$ associates the integral

$$\mathcal{F}[\varepsilon] = \int_{-h}^{h} E\, z^2\, \mathfrak{s}_{\frac{R}{E|z|}}[\varepsilon]\, dz = \frac{2R^3}{E^2} \int_{\frac{R}{Eh}}^{\infty} r^{-4}\, \mathfrak{s}_r[\varepsilon]\, dr$$



is a special case of the Prandtl-Ishlinskii operator $\mathcal{F}[\varepsilon] = \int_0^{\infty} \varphi(r)\, \mathfrak{s}_r[\varepsilon]\, dr$

with a given distribution function $\varphi(r)$.

The associated PDE

Given a right-hand side f, we look for a solution w of the problem

$$\rho\, w_{tt} - \frac{h^2\rho}{3} w_{xxtt} + \frac{1}{2h}\mathcal{F}[w_{xx}]_{xx} \;=\; \frac{1}{2h} f$$

with boundary conditions

$$w(0,t) = w(L,t) = \mathcal{F}[w_{xx}](0,t) = \mathcal{F}[w_{xx}](L,t) = 0 \,.$$

Prandtl-Ishlinskii operators are not differentiable in general; hence, for the existence and uniqueness analysis, we rewrite the PDE as system (rescaling all constants to 1)

$$\begin{aligned} u_t &= \mathcal{F}[w_{xx}] \\ w_t - w_{xxt} &= -u_{xx} + g \end{aligned}$$

with boundary conditions $u(0,t) = w(0,t) = u(L,t) = w(L,t) = 0$.

Theorem 1

Let the initial conditions $u_0 \in H^2(0,L)$, $w_0 \in H^3(0,L)$ *satisfy natural compatibility conditions, and let* g *be given,* $g_t \in L^2((0,T)\times(0,L))$. *Then the PDE system admits a unique solution* u, w *with the regularity*

$$u_{tt}, u_{xx}, w_{xxt} \in L^\infty(0,T;L^2(0,L)),\ u_{xt}, w_{xtt} \in L^2((0,T)\times(0,L)).$$

Method of proof: Galerkin approximations, energy estimates, Minty trick.

$$\begin{aligned} u_t &= \mathcal{F}[w_{xx}] \\ w_t - w_{xxt} &= -u_{xx} + g \\ u(0,t) = w(0,t) &= u(L,t) = w(L,t) = 0 \end{aligned}$$

Theorem 2: The clamped beam

Let the initial conditions $u_0 \in H^2(0,L)$, $w_0 \in H^3(0,L)$ *satisfy natural compatibility conditions, and let* g *be given,* $g_t, g_{tt} \in L^2((0,T)\times(0,L))$. *Then the PDE system admits a unique solution* u, w *with the regularity*

$$u_{tt}, u_{xx}, w_{xxt} \in L^\infty(0,T;L^2(0,L)),\ u_{xt}, w_{xtt} \in L^\infty(0,T;L^2(0,L)).$$

Method of proof: space semidiscretization, energy estimates.

$$\begin{aligned} u_t &= w_{xx} + \mathcal{F}[w_{xx}] \\ w_t - w_{xxt} &= -u_{xx} + g \\ w(0,t) = w_x(0,t) &= w(L,t) = w_x(L,t) = 0 \end{aligned}$$

Plates

Let $\Omega_0 \subset \mathbb{R}^2$ be the reference shape of the plate. In a similar way as in case of beams, after rescaling all physical constants, we derive the variational equation

$$\int_{\Omega_0} \Big(w_{tt}\, \hat{w} + w_{xtt}\, \hat{w}_y + w_{ytt}\, \hat{w}_y + \mathbf{B}\mathbf{D}_2 w \cdot \mathbf{D}_2 \hat{w} \Big) dx\, dy$$
$$+ \int_{\Omega_0} \langle \mathcal{F}[\mathbf{D}_2 w], \mathbf{D}_2 \hat{w} \rangle\, dx\, dy \;=\; \int_{\Omega_0} g\, \hat{w}\, dx\, dy$$

for every $\hat{w} \in H^2(\Omega_0) \cap H^1_0(\Omega_0)$, where

$\mathbf{D}_2$ is the differential operator $(\partial^2_{xx}, \partial^2_{xy}, \partial^2_{yy})$,

$\mathbf{B}$ is a positive definite kinematic hardening matrix, and

$\mathcal{F}$ is a 3D Prandtl-Ishlinskii operator.

The multidimensional stop and Prandtl-Ishlinskii operators

Given a convex closed set Z containing 0 in its interior in a real separable Hilbert space X, a function $\varepsilon : [0,T] \to X$, and an initial condition $\sigma^0 \in Z$, we look for functions $\sigma, \xi : [0,T] \to X$ such that $\sigma(0) = \sigma^0$, and

$$\sigma(t) + \xi(t) = \varepsilon(t)$$

$$\sigma(t) \in Z$$

$$\left\langle \dot{\xi}(t), \sigma(t) - \tilde{\sigma} \right\rangle \geq 0 \quad \forall \tilde{\sigma} \in Z\,.$$

$$\mathcal{F}[\varepsilon] = \int_0^\infty \varphi(r)\, \mathfrak{s}_{rZ}[\varepsilon]\, dr$$

Prandtl-Ishlinskii operator

For every $\varepsilon \in W^{1,1}(0,T;X)$ and $\sigma^0 \in Z$, there exists a unique solution $\sigma = \mathfrak{s}_Z[\sigma^0, \varepsilon] \in W^{1,1}(0,T;X)$.

The mapping $\mathfrak{s}_Z : Z \times W^{1,1}(0,T;X) \to W^{1,1}(0,T;X)$ is continuous and admits continuous extension to $\mathfrak{s}_Z : Z \times C([0,T];X) \to C([0,T];X)$. Under natural assumptions on $\varphi(r)$, the same holds for $\mathcal{F}$.

Theorem 3: Plate

Let the initial conditions $w_0, w_1 \in H^3(\Omega_0)$ *satisfy natural compatibility conditions, and let* $g_t \in L^2((0,T) \times \Omega_0)$. *Then the variational plate equation admits a unique solution* w, $w\big|_{t=0} = w_0$, $w_t\big|_{t=0} = w_1$, *with the regularity* $\Delta w_t, \nabla w_{tt} \in L^\infty(0,T; L^2(\Omega_0))$.

Method of proof: vanishing viscosity, energy estimates, Minty trick

$$\int_{\Omega_0} \Big(w_{tt}\, \hat{w} + w_{xtt}\, \hat{w}_y + w_{ytt}\, \hat{w}_y + \mathbf{B}\mathbf{D}_2 w \cdot \mathbf{D}_2 \hat{w} \Big) dx\, dy$$

$$+ \int_{\Omega_0} \langle \mathcal{F}[\mathbf{D}_2 w], \mathbf{D}_2 \hat{w} \rangle \, dx\, dy \;=\; \int_{\Omega_0} g\, \hat{w}\, dx\, dy$$

for every $\hat{w} \in H^2(\Omega_0) \cap H^1_0(\Omega_0)$.

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Method of proof: vanishing viscosity, energy estimates, Minty trick

$$\int_{\Omega_0} \Big(w_{tt}\,\hat{w} + w_{xtt}\,\hat{w}_y + w_{ytt}\,\hat{w}_y + \mathbf{B}\mathbf{D}_2 w \cdot \mathbf{D}_2 \hat{w} + \gamma \mathbf{D}_2 w_t \cdot \mathbf{D}_2 \hat{w} \Big) dx\,dy$$

$$+ \int_{\Omega_0} \langle \mathcal{F}[\mathbf{D}_2 w], \mathbf{D}_2 \hat{w} \rangle\, dx\,dy \;=\; \int_{\Omega_0} g\,\hat{w}\,dx\,dy$$

for every $\hat{w} \in H^2(\Omega_0) \cap H^1_0(\Omega_0)$, letting γ tend to $0+$.

Conclusions

- Reducing the space dimension in elastoplasticity, we lose control of the interface between the elastic and the plastic zones.
- For a "lower dimensional" observer, the transition from perfectly elastic to perfectly plastic behavior looks smooth.
- The reduced momentum balance equation contains a Prandtl-Ishlinskii operator as a linear superposition of solution operators to a family of variational inequalities parameterized by the transversal variable.
- PDEs with Prandtl-Ishlinskii operators can be solved by conventional methods.