RATE INDEPENDENT PROCESSES IN VISCOUS SOLIDS AND THEIR THERMODYNAMICS

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Combination of rate-independent vs. rate-dependent proceeses.

Example: plasticity or damage etc. vs. inertia, viscosity, or thermo-coupling.

Generalized standard materials (Halphen and Nguen) at small strains:

$$\varrho\frac{\partial^2 u}{\partial t^2} - \mathrm{div}(\sigma) = f, \qquad \underbrace{\sigma = \zeta_2'\big(e(\tfrac{\partial u}{\partial t})\big) + \sigma_{\rm el}}_{\text{Kelvin-Voigt rheology}}, \qquad \underbrace{e(u) = \frac{\nabla u^\top + \nabla u}{2}}_{\text{small-strain tensor}},$$

$$\partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big) + \sigma_{\rm in} \ni 0, \qquad \underbrace{\sigma_{\rm el} = \varphi_e'(e(u), z)}_{\text{elastic stress}}, \qquad \underbrace{\sigma_{\rm in} = \varphi_z'(e(u), z)}_{\text{"inelastic" stress}},$$

where

- $u : \Omega \to \mathbb{R}^n$ is a displacement,
- $z : \Omega \to \mathbb{R}^m$ a vector of internal parameters,
- σ the stress tensor,
- $\varphi : \mathbb{R}^{n\times n} \times \mathbb{R}^m \to \mathbb{R}$ is a stored energy, and
- $\zeta_2 : \mathbb{R}^{n\times n} \to [0, +\infty)$ and
  $\zeta_1 : \mathbb{R}^m \to [0, +\infty]$ are (pseudo)potentials of dissipative forces.

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- $\zeta_2 : \mathbb{R}^{n\times n} \to [0, +\infty)$ homogenoeus degree-2 (quadratic, viscosity),
  $\zeta_1 : \mathbb{R}^m \to [0, +\infty]$ homogenoeus degree-1 (activated response).

Energetics (at least formally):

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}\Big|^2 + \varphi(e(u),z)\,\mathrm{d}x}_{\text{kinetic and stored energy}} + \underbrace{\int_\Omega \xi\Big(e(\frac{\partial u}{\partial t}),\frac{\partial z}{\partial t}\Big)\mathrm{d}x}_{\text{rate of dissipation}} = \underbrace{\int_\Omega f\cdot\frac{\partial u}{\partial t}\,\mathrm{d}x}_{\text{power of external force}}$$

where

$$\xi\Big(e(\frac{\partial u}{\partial t}),\frac{\partial z}{\partial t}\Big) := \partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big)\frac{\partial z}{\partial t} + \zeta_2'\Big(e(\frac{\partial u}{\partial t})\Big):e(\frac{\partial u}{\partial t}).$$

Homogeneity degree-ℓ:

$$\forall \ell = 1,2\ \forall r\geq 0\ \forall v: \qquad \zeta_\ell(rv) = r^\ell\zeta_\ell(v).$$

Elementrary calculus shows the formula for the directional derivative $\zeta_\ell'(v)v$:

$$\zeta_\ell'(v)v = \lim_{\varepsilon\to 0+}\frac{\zeta_\ell(v+\varepsilon v)-\zeta_\ell(v)}{\varepsilon} = \lim_{\varepsilon\to 0+}\frac{(1+\varepsilon)^\ell-1}{\varepsilon}\zeta_\ell(v) = \ell\,\zeta_\ell(v).$$

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where

$$\xi\Big(e(\frac{\partial u}{\partial t}),\frac{\partial z}{\partial t}\Big) = \zeta_1\Big(\frac{\partial z}{\partial t}\Big) + 2\zeta_2\Big(e(\frac{\partial u}{\partial t})\Big).$$

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Without loss of generality, we may consider

$$\zeta_1(\dot{z}) := \delta_K^*(\dot{z}), \qquad \text{with } K \subset \mathbb{R}^m \text{ convex closed,}$$

where δ_K^* is the conjugate function to the indicator function δ_K of K.

- Assuming K bounded (resp. containing 0 in its interior) makes ζ_1 coercive (resp. bounded). Also, $K = \partial\zeta_1(0)$.
- ζ_1 nonsmooth at 0, which follows from its homogeneity of degree 1 (except the trivial case where ζ_1 is linear),
- activated processes: to trigger z evolving, the driving force $\varphi_z'(e(u), z)$ must reach an activation threshold, namely the boundary of K.

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Using the convex-analysis calculus $(\partial\delta_K^*)^{-1} = \partial\delta_K$, the evolution rule for z

$$\partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big) + \varphi_z'(e(u), z) \ni 0,$$

can be rewritten into the form of the so-called sweeping process

$$\frac{\partial z}{\partial t} \in \partial\delta_K\big(- \varphi_z'(e(u), z)\big).$$

Using the maximal monotonicity of $\partial\zeta_1$ and Colli-Visintin concept,

$$\partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big) + \varphi_z'(e(u), z) \ni 0,$$

can be rewritten into the form of an inequality

$$\int_Q \big(\omega + \varphi_z'(e(u), z)\big)\cdot\Big(v - \frac{\partial z}{\partial t}\Big)\mathrm{d}x\mathrm{d}t \geq 0$$

for all v, ω such that $\omega \in \partial\zeta_1(v)$ a.e. on $Q := (0, T) \times \Omega,\ T > 0$.

Using the maximal monotonicity of $\partial\zeta_1$ and its potentiality,

$$\partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big) + \varphi_z'(e(u),z) \ni 0,$$

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$$\int_Q \varphi_z'(e(u),z)\cdot\Big(v - \frac{\partial z}{\partial t}\Big) + \zeta_1(v)\,\mathrm{d}x\mathrm{d}t \geq \int_Q \zeta_1\Big(\frac{\partial z}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t$$

for all v.

The last integral is the total variation of $z:[0,T]\to L^1(Q;\mathbb{R}^m)$ w.r.t. δ_K^*:

$$\int_Q \zeta_1\Big(\frac{\partial z}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t = \mathrm{Var}_K(z;0,T) := \sup\sum_{i=1}^k \int_\Omega \delta_K^*\big(z(t_i,x)-z(t_{i-1},x)\big)\,\mathrm{d}x$$

where the supremum is taken over all partitions
$0=t_0<t_1<\ldots<t_k=T$, $k\in\mathbb{N}$.

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Using the maximal monotonicity of $\partial\zeta_1$ and its potentiality,

$$\partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big)+\varphi_z'(e(u),z)\ni 0,$$

can be rewritten into the form of an inequality: $\forall v$:

$$\int_Q \varphi_z'(e(u),z)\cdot\Big(v-\frac{\partial z}{\partial t}\Big)+\zeta_1(v)\,\mathrm{d}x\mathrm{d}t\geq\int_Q\zeta_1\Big(\frac{\partial z}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t.$$

The integral $\displaystyle\int_Q \varphi_z'(e(u),z)\cdot\frac{\partial z}{\partial t}\,\mathrm{d}x\mathrm{d}t=\int_Q\sigma_{\rm in}\cdot\frac{\partial z}{\partial t}\mathrm{d}x\mathrm{d}t$ makes troubles: only L^1-bounds for $\dfrac{\partial z}{\partial t}$ and no L^∞-compactness of $\varphi_z'(e(u),z)$. We use

$$\int_Q\sigma_{\rm in}\frac{\partial z}{\partial t}\mathrm{d}x\mathrm{d}t=\int_\Omega\varphi(e(u(T)),z(T))-\varphi(e(u_0),z_0)\,\mathrm{d}x-\int_Q\sigma_{\rm el}:e\Big(\frac{\partial u}{\partial t}\Big)\mathrm{d}x\mathrm{d}t,$$

counting also the initial condition

$$u(0,\cdot)=u_0,\qquad\qquad \Big(\text{later also }\ \frac{\partial u}{\partial t}(0,\cdot)=\dot u_0\quad\&\quad z(0,\cdot)=z_0\Big).$$

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$$\int_Q \sigma_{\rm in}\frac{\partial z}{\partial t}\mathrm{d}x\mathrm{d}t = \int_\Omega \varphi(e(u(T)),z(T)) - \varphi(e(u_0),z_0)\,\mathrm{d}x - \int_Q \sigma_{\rm el} : e\Big(\frac{\partial u}{\partial t}\Big)\mathrm{d}x\mathrm{d}t,$$

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$u(0,\cdot) = u_0,$ $\quad\Big($later also $\dfrac{\partial u}{\partial t}(0,\cdot) = \dot{u}_0$ & $z(0,\cdot) = z_0\Big).$

Using the maximal monotonicity of $\partial\zeta_1$ and its potentiality,

$$\partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big) + \varphi'_z(e(u),z) \ni 0,$$

can be rewritten into the form of an inequality: $\forall v$:

$$\int_Q \sigma_{\rm in} v + \zeta_1(v)\,\mathrm{d}x\mathrm{d}t \ge \mathrm{Var}_K(z;0,T) + \int_\Omega \varphi(e(u(T)),z(T))\,\mathrm{d}x$$
$$- \int_\Omega \varphi(e(u_0),z_0)\,\mathrm{d}x - \int_Q \sigma_{\rm el} : e(\frac{\partial u}{\partial t})\mathrm{d}x\mathrm{d}t.$$

We can base a definition of a weak solution on this inequality;

$e(\frac{\partial u}{\partial t})$ has to be controlled in the norm topology of $L^2(Q;\mathbb{R}^{n\times n})$, coercivity of viscosity potential ζ_2 (uniform monotonicity of ζ_2') needed.

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Otherwise, we can still use the force equlibrium tested by $\frac{\partial u}{\partial t}$:

$$\int_Q \sigma_{\rm el}{:}e\big(\frac{\partial u}{\partial t}\big)\mathrm{d}x\mathrm{d}t = \int_Q f\cdot\frac{\partial u}{\partial t} - 2\zeta_2\Big(e\big(\frac{\partial u}{\partial t}\big)\Big)\mathrm{d}x\mathrm{d}t + \frac{\varrho}{2}\int_\Omega |\dot u_0|^2 - \Big|\frac{\partial u}{\partial t}(T)\Big|^2\mathrm{d}x.$$

Using the monotonicity and potentiality of $\partial\zeta_1$ and the force equlibrium,

$$\partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big) + \varphi_z'(e(u), z) \ni 0,$$

can be rewritten into the form of an inequality: $\forall v$:

$$\int_Q \sigma_{\rm in} v + \zeta_1(v)\,{\rm d}x{\rm d}t \geq {\rm Var}_K(z; 0, T) + 2\int_Q \zeta_2\Big(e(\frac{\partial u}{\partial t})\Big){\rm d}x{\rm d}t$$
$$+ \int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}(T)\Big|^2 + \varphi(e(u(T)), z(T))\,{\rm d}x$$
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We can base a definition of a weak solution alternatively on this inequality;

$e(\frac{\partial u}{\partial t})$ suffices to be controlled in the weak topology of $L^2(Q;\mathbb{R}^{n\times n})$.

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$$\partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big)+\varphi_z'(e(u),z)\ni 0,$$

can be rewritten into the form of an inequality: $\forall v$:

$$\begin{aligned}\int_Q \sigma_{\rm in}v+\zeta_1(v)\,\mathrm{d}x\mathrm{d}t \geq \mathrm{Var}_K(z;0,T)+2\int_Q\zeta_2\Big(e(\frac{\partial u}{\partial t})\Big)\mathrm{d}x\mathrm{d}t\\ +\int_\Omega\frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}(T)\Big|^2+\varphi(e(u(T)),z(T))\,\mathrm{d}x\\ -\int_\Omega\frac{\varrho}{2}|\dot u_0|^2+\varphi(e(u_0),z_0)\,\mathrm{d}x+\int_Q f\cdot\frac{\partial u}{\partial t}\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

We can base a definition of a weak solution alternatively on this inequality;

in fact, both definitions are equivalent to each other

if $\frac{\partial^2 u}{\partial t^2}\in L^2(I;W^{1,2}(\Omega;\mathbb{R}^n)^*)$.

Assumptions:

φ quadratic coercive on $\mathbb{R}^{n\times n}\times\mathbb{R}^m$,
$\zeta_1:\mathbb{R}^m\to\mathbb{R}$ be coercive, and
$\zeta_2:\mathbb{R}^{n\times n}\to\mathbb{R}$ quadratic nonnegative,
$f\in L^1(I;L^2(\Omega;\mathbb{R}^n))$.

A time step $\tau>0$ and
the recursive increment formula (Rothe method, implicit Euler method):

$$\varrho\frac{u_\tau^k-2u_\tau^{k-1}+u_\tau^{k-2}}{\tau^2}-\mathrm{div}\Big(\zeta_2'\big(e\big(\frac{u_\tau^k-u_\tau^{k-1}}{\tau}\big)\big)+\varphi_e'(e(u_\tau^k),z_\tau^k)\Big)=f_\tau^k,$$

$$\partial\zeta_1\Big(\frac{z_\tau^k-z_\tau^{k-1}}{\tau}\Big)+\varphi_z'(e(u_\tau^k),z_\tau^k)\ni 0,$$

starting for $k=1$ with the intitial conditions $u_\tau^0=u_0$, $u_\tau^{-1}=u_0-\tau\dot u_0$, and $z_\tau^0=z_0$. Here $f_\tau^k=\frac{1}{\tau}\int_{(k-1)\tau}^{k\tau}f(t)\,\mathrm{d}t$.

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Existence of approximate solution (u_τ^k, z_τ^k): minimization of the functional

$$\int_\Omega \frac{\tau^2\varrho}{2}\Big|\frac{u-2u_\tau^{k-1}+u_\tau^{k-2}}{\tau^2}\Big|^2 + \tau\zeta_1\Big(\frac{z-z_\tau^{k-1}}{\tau}\Big) + \tau\zeta_2\Big(e\big(\frac{u-u_\tau^{k-1}}{\tau}\big)\Big) + \varphi(e(u),z) - f_\tau^k\cdot u\,\mathrm{d}x$$

for $(u,z)\in W^{1,2}(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{R}^m)$.

A-priori estimates for the piece-wise affine interpolant u_τ and the piece-wise constant interpolant $\bar z_\tau$:

1) test by $\frac{u_\tau^k-u_\tau^{k-1}}{\tau}$ and $\frac{z_\tau^k-z_\tau^{k-1}}{\tau}$:

$$\big\|u_\tau\big\|_{W^{1,\infty}(I;L^2(\Omega;\mathbb{R}^n))\cap W^{1,2}(I;W^{1,2}(\Omega;\mathbb{R}^n))}\le C,$$
$$\big\|\bar z_\tau\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^m))\cap \mathrm{BV}(\bar I;L^1(\Omega;\mathbb{R}^m))}\le C.$$

2) test by $v\in L^2(I;W^{1,2}(\Omega;\mathbb{R}^n))$:

$$\Big\|\frac{\mathrm{d}u_\tau}{\mathrm{d}t}\Big\|_{\mathrm{BV}(\bar I;(I;W^{1,2}(\Omega;\mathbb{R}^n)^*)}\le C.$$

Existence of approximate solution (u_τ^k, z_τ^k): minimization of the functional

$$\int_\Omega \frac{\tau^2\varrho}{2}\Big|\frac{u - 2u_\tau^{k-1} + u_\tau^{k-2}}{\tau^2}\Big|^2 + \tau\zeta_1\Big(\frac{z - z_\tau^{k-1}}{\tau}\Big) + \tau\zeta_2\Big(e\big(\frac{u - u_\tau^{k-1}}{\tau}\big)\Big) + \varphi(e(u), z) - f_\tau^k \cdot u \,\mathrm{d}x$$

for $(u, z) \in W^{1,2}(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{R}^m)$.

A-priori estimates for the piece-wise affine interpolant u_τ and the piece-wise constant interpolant $\bar{z}_\tau$:

1) test by $\frac{u_\tau^k - u_\tau^{k-1}}{\tau}$ and $\frac{z_\tau^k - z_\tau^{k-1}}{\tau}$:

$$\big\|u_\tau\big\|_{W^{1,\infty}(I;L^2(\Omega;\mathbb{R}^n))\cap W^{1,2}(I;W^{1,2}(\Omega;\mathbb{R}^n))} \le C,$$

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Convergence for $\tau \to 0$:

Banach's selection principle:

$u_\tau \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(I; L^2(\Omega; \mathbb{R}^n)) \cap W^{1,2}(I; W^{1,2}(\Omega; \mathbb{R}^n))$,

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$\frac{\mathrm{d}u_\tau}{\mathrm{d}t}(t) \overset{*}{\rightharpoonup} \frac{\mathrm{d}u}{\mathrm{d}t}(t)$ in $W^{1,2}(\Omega; \mathbb{R}^n)^*$ (and even in $L^2(\Omega; \mathbb{R}^n)$),

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$$\int_Q v + \zeta_1(v)\,\mathrm{d}x\mathrm{d}t \geq \int_Q \zeta_1\Big(\frac{\partial z_\tau}{\partial t}\Big) + 2\zeta_2\Big(e\Big(\frac{\partial u_\tau}{\partial t}\Big)\Big)\,\mathrm{d}x\mathrm{d}t$$
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Convergence in the alternative weak formulation:

$$\int_Q \varphi_z'(e(\bar{u}_\tau),\bar{z}_\tau)v+\zeta_1(v)\,\mathrm{d}x\mathrm{d}t$$

$$\geq \int_Q \zeta_1\Big(\frac{\partial z_\tau}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t+\int_\Omega \varphi(e(u_\tau(T)),z_\tau(T))\,\mathrm{d}x$$

$$-\int_\Omega \varphi(e(u_0),z_0)\,\mathrm{d}x-\int_Q \varphi_e'(e(\bar{u}_\tau),\bar{z}_\tau):e(\frac{\partial u_\tau}{\partial t})\mathrm{d}x\mathrm{d}t$$

needs additionally $e(\frac{\partial u_\tau}{\partial t})\to e(\frac{\partial u}{\partial t})$ strongly in $L^2(Q;\mathbb{R}^{n\times n})$

...to be proved later.

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Example: Viscoplasticity with hardening (for $\zeta_2 = 0$ see H.-D.Alber).

The internal variables $z = (\pi, \eta) \in \mathbb{R}^{n\times n} \times \mathbb{R}$:

- π the plastic strain,
- η the hardening parameter.

The stored energy:

$$\varphi(e, z) = \varphi(e, \pi, \eta) = \frac{1}{2}\mathbb{C}(e - \pi) : (e - \pi) + b\eta^2$$

The homogeneous degree-1 dissipation potential is

$$\zeta_1(\dot\pi, \dot\eta) = \delta_P^*(\dot\pi) + \delta_K(\dot\pi, \dot\eta)$$

where

- $\mathbb{C}$ is the positive-definite elasticity tensor,
- $b > 0$ is a hardening parameter,
- $P \subset \mathbb{R}^{3\times 3}$ a convex closed neighbourhood of the origin,
- δ_P is its indicator function, and δ_P^* its conjugate functional,
- $K := \{z = (\pi, \eta);\ \eta \geq \delta_P^*(\pi)\ \text{ a.e. on } \Omega\}$.

So far, degree-1 homogeneity of ζ_1 was not exploited.

An alternative way to treat the inclusion $-\sigma_{\rm in} \in \partial\zeta_1(\frac{\partial z}{\partial t})$:

$$-\varphi_z'(e(u),z) =: \sigma_{\rm in} \in \partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big) \overset{\text{homogeneity degree-1}}{\subset} \partial\zeta_1(0) = K.$$

Definition of the subdifferential $\partial\zeta_1(0)$, this is just:

$$\zeta_1(0) \le \zeta_1(\ v\ ) + \varphi_z'(e(u),z)\ v \qquad \forall v \in \mathbb{R}^m.$$

Convexity of $\varphi(e,\cdot)$ $\Rightarrow$

$$\varphi(e,z) \le \varphi(e,v) - \varphi_z'(e,z)(v-z).$$

In sum (we have cancelling):

$$\varphi(e,z) \le \varphi(e,v) + \zeta_1(v-z)$$

We use it for $e = e(u)$, integrate over Ω, and call z (partially) stable at t iff:

$$\int_\Omega \varphi(e(u(t)),z(t))\mathrm{d}x \le \int_\Omega \varphi(e(u(t)),v)+\zeta_1(v-z(t))\mathrm{d}x \quad \forall v\in L^2(\Omega;\mathbb{R}^m).$$

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We call (u,z) with $u \in W^{1,2}(I;W^{1,2}(\Omega;\mathbb{R}^n))$
and $z \in \mathrm{BV}(\bar{I};L^1(\Omega;\mathbb{R}^m)) \cap \mathrm{BM}(\bar{I};L^2(\Omega;\mathbb{R}^m))$ (bounded measurable)
an weak/energetic solution iff

- the mechanical equilibrium

$$\varrho\frac{\partial^2 u}{\partial t^2} - \mathrm{div}\Big(\zeta_2'\big(e(\frac{\partial u}{\partial t})\big) + \varphi_e'(e(u),z)\Big) = f$$

  holds in a weak sense,
- the energy inequality holds (involving Helmholtz' stored energy):

$$\int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}(T)\Big|^2 + \varphi(e(u(T)),z(T))\,\mathrm{d}x$$
$$+ \mathrm{Var}_K(z;0,T) + 2\int_Q \zeta_2\Big(e(\frac{\partial u}{\partial t})\Big)\mathrm{d}x\mathrm{d}t$$
$$\le \int_\Omega \frac{\varrho}{2}|\dot{u}_0|^2 + \varphi(e(u_0),z_0)\,\mathrm{d}x + \int_Q f\cdot\frac{\partial u}{\partial t}\,\mathrm{d}x\mathrm{d}t,$$

- $z(t)$ is (partially) stable for all $t \in [0,T]$,
- the initial conditions are satisfied.

Important: any weak/energetic solution is also a weak solution:

- by the definition of the directional derivative $\mathrm{D}\Phi(e(u(t)), z(t), v)$ of $\Phi(e(u(t)), \cdot) : L^2(\Omega; \mathbb{R}^m) \to \mathbb{R}$ at $z(t)$ in the direction v with $\Phi(e, z) := \int_\Omega \varphi(e, z)\,\mathrm{d}x$,
- by the stability of z at time t with respect to $z(t) + \varepsilon v$ and
- by degree-1 homogeneity of ζ_1,

we obtain

$$\int_\Omega \varphi_z'(e(u(t)), z(t))v\,\mathrm{d}x = \mathrm{D}\Phi(e(u(t)), z(t), v)$$

$$:= \lim_{\varepsilon\downarrow 0} \frac{\Phi(e(u(t)), z(t)) - \Phi(e(u(t)), z(t){+}\varepsilon v)}{\varepsilon}$$

$$= \lim_{\varepsilon\downarrow 0} \int_\Omega \frac{\varphi(e(u(t)), z(t){+}\varepsilon v) - \varphi(e(u(t)), z(t))}{\varepsilon}\mathrm{d}x$$

$$\geq -\lim_{\varepsilon\downarrow 0} \int_\Omega \frac{\zeta_1(z(t){+}\varepsilon v{-}z(t))}{\varepsilon}\mathrm{d}x = -\int_\Omega \zeta_1(v)\,\mathrm{d}x.$$

Integrating it over $[0, T]$ and summing with energy inequality, we get just the inequality used in the definition of the weak solution.

Conversely, if $\varphi(e,\cdot)$ is convex and if $\frac{\partial z}{\partial t} \in L^1(Q;\mathbb{R}^m)$, then any weak solution is also an energetic solution.

We have $\displaystyle\int_Q \zeta_1(\frac{\partial z}{\partial t}) + \varphi_z'(e(u),z)\frac{\partial z}{\partial t}\mathrm{d}x\mathrm{d}t = 0$;
here we use that the weak solution now satisfies $\varphi_z'(e(u),z) \in -\partial\zeta_1(\frac{\partial z}{\partial t})$ a.e. on Q and, by the definition of the subdifferential of ζ_1 used at $\frac{\partial z}{\partial t}$, we have $\int_Q \zeta_1(\frac{\partial z}{\partial t}) - \xi\frac{\partial z}{\partial t}\mathrm{d}x\mathrm{d}t \le 0$ for any $\xi \in \partial\zeta_1(\frac{\partial z}{\partial t})$, in particular for $\xi = -\varphi_z'(e(u),z)$, while the opposite inequality follows from the degree-1 homogeneity of ζ_1 and definition of its subdifferential used at 0

$$\xi\frac{\partial z}{\partial t} = \zeta_1(0) + \xi\Big(\frac{\partial z}{\partial t} - 0\Big) \le \zeta_1\Big(\frac{\partial z}{\partial t}\Big)$$

for any $\xi \in \partial\zeta_1(0) = K$ but $K \supset \partial\zeta_1(\frac{\partial z}{\partial t})$ because of the degree-1 homogeneity of ζ_1 hence again we can take $\xi = -\varphi_z'(e(u),z)$.

Existence of the energetic solution (by approximation by Rothe's method):

Direct merging of the above assertions is not optimal: it would rely on $\varphi(e,\cdot)$ convex and $\frac{\partial z}{\partial t}$ absolutely continuous.

Yet, we can make a direct proof:

Step 1: A-priori estimates for approximate solutions as before,

Step 2: selection of subsequences by Banach's and Helly's principles, in particular:

$$e(\bar{u}_\tau(t)) \to e(u(t)) \qquad \text{weakly in } L^2(\Omega;\mathbb{R}^{n\times n}),$$
$$\bar{z}_\tau(t) \to z(t) \qquad \text{weakly in } L^2(\Omega;\mathbb{R}^m)$$

for any $t \in [0,T]$.

Note that, because of viscosity, we have $e(\bar{u}_\tau) \in \mathrm{BV}(I; L^2(\Omega;\mathbb{R}^{n\times n}))$.

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Direct merging of the above assertions is not optimal: it would rely on $\varphi(e,\cdot)$ convex and $\frac{\partial z}{\partial t}$ absolutely continuous.

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for any $t \in [0, T]$.

Note that, because of viscosity, we have $e(\bar{u}_\tau) \in \mathrm{BV}(I; L^2(\Omega;\mathbb{R}^{n\times n}))$.

Step 3: stability: (minimization property & degree-1 homogeneity):

$$\int_\Omega \varphi(e(\bar{u}_\tau(t)), \bar{z}_\tau(t))\,\mathrm{d}x \le \int_\Omega \varphi(e(\bar{u}_\tau(t)), v) + \zeta_1(v - \bar{z}_\tau(t))\,\mathrm{d}x \qquad (*)$$

for all $v \in L^2(\Omega;\mathbb{R}^m)$ and $t \in [0,T]$.

A limit passage:

for any $\tilde{v} \in L^2(\Omega;\mathbb{R}^m)$, use $v_\tau = \tilde{v} - z(t) + \bar{z}_\tau(t)$ as a test function,

as φ is assumed quadratic, use the binomial formula "$a^2 - b^2 = (a-b)(a+b)$"

$$\begin{aligned}\varphi\big(e(\bar{u}_\tau(t)), \bar{z}_\tau(t)\big) - \varphi\big(e(\bar{u}_\tau(t)), v_\tau\big) &= \frac{1}{2}\varphi'\big(0, \bar{z}_\tau(t) - v_\tau\big)\big(2e(\bar{u}_\tau(t)), \bar{z}_\tau(t) + v_\tau\big)\\ &= \frac{1}{2}\varphi'\big(0, z(t) - \tilde{v}\big)\big(2e(\bar{u}_\tau(t)), \bar{z}_\tau(t) + v_\tau\big)\\ &\to \frac{1}{2}\varphi'\big(0, z(t) - \tilde{v}\big)\big(2e(u(t)), z(t) + v\big)\\ &= \varphi\big(e(u(t)), z(t)\big) - \varphi\big(e(u(t)), \tilde{v}\big)\end{aligned}$$

weakly in $L^1(\Omega)$. Moreover, we have simply $\zeta_1(v_\tau - \bar{z}_\tau(t)) = \zeta_1(\tilde{v} - z(t))$,

so that the limit passage in (*) is proved.

Step 4: a limit passage in the energy inequality as before.

Step 3: stability: (minimization property & degree-1 homogeneity):

$$\int_\Omega \varphi(e(\bar{u}_\tau(t)), \bar{z}_\tau(t))\,\mathrm{d}x \le \int_\Omega \varphi(e(\bar{u}_\tau(t)), v) + \zeta_1(v - \bar{z}_\tau(t))\,\mathrm{d}x \qquad (*)$$

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A limit passage:
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weakly in $L^1(\Omega)$. Moreover, we have simply $\zeta_1(v_\tau - \bar{z}_\tau(t)) = \zeta_1(\tilde{v} - z(t))$,
so that the limit passage in (*) is proved.

Step 4: a limit passage in the energy inequality as before.

In addition: energy equality (so-called *Step 5*):

consider $\varepsilon > 0$, and a partition $0 = t_0^\varepsilon < t_1^\varepsilon < \dots < t_{k_\varepsilon}^\varepsilon \le T$ with $\max_{i=1,\dots,k_\varepsilon}(t_i^\varepsilon - t_{i-1}^\varepsilon) \le \varepsilon$ and also $T - t_{k_\varepsilon}^\varepsilon \le \varepsilon$.

stability of z at time t_{i-1}^ε gives, when tested by $v := z(t_i^\varepsilon)$, the estimate

$$\int_\Omega \varphi(e(u(t_{i-1}^\varepsilon)), z(t_{i-1}^\varepsilon))\,\mathrm{d}x \le \int_\Omega \varphi(e(u(t_{i-1}^\varepsilon)), z(t_i^\varepsilon)) + \zeta_1(z(t_i^\varepsilon) - z(t_{i-1}^\varepsilon))\,\mathrm{d}x$$
$$= \int_\Omega \bigg(\varphi(e(u(t_i^\varepsilon)), z(t_i^\varepsilon)) + \zeta_1(z(t_i^\varepsilon) - z(t_{i-1}^\varepsilon)) - \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \varphi'_e(e(u(s)), z(t_i^\varepsilon)) : e\Big(\frac{\partial u}{\partial t}(s)\Big)\,\mathrm{d}s \bigg)\mathrm{d}x.$$

Also we use stability of z at time $t_{k_\varepsilon}^\varepsilon$ tested by $v := z(T)$.

Summing it for $i = 1, ..., k_\varepsilon$ and assuming that $\{t_i^\varepsilon\}_{i=1}^{k_\varepsilon}$ are chosen so that $\frac{\partial}{\partial t}u(t_i^\varepsilon) \in W^{1,2}(\Omega;\mathbb{R}^n)$ are well defined, we obtain

$$
\begin{aligned}
&\int_\Omega \varphi(e(u(T)),z(T)) - \int_\Omega \varphi(e(u_0)z_0))\,\mathrm{d}x + \mathrm{Var}_K(z;0,T) \\
&\geq \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \varphi_e'(e(u(s)),z(t_i^\varepsilon)) : e\Big(\frac{\partial u}{\partial t}(s)\Big)\,\mathrm{d}x\mathrm{d}s - \delta \\
&= \sum_{i=1}^{k_\varepsilon} (t_i^\varepsilon - t_{i-1}^\varepsilon) \int_\Omega \varphi_e'(e(u(t_i^\varepsilon)),z(t_i^\varepsilon)) : e\Big(\frac{\partial u}{\partial t}(t_i^\varepsilon)\Big)\,\mathrm{d}x \\
&+ \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \Big(\varphi_e'(e(u(s)),z(t_i^\varepsilon)) - \varphi_e'(e(u(t_i^\varepsilon)),z(t_i^\varepsilon))\Big) : e\Big(\frac{\partial u}{\partial t}(s)\Big)\,\mathrm{d}x\mathrm{d}s \\
&+ \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \varphi_e'(e(u(t_i^\varepsilon)),z(t_i^\varepsilon)) : e\Big(\frac{\partial u}{\partial t}(s) - \frac{\partial u}{\partial t}(t_i^\varepsilon)\Big)\,\mathrm{d}x\mathrm{d}s - \delta.
\end{aligned}
$$

Summing it for $i = 1, ..., k_\varepsilon$ and assuming that $\{t_i^\varepsilon\}_{i=1}^{k_\varepsilon}$ are chosen so that $\frac{\partial}{\partial t}u(t_i^\varepsilon) \in W^{1,2}(\Omega;\mathbb{R}^n)$ are well defined, we obtain

$$\int_\Omega \varphi(e(u(T)),z(T)) - \int_\Omega \varphi(e(u_0)z_0))\,\mathrm{d}x + \mathrm{Var}_K(z;0,T)$$

$$\geq \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \varphi_e'(e(u(s)),z(t_i^\varepsilon)) : e\Big(\frac{\partial u}{\partial t}(s)\Big)\,\mathrm{d}x\mathrm{d}s - \delta$$

$$= \sum_{i=1}^{k_\varepsilon} (t_i^\varepsilon - t_{i-1}^\varepsilon) \int_\Omega \varphi_e'(e(u(t_i^\varepsilon)),z(t_i^\varepsilon)) : e\Big(\frac{\partial u}{\partial t}(t_i^\varepsilon)\Big)\,\mathrm{d}x$$

$$+ \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \Big(\varphi_e'(e(u(s)),z(t_i^\varepsilon)) - \varphi_e'(e(u(t_i^\varepsilon)),z(t_i^\varepsilon))\Big) : e\Big(\frac{\partial u}{\partial t}(s)\Big)\,\mathrm{d}x\mathrm{d}s$$

$$+ \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \varphi_e'(e(u(t_i^\varepsilon)),z(t_i^\varepsilon)) : e\Big(\frac{\partial u}{\partial t}(s) - \frac{\partial u}{\partial t}(t_i^\varepsilon)\Big)\,\mathrm{d}x\mathrm{d}s - \delta.$$

Approximation of Lebesgue integral $\int_0^T \int_\Omega \varphi_e'(e(u(t)),z(t)) : e(\frac{\partial u}{\partial t})\,\mathrm{d}x\mathrm{d}t$ by Riemann sums (Hahn, 1914, Francfort, Mielke, Dal Maso at al.)

Summing it for $i = 1, ..., k_\varepsilon$ and assuming that $\{t_i^\varepsilon\}_{i=1}^{k_\varepsilon}$ are chosen so that $\frac{\partial}{\partial t}u(t_i^\varepsilon) \in W^{1,2}(\Omega;\mathbb{R}^n)$ are well defined, we obtain

$$\int_\Omega \varphi(e(u(T)), z(T)) - \int_\Omega \varphi(e(u_0)z_0))\,\mathrm{d}x + \mathrm{Var}_K(z;0,T)$$

$$\geq \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \varphi_e'(e(u(s)), z(t_i^\varepsilon)) : e\Big(\frac{\partial u}{\partial t}(s)\Big)\,\mathrm{d}x\mathrm{d}s - \delta$$

$$= \sum_{i=1}^{k_\varepsilon} (t_i^\varepsilon - t_{i-1}^\varepsilon) \int_\Omega \varphi_e'(e(u(t_i^\varepsilon)), z(t_i^\varepsilon)) : e\Big(\frac{\partial u}{\partial t}(t_i^\varepsilon)\Big)\,\mathrm{d}x$$

$$+ \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \Big(\varphi_e'(e(u(s)), z(t_i^\varepsilon)) - \varphi_e'(e(u(t_i^\varepsilon)), z(t_i^\varepsilon))\Big) : e\Big(\frac{\partial u}{\partial t}(s)\Big)\mathrm{d}x\mathrm{d}s$$

$$+ \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \varphi_e'(e(u(t_i^\varepsilon)), z(t_i^\varepsilon)) : e\Big(\frac{\partial u}{\partial t}(s) - \frac{\partial u}{\partial t}(t_i^\varepsilon)\Big)\,\mathrm{d}x\mathrm{d}s - \delta.$$

Convergence to 0 as $|S_2^\varepsilon| \leq |\varphi_e'|\,\big\|e(u)\big\|_{W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^{n\times n}))}$
$\max_{i=1,\dots,k_\varepsilon} \max_{s\in[t_{i-1}^\varepsilon, t_i^\varepsilon]} \Big\|e\big(u(s) - u(t_i^\varepsilon)\big)\Big\|_{L^2(\Omega;\mathbb{R}^{n\times n})}.$

Summing it for $i = 1, ..., k_\varepsilon$ and assuming that $\{t_i^\varepsilon\}_{i=1}^{k_\varepsilon}$ are chosen so that $\frac{\partial}{\partial t}u(t_i^\varepsilon) \in W^{1,2}(\Omega;\mathbb{R}^n)$ are well defined, we obtain

$$\int_\Omega \varphi(e(u(T)), z(T)) - \int_\Omega \varphi(e(u_0)z_0))\,\mathrm{d}x + \mathrm{Var}_K(z;0,T)$$

$$\geq \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \varphi_e'(e(u(s)), z(t_i^\varepsilon)) : e\Big(\frac{\partial u}{\partial t}(s)\Big)\,\mathrm{d}x\mathrm{d}s - \delta$$

$$= \sum_{i=1}^{k_\varepsilon} (t_i^\varepsilon - t_{i-1}^\varepsilon) \int_\Omega \varphi_e'(e(u(t_i^\varepsilon)), z(t_i^\varepsilon)) : e\Big(\frac{\partial u}{\partial t}(t_i^\varepsilon)\Big)\,\mathrm{d}x$$

$$+ \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \Big(\varphi_e'(e(u(s)), z(t_i^\varepsilon)) - \varphi_e'(e(u(t_i^\varepsilon)), z(t_i^\varepsilon))\Big) : e\Big(\frac{\partial u}{\partial t}(s)\Big)\mathrm{d}x\mathrm{d}s$$

$$+ \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \varphi_e'(e(u(t_i^\varepsilon)), z(t_i^\varepsilon)) : e\Big(\frac{\partial u}{\partial t}(s) - \frac{\partial u}{\partial t}(t_i^\varepsilon)\Big)\,\mathrm{d}x\mathrm{d}s - \delta.$$

Convergence to 0 as $|S_3^\varepsilon| \leq \big\|\varphi_e'(e(u), z)\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^{n\times n}))}$
$\sum_{i=1}^{k_\varepsilon} \big\|e\big(u(t_i^\varepsilon) - u(t_{i-1}^\varepsilon) - (t_i^\varepsilon - t_{i-1}^\varepsilon)\frac{\partial u}{\partial t}(t_i^\varepsilon)\big)\big\|_{L^2(\Omega;\mathbb{R}^{n\times n})}$.

Summing it for $i = 1, ..., k_\varepsilon$ and assuming that $\{t_i^\varepsilon\}_{i=1}^{k_\varepsilon}$ are chosen so that $\frac{\partial}{\partial t}u(t_i^\varepsilon) \in W^{1,2}(\Omega;\mathbb{R}^n)$ are well defined, we obtain

$$\begin{aligned}
&\int_\Omega \varphi(e(u(T)), z(T)) - \int_\Omega \varphi(e(u_0)z_0))\,\mathrm{d}x + \mathrm{Var}_K(z;0,T) \\
&\geq \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \varphi'_e(e(u(s)), z(t_i^\varepsilon)) : e\Big(\frac{\partial u}{\partial t}(s)\Big)\,\mathrm{d}x\mathrm{d}s - \delta \\
&= \sum_{i=1}^{k_\varepsilon} (t_i^\varepsilon - t_{i-1}^\varepsilon) \int_\Omega \varphi'_e(e(u(t_i^\varepsilon)), z(t_i^\varepsilon)) : e\Big(\frac{\partial u}{\partial t}(t_i^\varepsilon)\Big)\,\mathrm{d}x \\
&+ \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \Big(\varphi'_e(e(u(s)), z(t_i^\varepsilon)) - \varphi'_e(e(u(t_i^\varepsilon)), z(t_i^\varepsilon))\Big) : e\Big(\frac{\partial u}{\partial t}(s)\Big)\mathrm{d}x\mathrm{d}s \\
&+ \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \varphi'_e(e(u(t_i^\varepsilon)), z(t_i^\varepsilon)) : e\Big(\frac{\partial u}{\partial t}(s) - \frac{\partial u}{\partial t}(t_i^\varepsilon)\Big)\,\mathrm{d}x\mathrm{d}s - \delta.
\end{aligned}$$

$$\delta := \Big| \int_{t_{k_\varepsilon}^\varepsilon}^T \int_\Omega \varphi'_e(e(u(s)), z(T)) : e\Big(\frac{\partial u}{\partial t}(s)\Big)\,\mathrm{d}x\mathrm{d}s \Big| \to 0 \quad \text{as } t_{k_\varepsilon}^\varepsilon \uparrow T \text{ for } \varepsilon \to 0.$$

Advantages of the energetic approach:

- possibility to obtain balance of energy as an equality,
- φ'_z has been eliminated from the definition of the energetic solution, which allows for a further generalization for cases where $\varphi(e,\cdot)$ is nonsmooth and nonconvex, or even the set of z does not have a convex structure (gradient theories).
- Also Var_K can be generalized to more general distances that no longer relies on any linear structure on the set of z (quasi-variational inequalities of the type $\partial\zeta_1(z, \frac{\partial z}{\partial t}) + \sigma_{\rm in} \ni 0$).
- In the quasistatic case ($\varrho = 0$, $\zeta_2 = 0$), also φ'_e can be eliminated, and no linear structure for u is needed any more:

Recall that (u,z) with $u \in W^{1,2}(I;W^{1,2}(\Omega;\mathbb{R}^n))$
and $z \in \mathrm{BV}(\bar{I};L^1(\Omega;\mathbb{R}^m)) \cap \mathrm{BM}(\bar{I};L^2(\Omega;\mathbb{R}^m))$ (bounded measurable)
an weak/energetic solution iff

- the mechanical equilibrium

$$\varrho\frac{\partial^2 u}{\partial t^2} - \mathrm{div}\Big(\zeta_2'\big(e(\frac{\partial u}{\partial t})\big) + \varphi_e'(e(u),z)\Big) = f$$

  holds in a weak sense,
- the energy inequality holds (involving Helmholtz' stored energy):

$$\int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}(T)\Big|^2 + \varphi(e(u(T)),z(T))\,\mathrm{d}x$$
$$+\,\mathrm{Var}_K(z;0,T) + 2\int_Q \zeta_2\Big(e(\frac{\partial u}{\partial t})\Big)\mathrm{d}x\mathrm{d}t$$
$$\le \int_\Omega \frac{\varrho}{2}|\dot{u}_0|^2 + \varphi(e(u_0),z_0)\,\mathrm{d}x \qquad + \int_Q f\cdot\frac{\partial u}{\partial t}\,\mathrm{d}x\mathrm{d}t,$$

- $z(t)$ is (partially) stable for all $t \in [0,T]$,
- the initial conditions are satisfied.

It needs $f \in L^1(I;L^2(\Omega;\mathbb{R}^n))$.

We call alternatively (u,z) with $u\in W^{1,2}(I;W^{1,2}(\Omega;\mathbb{R}^n))$
and $z\in \mathrm{BV}(\bar{I};L^1(\Omega;\mathbb{R}^m))\cap \mathrm{BM}(\bar{I};L^2(\Omega;\mathbb{R}^m))$ (bounded measurable)
an weak/energetic solution iff

- the mechanical equilibrium

$$\varrho\frac{\partial^2 u}{\partial t^2}-\mathrm{div}\Big(\zeta_2'\big(e(\frac{\partial u}{\partial t})\big)+\varphi_e'(e(u),z)\Big)=f$$

  holds in a weak sense,
- the energy inequality holds (involving Gibbs' stored energy):

$$\int_\Omega\frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}(T)\Big|^2+\varphi(e(u(T)),z(T))-f(T)\cdot u(T)\,\mathrm{d}x$$
$$+\mathrm{Var}_K(z;0,T)+2\int_Q\zeta_2\Big(e(\frac{\partial u}{\partial t})\Big)\mathrm{d}x\mathrm{d}t$$
$$\le\int_\Omega\frac{\varrho}{2}|\dot{u}_0|^2+\varphi(e(u_0),z_0)-f(0)\cdot u_0\,\mathrm{d}x-\int_Q\frac{\partial f}{\partial t}\cdot u\,\mathrm{d}x\mathrm{d}t,$$

- $z(t)$ is (partially) stable for all $t\in[0,T]$,
- the initial conditions are satisfied.

It needs $f\in W^{1,1}(I;L^2(\Omega;\mathbb{R}^n))$.

The quasistatic case ($\varrho = 0$, $\zeta_2 = 0$): we call $u \in L^\infty(I; W^{1,2}(\Omega; \mathbb{R}^n))$ and $z \in \mathrm{BV}(\bar{I}; L^1(\Omega; \mathbb{R}^m)) \cap \mathrm{BM}(\bar{I}; L^2(\Omega; \mathbb{R}^m))$ (bounded measurable) an weak/energetic solution iff

- the mechanical equilibrium

$$\varrho \frac{\partial^2 u}{\partial t^2} - \mathrm{div}\Big(\zeta_2'\big(e(\frac{\partial u}{\partial t})\big) + \varphi_e'(e(u), z)\Big) = f$$

  holds in a weak sense,

- the energy inequality holds (involving Gibbs' stored energy):

$$\int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}(T)\Big|^2 + \varphi(e(u(T)), z(T)) - f(T) \cdot u(T)\,\mathrm{d}x$$

$$+ \mathrm{Var}_K(z; 0, T) + 2\int_Q \zeta_2\Big(e(\frac{\partial u}{\partial t})\Big)\mathrm{d}x\mathrm{d}t$$

$$\leq \int_\Omega \frac{\varrho}{2}|\dot{u}_0|^2 + \varphi(e(u_0), z_0) - f(0) \cdot u_0\,\mathrm{d}x - \int_Q \frac{\partial f}{\partial t} \cdot u\,\mathrm{d}x\mathrm{d}t,$$

- $z(t)$ is (partially) stable for all $t \in [0, T]$,
- the initial conditions are satisfied.

It needs $f \in W^{1,1}(I; L^2(\Omega; \mathbb{R}^n))$.

The quasistatic case ($\varrho = 0$, $\zeta_2 = 0$): we call $u \in L^\infty(I; W^{1,2}(\Omega; \mathbb{R}^n))$ and $z \in \mathrm{BV}(\bar{I}; L^1(\Omega; \mathbb{R}^m)) \cap \mathrm{BM}(\bar{I}; L^2(\Omega; \mathbb{R}^m))$ (bounded measurable) an weak/energetic solution iff

- the mechanical equilibrium (minimal-energy principle)

$$\int_\Omega \varphi(e(u(t)), z(t)) - f(t) \cdot u(t)\,\mathrm{d}x \le \int_\Omega \varphi(e(w), z(t)) - f(t) \cdot w\,\mathrm{d}x$$

holds for all $w \in W^{1,2}(\Omega; \mathbb{R}^n)$,

- the energy inequality holds (involving Gibbs' stored energy):

$$\int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}(T)\Big|^2 + \varphi(e(u(T)), z(T)) - f(T) \cdot u(T)\,\mathrm{d}x$$

$$+ \mathrm{Var}_K(z; 0, T) + 2\int_Q \zeta_2\Big(e\Big(\frac{\partial u}{\partial t}\Big)\Big)\mathrm{d}x\mathrm{d}t$$

$$\le \int_\Omega \frac{\varrho}{2}|\dot{u}_0|^2 + \varphi(e(u_0), z_0) - f(0) \cdot u_0\,\mathrm{d}x - \int_Q \frac{\partial f}{\partial t} \cdot u\,\mathrm{d}x\mathrm{d}t,$$

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The quasistatic case ($\varrho = 0$, $\zeta_2 = 0$): we call $u \in L^\infty(I; W^{1,2}(\Omega;\mathbb{R}^n))$ and $z \in \mathrm{BV}(\bar{I}; L^1(\Omega;\mathbb{R}^m)) \cap \mathrm{BM}(\bar{I}; L^2(\Omega;\mathbb{R}^m))$ (bounded measurable) an energetic solution iff

- the mechanical equilibrium (full stability)

$$\int_\Omega \varphi(e(u(t)), z(t)) - f(t)\cdot u(t)\,\mathrm{d}x \le \int_\Omega \varphi(e(w), v) - f(t)\cdot w + \zeta_1(v - z(t))\,\mathrm{d}x$$

holds for all $w \in W^{1,2}(\Omega;\mathbb{R}^n)$ and $z \in L^2(\Omega;\mathbb{R}^m)$,

- the energy equality holds (involving stored energy):

$$\int_\Omega \varphi(e(u(T)), z(T)) - f(T)\cdot u(T)\,\mathrm{d}x + \mathrm{Var}_K(z; 0, T) = \int_\Omega \varphi(e(u_0), z_0) - f(0)\cdot u_0\,\mathrm{d}x - \int_Q \frac{\partial f}{\partial t}\cdot u\,\mathrm{d}x\mathrm{d}t,$$

- the initial conditions are satisfied.

It needs $f \in W^{1,1}(I; L^2(\Omega;\mathbb{R}^n))$. (Mielke, Theil, 2001).

energetic solution $\Rightarrow$ weak/energetic solution (simple)

1) strict convexity of $\varphi(\cdot, z)$ (and B.C., say homogenoeus Dirichlet):
$-\mathrm{div}\big(\varphi'_e(e(u), z)\big) = f(t)$ has the only solution $u = u(t, z)$.
Let us define the reduced stored energy $\hat\varphi(t, z) = \varphi(e(u(t, z))), z) - f(t)$.
It holds

$$\int_\Omega \hat\varphi(t, z)\mathrm{d}x = \min_{w\in W_0^{1,2}(\Omega)} \int_\Omega \varphi(w, z) - f(t)\cdot w\,\mathrm{d}x.$$

2) $\sigma_{\rm in} := \varphi'_z(e(u(t, z)), z)$ equals also $\hat\varphi'_z(t, z)$ ("complience" problem).

3) the reduced evolution $\partial\zeta_1(\frac{\partial z}{\partial t}) + \hat\varphi'_z(t, z) \ni 0$ leads to the stability condition $\forall v \in L^2(\Omega; \mathbb{R}^m)$:

energetic solution $\Leftarrow$ weak/energetic solution (more difficult)

1) strict convexity of $\varphi(\cdot, z)$ (and B.C., say homogenoeus Dirichlet): $-\mathrm{div}\big(\varphi_e'(e(u), z)\big) = f(t)$ has the only solution $u = u(t, z)$.
Let us define the reduced stored energy $\hat{\varphi}(t, z) = \varphi(e(u(t, z))), z) - f(t)$.
It holds

$$\int_\Omega \hat{\varphi}(t, z)\mathrm{d}x = \min_{w \in W_0^{1,2}(\Omega)} \int_\Omega \varphi(w, z) - f(t) \cdot w \, \mathrm{d}x.$$

2) $\sigma_{\rm in} := \varphi_z'(e(u(t, z)), z)$ equals also $\hat{\varphi}_z'(t, z)$ ("complience" problem).

3) the reduced evolution $\partial\zeta_1(\frac{\partial z}{\partial t}) + \hat{\varphi}_z'(t, z) \ni 0$ leads to the stability condition $\forall v \in L^2(\Omega; \mathbb{R}^m)$:

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1) strict convexity of $\varphi(\cdot, z)$ (and B.C., say homogenoeus Dirichlet):
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Let us define the reduced stored energy $\hat\varphi(t, z) = \varphi(e(u(t, z))), z) - f(t)$.
It holds

$$\int_\Omega \hat\varphi(t, z)\mathrm{d}x = \min_{w\in W^{1,2}_0(\Omega)} \int_\Omega \varphi(w, z) - f(t)\cdot w\,\mathrm{d}x.$$

2) $\sigma_{\rm in} := \varphi'_z(e(u(t, z)), z)$ equals also $\hat\varphi'_z(t, z)$ ("complience" problem).

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energetic solution $\Leftarrow$ weak/energetic solution (more difficult)

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Let us define the reduced stored energy $\hat\varphi(t,z)=\varphi(e(u(t,z))),z)-f(t)$.
It holds

$$\int_\Omega \hat\varphi(t,z)\mathrm{d}x = \min_{w\in W_0^{1,2}(\Omega)} \int_\Omega \varphi(w,z)-f(t)\cdot w\,\mathrm{d}x.$$

2) $\sigma_{\rm in}:=\varphi'_z(e(u(t,z)),z)$ equals also $\hat\varphi'_z(t,z)$ ("complience" problem).

3) the reduced evolution $\partial\zeta_1(\frac{\partial z}{\partial t})+\hat\varphi'_z(t,z)\ni 0$ leads to the stability condition $\forall v\in L^2(\Omega;\mathbb{R}^m)$:

$$\int_\Omega \hat\varphi(t,z(t))\,\mathrm{d}x \le \int_\Omega \hat\varphi(t,v)+\zeta_1(v-z(t))\,\mathrm{d}x.$$

energetic solution $\Leftarrow$ weak/energetic solution (more difficult)

1) strict convexity of $\varphi(\cdot,z)$ (and B.C., say homogenoeus Dirichlet): $-\mathrm{div}\big(\varphi'_e(e(u),z)\big)=f(t)$ has the only solution $u=u(t,z)$.
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3) the reduced evolution $\partial\zeta_1(\frac{\partial z}{\partial t})+\hat\varphi'_z(t,z)\ni0$ leads to the stability condition $\forall v\in L^2(\Omega;\mathbb{R}^m)$:

$$\int_\Omega\varphi(e(u(t),z(t))-f(t)\cdot u(t)\,\mathrm{d}x\le\min_{w\in W^{1,2}(\Omega)}\int_\Omega\varphi(w,v)-f(t)\cdot w\,\mathrm{d}x+\int_\Omega\zeta_1(v-z(t))\,\mathrm{d}x.$$

energetic solution $\Leftarrow$ weak/energetic solution (more difficult)

1) strict convexity of $\varphi(\cdot,z)$ (and B.C., say homogenoeus Dirichlet): $-\mathrm{div}\big(\varphi'_e(e(u),z)\big)=f(t)$ has the only solution $u=u(t,z)$.
Let us define the reduced stored energy $\hat\varphi(t,z)=\varphi(e(u(t,z))),z)-f(t)$.
It holds

$$\int_\Omega \hat\varphi(t,z)\mathrm{d}x=\min_{w\in W_0^{1,2}(\Omega)}\int_\Omega \varphi(w,z)-f(t)\cdot w\,\mathrm{d}x.$$

2) $\sigma_{\rm in}:=\varphi'_z(e(u(t,z)),z)$ equals also $\hat\varphi'_z(t,z)$ ("complience" problem).

3) the reduced evolution $\partial\zeta_1(\frac{\partial z}{\partial t})+\hat\varphi'_z(t,z)\ni 0$ leads to the stability condition $\forall v\in L^2(\Omega;\mathbb{R}^m)$:

$$\int_\Omega \varphi(e(u(t),z(t))-f(t)\cdot u(t)\,\mathrm{d}x\le\int_\Omega \varphi(w,v)-f(t)\cdot w+\zeta_1(v-z(t))\,\mathrm{d}x.$$

$\forall w\in W_0^{1,2}(\Omega)$, i.e. full stability of $(u(t),z(t))$.

Convergence for slow-loading regimes to fully rate-independent evolution:
scalling time as εt:
scalling ϱ as $\varepsilon^2\varrho$ and ζ_2 as $\varepsilon\zeta_2$ (while ζ_1 unchanged).
a-priori estimates for the weak/energetic solution $(u_\varepsilon, z_\varepsilon)$:

$$\Big\|\frac{\partial u_\varepsilon}{\partial t}\Big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^n))} \le \frac{C}{\varepsilon},$$

$$\Big\|e\Big(\frac{\partial u_\varepsilon}{\partial t}\Big)\Big\|_{L^2(Q;\mathbb{R}^{n\times n})} \le \frac{C}{\sqrt{\varepsilon}},$$

$$\big\|e(u_\varepsilon)\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^{n\times n}))} \le C,$$

$$\big\|z_\varepsilon\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^m))\cap \mathrm{BV}(\bar I;L^1(\Omega;\mathbb{R}^m))} \le C.$$

Convergence for $\varepsilon \to 0$ in the (weakly formulated) force equilibrium:

$$\int_Q \underbrace{\varepsilon\zeta_2'\Big(e\Big(\frac{\partial u_\varepsilon}{\partial t}\Big)\Big) : e(v)}_{=\mathcal{O}(\sqrt{\varepsilon})} - \underbrace{\varepsilon^2\varrho\frac{\partial u_\varepsilon}{\partial t}\cdot\frac{\partial v}{\partial t}}_{=\mathcal{O}(\varepsilon)} + \varphi_e'(e(u_\varepsilon), z_\varepsilon) : e(v) - f\cdot v\,\mathrm{d}x\mathrm{d}t = 0$$

Convergence for slow-loading regimes to fully rate-independent evolution:
scalling time as εt:
scalling ϱ as $\varepsilon^2\varrho$ and ζ_2 as $\varepsilon\zeta_2$ (while ζ_1 unchanged).
a-priori estimates for the weak/energetic solution $(u_\varepsilon, z_\varepsilon)$:

$$\Big\|\frac{\partial u_\varepsilon}{\partial t}\Big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^n))} \le \frac{C}{\varepsilon},$$

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Convergence for slow-loading regimes to fully rate-independent evolution:
scalling time as εt:
scalling ϱ as $\varepsilon^2\varrho$ and ζ_2 as $\varepsilon\zeta_2$ (while ζ_1 unchanged).
a-priori estimates for the weak/energetic solution $(u_\varepsilon, z_\varepsilon)$:

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$$\big\|z_\varepsilon\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^m))\cap \mathrm{BV}(\bar{I};L^1(\Omega;\mathbb{R}^m))} \le C.$$

Convergence for $\varepsilon \to 0$ in the partial stability:

$$\int_\Omega \varphi(e(u_\varepsilon(t)), z_\varepsilon(t))\mathrm{d}x \le \int_\Omega \varphi(e(u_\varepsilon(t)), v) + \zeta_1(v - z_\varepsilon(t))\mathrm{d}x$$

by "binomial trick" relying on φ quadratic.

Convergence for slow-loading regimes to fully rate-independent evolution:
scalling time as εt:
scalling ϱ as $\varepsilon^2\varrho$ and ζ_2 as $\varepsilon\zeta_2$ (while ζ_1 unchanged).
a-priori estimates for the weak/energetic solution $(u_\varepsilon, z_\varepsilon)$:

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$$\Big\|e\Big(\frac{\partial u_\varepsilon}{\partial t}\Big)\Big\|_{L^2(Q;\mathbb{R}^{n\times n})} \le \frac{C}{\sqrt{\varepsilon}},$$
$$\big\|e(u_\varepsilon)\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^{n\times n}))} \le C,$$
$$\big\|z_\varepsilon\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^m))\cap\mathrm{BV}(\bar I;L^1(\Omega;\mathbb{R}^m))} \le C.$$

Convergence for $\varepsilon \to 0$ in the Gibbs-type energy inequality:

$$\int_\Omega \varepsilon^2\varrho\Big|\frac{\partial u_\varepsilon}{\partial t}(T)\Big|^2 + \varphi(u_\varepsilon(T), z_\varepsilon(T)) - f(T)\cdot u_\varepsilon(T)\,\mathrm{d}x + \int_Q \varepsilon\zeta_2\Big(\frac{\partial u_\varepsilon}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t$$
$$+\mathrm{Var}_K(z_\varepsilon;0,T) = \int_\Omega \varepsilon^2\varrho\big|\dot u_0\big|^2 + \varphi(u_0,z_0) - f(0)\cdot u_0\,\mathrm{d}x - \int_Q \frac{\partial f}{\partial t}\cdot u_\varepsilon\,\mathrm{d}x\mathrm{d}t.$$

Convergence for slow-loading regimes to fully rate-independent evolution:
scalling time as εt:
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Convergence for $\varepsilon \to 0$ in the Gibbs-type energy inequality:

$$\int_\Omega \varepsilon^2\varrho\Big|\frac{\partial u_\varepsilon}{\partial t}(T)\Big|^2 + \varphi(u_\varepsilon(T), z_\varepsilon(T)) - f(T)\cdot u_\varepsilon(T)\,\mathrm{d}x + \int_Q \varepsilon\zeta_2\Big(\frac{\partial u_\varepsilon}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t$$

$$+\mathrm{Var}_K(z_\varepsilon; 0, T) \le \int_\Omega \varepsilon^2\varrho\big|\dot u_0\big|^2 + \varphi(u_0, z_0) - f(0)\cdot u_0\,\mathrm{d}x - \int_Q \frac{\partial f}{\partial t}\cdot u_\varepsilon\,\mathrm{d}x\mathrm{d}t.$$

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Convergence for $\varepsilon \to 0$ in the Gibbs-type energy inequality:

$$\int_\Omega \varepsilon^2\varrho\Big|\frac{\partial u_\varepsilon}{\partial t}(T)\Big|^2 + \varphi(u_\varepsilon(T), z_\varepsilon(T)) - f(T)\cdot u_\varepsilon(T)\,\mathrm{d}x + \int_Q \varepsilon\zeta_2\Big(\frac{\partial u_\varepsilon}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t$$

$$+\mathrm{Var}_K(z_\varepsilon; 0, T) \le \int_\Omega \varepsilon^2\varrho|\dot{u}_0|^2 + \varphi(u_0, z_0) - f(0)\cdot u_0\,\mathrm{d}x - \int_Q \frac{\partial f}{\partial t}\cdot u_\varepsilon\,\mathrm{d}x\mathrm{d}t.$$

Definition of the weak solution now reduces to:

- force equilibrium $-\mathrm{div}\big(\varphi'_e(e(u),z)\big) = f(t)$ (formulated weakly)
- variational inequality:

$$\int_Q \sigma_{\rm in} v + \zeta_1(v) \geq \mathrm{Var}_K(z;0,T) + \int_\Omega \varphi(e(u(T)),z(T)) - \varphi(e(u_0),z_0) - f(T)\cdot u(T) + f(0)\cdot u_0\,\mathrm{d}x - \int_Q \frac{\partial f}{\partial t}\cdot u\,\mathrm{d}x\mathrm{d}t.$$

for all v.

Energetic solution $\Rightarrow$ weak solution.

Energetic solution $\Leftarrow$ weak solution
if $\frac{\partial u}{\partial t}$ and $\frac{\partial z}{\partial t}$ integrable and $\varphi(e,\cdot)$ convex.

In addition, a "viscous" regularization of ζ_1 in the form

$$\zeta_1(\dot z) = \delta_K^*(\dot z) + \frac{1}{2}|\dot z|^2.$$

The evolution of z would then looks as

$$\partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big) + \sigma_{\rm in} = \partial\delta_K^*\Big(\frac{\partial z}{\partial t}\Big) + \frac{\partial z}{\partial t} + \sigma_{\rm in} \ni 0$$

and the additional estimate

$$\Big\|\frac{\partial z_\varepsilon}{\partial t}\Big\|_{L^2(Q;\mathbb{R}^m)} \le \frac{C}{\sqrt{\varepsilon}}.$$

Convergence to the weak solution is simple:

$$\int_Q \sigma_{{\rm in},\varepsilon} v + \zeta_1(v) + \frac{\varepsilon}{2}|v|^2 \,{\rm d}x{\rm d}t \ge$$

$$+ \int_\Omega \frac{\varepsilon^2\varrho}{2}\Big|\frac{\partial u_\varepsilon}{\partial t}(T)\Big|^2 + \varphi(e(u_\varepsilon(T)), z_\varepsilon(T)) - f(T)\cdot u_\varepsilon(T)\,{\rm d}x$$

$$- \int_\Omega \frac{\varepsilon^2\varrho}{2}|\dot u_0|^2 + \varphi(e(u_0), z_0) - f(0)\cdot u_0\,{\rm d}x - \int_Q \frac{\partial f}{\partial t}\cdot u_\varepsilon\,{\rm d}x{\rm d}t.$$

Yet convergence to energetic solution is delicate:

In addition, a "viscous" regularization of ζ_1 in the form

$$\zeta_1(\dot z) = \delta_K^*(\dot z) + \frac{1}{2}|\dot z|^2.$$

Scalling for slow-loading εt would lead to

$$\partial\delta_K^*\Big(\frac{\partial z}{\partial t}\Big) + \varepsilon\frac{\partial z}{\partial t} + \sigma_{\rm in} \ni 0$$

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$$\Big\|\frac{\partial z_\varepsilon}{\partial t}\Big\|_{L^2(Q;\mathbb{R}^m)} \le \frac{C}{\sqrt{\varepsilon}}.$$

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$$-\int_\Omega \frac{\varepsilon^2\varrho}{2}|\dot u_0|^2 + \varphi(e(u_0), z_0) - f(0)\cdot u_0\,{\rm d}x - \int_Q \frac{\partial f}{\partial t}\cdot u_\varepsilon\,{\rm d}x{\rm d}t.$$

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Convergence to the weak solution is simple:

$$\int_Q \sigma_{{\rm in},\varepsilon} v + \zeta_1(v) + \frac{\varepsilon}{2}|v|^2\,{\rm d}x{\rm d}t \ge \int_Q \delta_K^*\Big(\frac{\partial z_\varepsilon}{\partial t}\Big) + \frac{\varepsilon}{2}\Big|\frac{\partial z_\varepsilon}{\partial t}\Big|^2 + 2\varepsilon\zeta_2\Big(e(\frac{\partial u_\varepsilon}{\partial t})\Big){\rm d}x{\rm d}t$$
$$+\int_\Omega \frac{\varepsilon^2\varrho}{2}\Big|\frac{\partial u_\varepsilon}{\partial t}(T)\Big|^2 + \varphi(e(u_\varepsilon(T)), z_\varepsilon(T)) - f(T)\cdot u_\varepsilon(T)\,{\rm d}x$$
$$-\int_\Omega \frac{\varepsilon^2\varrho}{2}|\dot u_0|^2 + \varphi(e(u_0), z_0) - f(0)\cdot u_0\,{\rm d}x - \int_Q \frac{\partial f}{\partial t}\cdot u_\varepsilon\,{\rm d}x{\rm d}t.$$

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Convergence to the weak solution is simple:

$$\int_Q \sigma_{{\rm in},\varepsilon} v + \zeta_1(v) + \frac{\varepsilon}{2}|v|^2\,{\rm d}x{\rm d}t \ge \int_Q \delta_K^*\Big(\frac{\partial z_\varepsilon}{\partial t}\Big) + \frac{\varepsilon}{2}\Big|\frac{\partial z_\varepsilon}{\partial t}\Big|^2 + 2\varepsilon\zeta_2\Big(e\big(\frac{\partial u_\varepsilon}{\partial t}\big)\Big)\,{\rm d}x{\rm d}t$$
$$+ \int_\Omega \frac{\varepsilon^2\varrho}{2}\Big|\frac{\partial u_\varepsilon}{\partial t}(T)\Big|^2 + \varphi(e(u_\varepsilon(T)), z_\varepsilon(T)) - f(T)\cdot u_\varepsilon(T)\,{\rm d}x$$
$$- \int_\Omega \frac{\varepsilon^2\varrho}{2}|\dot u_0|^2 + \varphi(e(u_0), z_0) - f(0)\cdot u_0\,{\rm d}x - \int_Q \frac{\partial f}{\partial t}\cdot u_\varepsilon\,{\rm d}x{\rm d}t.$$

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$$\zeta_1(\dot z) = \delta^*_K(\dot z) + \frac{1}{2}|\dot z|^2.$$

Scalling for slow-loading εt would lead to

$$\partial\delta^*_K\Big(\frac{\partial z}{\partial t}\Big) + \varepsilon\frac{\partial z}{\partial t} + \sigma_{\rm in} \ni 0$$

and the additional estimate

$$\Big\|\frac{\partial z_\varepsilon}{\partial t}\Big\|_{L^2(Q;\mathbb{R}^m)} \le \frac{C}{\sqrt{\varepsilon}}.$$

Convergence to the weak solution is simple:

$$\int_Q \sigma_{{\rm in},\varepsilon} v + \zeta_1(v) + \frac{\varepsilon}{2}|v|^2\,{\rm d}x{\rm d}t \ge {\rm Var}_K(z_\varepsilon;0,T)$$

$$+\int_\Omega \frac{\varepsilon^2\varrho}{2}\Big|\frac{\partial u_\varepsilon}{\partial t}(T)\Big|^2 + \varphi(e(u_\varepsilon(T)),z_\varepsilon(T)) - f(T)\cdot u_\varepsilon(T)\,{\rm d}x$$

$$-\int_\Omega \frac{\varepsilon^2\varrho}{2}|\dot u_0|^2 + \varphi(e(u_0),z_0) - f(0)\cdot u_0\,{\rm d}x - \int_Q \frac{\partial f}{\partial t}\cdot u_\varepsilon\,{\rm d}x{\rm d}t.$$

Yet convergence to energetic solution is delicate:

In addition, a "viscous" regularization of ζ_1 in the form

$$\zeta_1(\dot z) = \delta_K^*(\dot z) + \frac{1}{2}|\dot z|^2.$$

and the additional estimate

$$\left\| \frac{\partial z_\varepsilon}{\partial t} \right\|_{L^2(Q;\mathbb{R}^m)} \le \frac{C}{\sqrt{\varepsilon}}.$$

Convergence to the weak solution is simple:

$$\int_Q \sigma_{\mathrm{in},\varepsilon} v + \zeta_1(v) + \frac{\varepsilon}{2}|v|^2 \,\mathrm{d}x\mathrm{d}t \ge \mathrm{Var}_K(z_\varepsilon;0,T)$$
$$+ \int_\Omega \frac{\varepsilon^2\varrho}{2}\left|\frac{\partial u_\varepsilon}{\partial t}(T)\right|^2 + \varphi(e(u_\varepsilon(T)),z_\varepsilon(T)) - f(T)\cdot u_\varepsilon(T)\,\mathrm{d}x$$
$$- \int_\Omega \frac{\varepsilon^2\varrho}{2}|\dot u_0|^2 + \varphi(e(u_0),z_0) - f(0)\cdot u_0\,\mathrm{d}x - \int_Q \frac{\partial f}{\partial t}\cdot u_\varepsilon\,\mathrm{d}x\mathrm{d}t.$$

Yet convergence to energetic solution is delicate:
(Dal Maso, Knees, Krejčí, Mielke, Ortner, Rossi, Savaré, Stefanelli, Zanini, etc.)

Gradient theories.

Stored energy: $\varphi = \varphi(e, z, \nabla z) + \delta_C(z)$ with $C \subset \mathbb{R}^m$ convex closed.

Then (functional Gateaux' derivative):

$$\sigma_{\rm el} = \Big(\int_\Omega \varphi(e(u), z, \nabla z), {\rm d}x \Big)'_e \quad \text{and} \quad \sigma_{\rm in} = \Big(\int_\Omega \varphi(e(u), z, \nabla z), {\rm d}x \Big)'_z$$

i.e. the system looks now as

$$\varrho \frac{\partial^2 u}{\partial t^2} - {\rm div}\Big(\zeta_2'\big(e(\frac{\partial u}{\partial t})\big) + \varphi_e'(e(u), z, \nabla z) \Big) = f,$$

$$\partial \zeta_1\Big(\frac{\partial z}{\partial t}\Big) + \varphi_z'(e(u), z, \nabla z) - {\rm div}\varphi_{\nabla z}'(e(u), z, \nabla z) + \partial \delta_C(z) \ni 0.$$

Nonconvex φ possible (compactness in z):

$$\varphi(e, z, \nabla z) = \varphi_1(z) + \varphi_2(e, z, \nabla z), \qquad \varphi_1 \text{ continuous, } \varphi_2 \text{ quadratic.}$$

If K bounded (i.e. ζ_1 continuous), then $C \neq \mathbb{R}^m$ allowed.

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$$\sigma_{\rm el} = \varphi'_e(e, z, \nabla z) \quad \text{and} \quad \sigma_{\rm in} = \varphi'_z(e(u), z, \nabla z) - {\rm div}\varphi'_{\nabla z}(e(u), z, \nabla z),$$

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More-gradient theories: $\varphi = \varphi(e, z, \nabla e, \nabla z)$.
$\Rightarrow$ more nonlinearities,
in particular, $\varphi(e, z, \nabla e, \nabla z) = \varphi_0(e, z, \nabla e, \nabla z) + \delta_C(e, z)$ with
$C \subset \mathbb{R}^{n\times n} \times \mathbb{R}^m$ convex closed possible if inertial effects neglected.

Nonlocal regularization of z: Instead of $\frac{\kappa}{2}\int_\Omega |\nabla z|^2 \mathrm{d}x$, one can use

$$|z|_\alpha^2 = \frac{\kappa}{4}\int_\Omega\int_\Omega \frac{|z(x) - z(\xi)|^2}{|x-\xi|^{n+2\alpha}}\,\mathrm{d}x\mathrm{d}\xi$$

with $0 < \alpha < 1$.
No boundary conditions on z.
Then, instead of $-\kappa\Delta z$, we would get

$$\kappa\int_\Omega \frac{z(\cdot) - z(\xi)}{|\cdot - \xi|^{n+2\alpha}}\,\mathrm{d}\xi.$$

Similarly for e and similarly also for $\zeta_2 = \zeta_2\Big(e\big(\frac{\partial u}{\partial t}\big), \nabla e\big(\frac{\partial u}{\partial t}\big)\Big)$.

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Example: phase transformation in shape-memory alloys: $m > 1$, K bounded.

A "mixture" of quadratic energies in the form

$$\varphi(e, z, \nabla z) := \sum_{\ell=1}^{m} z_\ell \frac{(e - e_\ell)^\top \mathbb{C}_\ell (e - e_\ell)}{2} + \psi(z) + \frac{|\nabla z|^2}{2} + \delta_C(z) \quad \text{with } e_\ell := \frac{U_\ell^\top + U_\ell}{2},$$

with

- U_ℓ distortion matrices of particular pure phases (or phase variants),
- z's are volume fractions.
- ψ reflects the differences between chemical energies,
- $C := \{z \in \mathbb{R}^m;\ z_\ell \geq 0\ \&\ \sum_{\ell=1}^m z_\ell = 1\}$ the Gibbs' simplex.
- $\varepsilon(z)$ is the transformation strain,

The philosophy of mixtures of austenite/martensite phases is by Frémond see also Colli, Hoffmann, Sprekels, Visintin, etc.

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$$\varphi(e, z, \nabla z) = \sum_{\ell=1}^{m} \frac{(e-\varepsilon(z))^\top \mathbb{C}(e-\varepsilon(z))}{2} + \tilde{\psi}(z) + \frac{|\nabla z|^2}{2} + \delta_C(z) \quad \text{with } \varepsilon(z) = \sum_{\ell=1}^{m} z_\ell e_\ell,$$

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cf. Auricchio, Boyd, Lagoudas, Lexcelent, Govindjee, Miehe, Jung, Papadopoulos, Ritchie, etc.

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Another way is the quasiconvexification under volume constrains, also called cross-quasiconvexification, see Mielke, Theil, Levitas.

Example: Damage in visco-elastic materials.

For a scalar model of damage, $m = 1$, $C = \mathbb{R}$, and $K := [-a, +\infty)$ where $a > 0$ is an activation threshold for damage development.
The stored energy (quadratic in $(z, \nabla z)$

$$\varphi(e, z, \nabla e, \nabla z) := z^2 \mathbb{C}e : e + \frac{\kappa_1}{2}|\nabla e|^2 + \frac{\kappa_2}{2}|\nabla z|^2 \qquad ,$$

Natural initial condition is $z_0 = 1$, i.e. undamaged material.
The dissipation energy:

$$\zeta_1(\dot{z}) = \delta^*_K(\dot{z}) = \underbrace{\delta_{(-\infty,0]}(\dot{z})}_{\Longrightarrow\ z \text{ is non-increasing in time}} - a\dot{z} \ .$$

The coefficient $\kappa_2 > 0$ is related with a certain "hardening" effects: activation threshold a is effectively increased/decreased at a given point if its surrounding is less/more damaged, respectively.

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Alternatively, damage contributes to the stored energy
(Frémond, Nedjar).

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Or partly both in the stored and the dissipation energy;

$$a_0 + a_1 = a.$$

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Special techniques allows for $\varphi(e, z, \nabla z) := z\mathbb{C} e : e + \frac{\kappa_2}{2}|\nabla z|^2 + \delta_{[0,+\infty)}(z)$.

Thermodynamics:

The specific Helmholtz free energy

$$\psi = \psi(e, z, \nabla e, \nabla z, \theta)$$

with θ temperature.

Elastic stress: $\sigma_{\rm el} := \psi'_e - {\rm div}\psi'_{\nabla e}$.

Inelastic stress: $\sigma_{\rm in} := \psi'_z - {\rm div}\psi'_{\nabla z}$.

Entropy: $s := -\psi'_\theta$.

Internal energy: $w := \psi + \theta s$.

Entropy equation:

$$\theta\frac{\partial s}{\partial t} + {\rm div}(j) = \xi = \zeta_1\Big(\frac{\partial z}{\partial t}\Big) + 2\zeta_2\Big(\frac{\partial e}{\partial t}, \nabla\frac{\partial e}{\partial t}\Big), \qquad \underbrace{j = -\mathbb{K}\nabla\theta}_{\text{Fourier law}} .$$

Clausius-Duhem inequality: $\frac{\rm d}{{\rm d}t}\int_\Omega s(t,x){\rm d}x \geq 0$ (in an isolated system).

Energetics:

$$\frac{\rm d}{{\rm d}t}\int_\Omega \underbrace{\frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}\Big|^2 + w}_{\text{total energy}} \,{\rm d}x = \underbrace{\int_\Omega f\cdot\frac{\partial u}{\partial t}\,{\rm d}x + \int_\Gamma j\,{\rm d}S}_{\text{power of external load and heat}} .$$

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Partly linearized free energy:

$$\psi(e,z,\nabla e,\nabla z,\theta) := \varphi(e,z,\nabla e,\nabla z) + \theta\phi(e,z) - \psi_0(\theta).$$

$\Rightarrow$ entropy $s = s(e,z,\theta) = \psi_0'(\theta) - \phi(e,z)$ separates variables
$\Rightarrow$ heat equation:

$$c(\theta)\frac{\partial\theta}{\partial t} - \kappa\Delta\theta = \zeta_1\Big(\frac{\partial z}{\partial t}\Big) + 2\zeta_2\Big(\frac{\partial e}{\partial t},\nabla\frac{\partial e}{\partial t}\Big) + \theta\phi_e'(e,z){:}\frac{\partial e}{\partial t} + \theta\phi_z'(e,z)\frac{\partial z}{\partial t}$$

with $e = e(u)$ and the heat capacity $c(\theta) = \theta\psi_0''(\theta)$.

We will use $\psi_0(\theta) := c_{\rm v}\theta\ln(\theta)$ so that $c(\theta) = c_{\rm v} > 0$ constant.

A problem: L^1-estimates for $\dfrac{\partial z}{\partial t}$ but no L^∞-estimates for θ.

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Altogether, we thus will treat the system (for $e = e(u)$):

$$\varrho\frac{\partial^2 u}{\partial t^2} - \mathrm{div}\Big([\zeta_2]'_{\dot e}\big(\frac{\partial e}{\partial t},\nabla\frac{\partial e}{\partial t}\big) + \varphi'_e(e,z,\nabla e,\nabla z) + \theta\phi'(e) - \mathrm{div}[\zeta_2]'_{\nabla\dot e}\big(\frac{\partial e}{\partial t},\nabla\frac{\partial e}{\partial t}\big) - \mathrm{div}\varphi'_{\nabla e}(e,z,\nabla e,\nabla z)\Big) = f,$$

$$\partial\zeta_1\Big(\frac{\partial z}{\partial t}\Big) + \varphi'_z(e,z,\nabla e,\nabla z) - \mathrm{div}\varphi'_{\nabla z}(e,z,\nabla e,\nabla z) \ni 0,$$

$$c_{\rm v}\frac{\partial\theta}{\partial t} - \kappa\Delta\theta = \zeta_1\Big(\frac{\partial z}{\partial t}\Big) + 2\zeta_2\Big(\frac{\partial e}{\partial t},\nabla\frac{\partial e}{\partial t}\Big) + \theta\phi'(e) : \frac{\partial e}{\partial t}.$$

$\zeta_1\big(\frac{\partial z}{\partial t}\big)$ is now a measure; $\Big[\zeta_1\big(\frac{\partial z}{\partial t}\big)\Big]\big([t_1,t_2]\times A\big) := Var_K(z|_A;[t_1,t_2])$.

Altogether, we thus will treat the system (for $e = e(u)$):

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We call a triple (u, z, θ) with

$$u \in C_{\rm w}(I; W^{2,2}(\Omega;\mathbb{R}^n)) \cap W^{1,2}(I; W^{1,2}(\Omega;\mathbb{R}^n)) \cap W^{1,\infty}(I; L^2(\Omega;\mathbb{R}^n)),$$

$$z \in \mathrm{BM}(\bar{I}; W^{1,2}(\Omega;\mathbb{R}^m)) \cap \mathrm{BV}(\bar{I}; L^1(\Omega;\mathbb{R}^m)),$$

$$\theta \in L^r(I; W^{1,r}(\Omega)) \cap L^\infty(I; L^\omega(\Omega)) \quad \text{with any } 1 \le r < \frac{n+2\omega}{n+\omega},$$

an energetic solution if

- the weakly formulated balance of forces holds,
- the (very weakly formulated) heat equation holds,
- the total energy inequality holds

$$\int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u}{\partial t}(T)\Big|^2 + \varphi\big(e(T), z(T), \nabla e(T), \nabla z(T)\big) + c_{\rm v}\theta(T)\,\mathrm{d}x \le \int_\Omega \frac{\varrho}{2}\big|\dot{u}_0\big|^2 + \varphi\big(e_0, z_0, \nabla e_0, \nabla z_0\big) + c_{\rm v}\theta_0\,\mathrm{d}x + \int_Q f\cdot\frac{\partial u}{\partial t}\,\mathrm{d}x\mathrm{d}t,$$

- the (partial) stability holds for any $t \in [0, T]$: $\forall v \in L^2(\Omega;\mathbb{R}^m)$:

$$\int_\Omega \varphi\big(e(t), z(t), \nabla e(t), \nabla z(t)\big)\mathrm{d}x \le \int_\Omega \varphi\big(e(t), v, \nabla e(t), \nabla v\big) + \zeta_1\big(v - z(t)\big)\mathrm{d}x.$$

Important: any energetic solution is also a weak solution.

- as before, we use the stability of z at time t with respect to $z(t)+\varepsilon v$ and degree-1 homogeneity of ζ_1,
- then energy equality in the force equilibrium (because $\frac{\partial u^2}{\partial t^2}$ is in duality with $\frac{\partial u}{\partial t}$),
- and now also energy equality in the thermal part (because 1 is in duality with $\frac{\partial \theta}{\partial t}$).

The recursive increment formula (implicit Euler method):

$$\varrho\frac{u_\tau^k-2u_\tau^{k-1}+u_\tau^{k-2}}{\tau^2}-\mathrm{div}\Big([\zeta_2]'_{\dot e}\big(e(\frac{u_\tau^k-u_\tau^{k-1}}{\tau}),\nabla e(\frac{u_\tau^k-u_\tau^{k-1}}{\tau})\big)$$
$$+\varphi'_e(e(u_\tau^k),z_\tau^k,\nabla e(u_\tau^k),\nabla z_\tau^k)+\theta_\tau^k\phi'(e(u_\tau^k))$$
$$-\mathrm{div}\varphi'_{\nabla e}(e(u_\tau^k),z_\tau^k,\nabla e(u_\tau^k),\nabla z_\tau^k)$$
$$-\mathrm{div}[\zeta_2]'_{\nabla\dot e}\big(e(\frac{u_\tau^k-u_\tau^{k-1}}{\tau}),\nabla e(\frac{u_\tau^k-u_\tau^{k-1}}{\tau})\big)\Big)$$
$$+\varepsilon\Big|\frac{u_\tau^k-u_\tau^{k-1}}{\tau}\Big|^2\frac{u_\tau^k-u_\tau^{k-1}}{\tau}=f_\tau^k,$$

$$\partial\zeta_1\Big(\frac{z_\tau^k-z_\tau^{k-1}}{\tau}\Big)+\varphi'_z(e(u_\tau^k),z_\tau^k,\nabla z_\tau^k)$$
$$-\mathrm{div}\varphi'_{\nabla z}(e(u_\tau^k),z_\tau^k,\nabla e_\tau^k,\nabla z_\tau^k)\ni 0,$$

$$c_{\mathrm v}\frac{\theta_\tau^k-\theta_\tau^{k-1}}{\tau}-\kappa\Delta\theta_\tau^k+\varepsilon\big|\theta_\tau^k\big|^2\theta_\tau^k=\zeta_1\Big(\frac{z_\tau^k-z_\tau^{k-1}}{\tau}\Big)$$
$$+2\zeta_2\Big(e(\frac{u_\tau^k-u_\tau^{k-1}}{\tau}),\nabla e(\frac{u_\tau^k-u_\tau^{k-1}}{\tau})\Big)+\theta_\tau^k\phi'(e(u_\tau^k)):e(\frac{u_\tau^k-u_\tau^{k-1}}{\tau}).$$

By maximum principle $\theta_\tau^k\ge 0$.

Existence of $(u_\tau^k, z_\tau^k, \theta_\tau^k)$ by a suitable regularization and limit passage.

$$\varrho\frac{u_\tau^k-2u_\tau^{k-1}+u_\tau^{k-2}}{\tau^2} - \mathrm{div}\Big([\zeta_2]'_{\dot e}\big(e(\frac{u_\tau^k-u_\tau^{k-1}}{\tau}),\nabla e(\frac{u_\tau^k-u_\tau^{k-1}}{\tau})\big)$$
$$+\varphi'_e(e(u_\tau^k),z_\tau^k,\nabla e(u_\tau^k),\nabla z_\tau^k)+\theta_\tau^k\phi'(e(u_\tau^k))$$
$$-\mathrm{div}\varphi'_{\nabla e}(e(u_\tau^k),z_\tau^k,\nabla e(u_\tau^k),\nabla z_\tau^k)$$
$$-\mathrm{div}[\zeta_2]'_{\nabla\dot e}\big(e(\frac{u_\tau^k-u_\tau^{k-1}}{\tau}),\nabla e(\frac{u_\tau^k-u_\tau^{k-1}}{\tau})\big)\Big)$$
$$+\varepsilon\Big|\frac{u_\tau^k-u_\tau^{k-1}}{\tau}\Big|^2\frac{u_\tau^k-u_\tau^{k-1}}{\tau} = f_\tau^k,$$

$$\Big(1+\varepsilon\zeta_1\Big(\frac{z_\tau^k-z_\tau^{k-1}}{\tau}\Big)\Big)\partial\zeta_1\Big(\frac{z_\tau^k-z_\tau^{k-1}}{\tau}\Big)+\varphi'_z(e(u_\tau^k),z_\tau^k,\nabla z_\tau^k)$$
$$-\mathrm{div}\varphi'_{\nabla z}(e(u_\tau^k),z_\tau^k,\nabla e_\tau^k,\nabla z_\tau^k)\ni 0,$$

$$c_{\rm v}\frac{\theta_\tau^k-\theta_\tau^{k-1}}{\tau}-\kappa\Delta\theta_\tau^k+\varepsilon\big|\theta_\tau^k\big|^2\theta_\tau^k=\zeta_1\Big(\frac{z_\tau^k-z_\tau^{k-1}}{\tau}\Big)$$
$$+2\zeta_2\Big(e(\frac{u_\tau^k-u_\tau^{k-1}}{\tau}),\nabla e(\frac{u_\tau^k-u_\tau^{k-1}}{\tau})\Big)+\theta_\tau^k\phi'(e(u_\tau^k)):e(\frac{u_\tau^k-u_\tau^{k-1}}{\tau}).$$

By maximum principle $\theta_\tau^k\geq 0$.

Proof sketched:

1) Test the mechanical part as we did for energy estimate and add also the heat equation tested by a constant 1:
the energy estimate

$$\int_\Omega \frac{\varrho}{2}\left|\frac{\partial u_\tau}{\partial t}\right|^2 + \varphi(e(\bar u_\tau), \bar z_\tau, \nabla e(\bar u_\tau), \nabla \bar z_\tau)) + c_{\rm v}\bar\theta_\tau {\rm d}x \le C$$

uniformly in time.

Using $\theta_\tau \ge 0$, we get a-priori estimates:

$$\left\|\frac{\partial u_\tau}{\partial t}\right\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^n))} \le C,$$

$$\left\|e(u_\tau)\right\|_{L^\infty(I;W^{1,2}(\mathbb{R}^{n\times n})} \le C,$$

$$\left\|\bar z_\tau\right\|_{L^\infty(I;W^{1,2}(\Omega;\mathbb{R}^m))} \le C,$$

$$\|\bar\theta_\tau\|_{L^\infty(I;L^1(\Omega))} \le C.$$

2) estimation of $\nabla\theta_\tau$ as proposed by Boccardo, Gallouët, etc. Test the heat-transfer equation by $\chi_i(\theta_\tau)$ with

$$\chi_i(\theta) := \begin{cases} 0 & \text{if } \theta \le i, \\ \theta - i & \text{if } i \le \theta \le i+1, \\ 1 & \text{if } \theta \ge i+1. \end{cases}$$

- use Gagliardo-Nirenberg inequality several times,
- estimation of the mechanical part simultaneously with the heat equation
- precise interpolation of the adiabatic term

Then further a-priori estimates:

$$\Big\| e\Big(\frac{\partial u_\tau}{\partial t}\Big)\Big\|_{L^2(I;W^{1,2}(\mathbb{R}^{n\times n})} \le C,$$

$$\big\|\bar{z}_\tau\big\|_{\mathrm{BV}(\bar{I};L^1(\Omega;\mathbb{R}^m))} \le C,$$

$$\|\bar{\theta}_\tau\|_{L^r(I;W^{1,r}(\Omega))} \le C \quad \text{with any } 1 \le r < \tfrac{n+2}{n+1},$$

$$\Big\|\frac{\partial\theta_\tau}{\partial t}\Big\|_{L^1(I;W^{-1-n/2,2}(\Omega))} \le C.$$

Convergence:

- strong convergence of $\bar{\theta}_\tau$ by (generalized) Aubin-Lions' theorem,
- limit passage in the equilibrium-of-force equation,
- limit passage in the stability for any $t \in [0, T]$ as before (since θ is not involved as we assumed $\phi'_z = 0$),
- limit passage in the energy inequality by weak lower semicontinuity,
- limit passage in the heat equation: tricky:

By weak lower/upper semicontinuity that

$$
\begin{aligned}
&\mathrm{Var}_K(z;0,T) + 2\int_Q \zeta_2\big(e(\tfrac{\partial u}{\partial t}),\nabla e(\tfrac{\partial u}{\partial t})\big)\,\mathrm{d}x\mathrm{d}t \\
&\le \liminf_{\tau\downarrow 0}\int_Q \zeta_1(\tfrac{\partial z_\tau}{\partial t}) + 2\zeta_2\big(e(\tfrac{\partial u}{\partial t}),\nabla e(\tfrac{\partial u}{\partial t})\big)\,\mathrm{d}x\mathrm{d}t \le \\
&\le \limsup_{\tau\downarrow 0}\int_Q \zeta_1\big(\tfrac{\partial z_\tau}{\partial t}\big) + 2\zeta_2\big(e(\tfrac{\partial u_\tau}{\partial t}),\nabla e(\tfrac{\partial u_\tau}{\partial t})\big)\,\mathrm{d}x\mathrm{d}t \\
&\le \limsup_{\tau\downarrow 0}\bigg(-\int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u_\tau}{\partial t}(T)\Big|^2 \\
&+ \varphi(e(u_\tau(T)), z_\tau(T), \nabla e(u_\tau(T)), \nabla z_\tau(T)) - f(T)\cdot u_\tau(T)\mathrm{d}x \\
&+ \int_\Omega \frac{\varrho}{2}|\dot u_0|^2 - \varphi(e(u_0), z_0, \nabla e(u_0), \nabla z_0) + f(0)\cdot u_0\,\mathrm{d}x \\
&- \int_Q \frac{\partial f}{\partial t}\cdot u_\tau + \bar\theta_\tau\phi'(e(\bar u_\tau)){:}e\big(\frac{\partial u_\tau}{\partial t}\big)\,\mathrm{d}x\mathrm{d}t\bigg) \\
&= \mathrm{Var}_K(z;0,T) + 2\int_Q \zeta_2\big(e(\tfrac{\partial u}{\partial t}),\nabla e(\tfrac{\partial u}{\partial t})\big)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
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By weak lower/upper semicontinuity that

$$\mathrm{Var}_K(z;0,T)+2\int_Q \zeta_2\Big(e\big(\frac{\partial u}{\partial t}\big),\nabla e\big(\frac{\partial u}{\partial t}\big)\Big)\,\mathrm{d}x\mathrm{d}t$$

$$\leq \liminf_{\tau\downarrow 0}\int_Q \zeta_1\big(\frac{\partial z_\tau}{\partial t}\big)+2\zeta_2\Big(e\big(\frac{\partial u}{\partial t}\big),\nabla e\big(\frac{\partial u}{\partial t}\big)\Big)\,\mathrm{d}x\mathrm{d}t \leq$$

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$$\leq \limsup_{\tau\downarrow 0}\bigg(-\int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u_\tau}{\partial t}(T)\Big|^2$$

$$+\varphi(e(u_\tau(T)),z_\tau(T),\nabla e(u_\tau(T)),\nabla z_\tau(T))-f(T)\cdot u_\tau(T)\mathrm{d}x$$

$$+\int_\Omega \frac{\varrho}{2}|\dot u_0|^2-\varphi(e(u_0),z_0,\nabla e(u_0),\nabla z_0)+f(0)\cdot u_0\,\mathrm{d}x$$

$$-\int_Q \frac{\partial f}{\partial t}\cdot u_\tau+\bar\theta_\tau\phi'(e(\bar u_\tau)){:}e\big(\frac{\partial u_\tau}{\partial t}\big)\,\mathrm{d}x\mathrm{d}t\bigg)$$

$$=\mathrm{Var}_K(z;0,T)+2\int_Q \zeta_2\Big(e\big(\frac{\partial u}{\partial t}\big),\nabla e\big(\frac{\partial u}{\partial t}\big)\Big)\,\mathrm{d}x\mathrm{d}t.$$

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\end{aligned}
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We used:

energy balance as equality in z: by Riemann's sum argument,
energy balance as equality in u: (as $\frac{\partial u^2}{\partial t^2}$ is in duality with $\frac{\partial u}{\partial t}$),
and also adiabatic heat converges

$$\bar{\theta}_\tau \phi'(e(\bar{u}_\tau)){:}e\Big(\frac{\partial u_\tau}{\partial t}\Big) \to \theta\phi'(e(u)){:}e\Big(\frac{\partial u}{\partial t}\Big) \quad \text{strongly in } L^1(Q)$$

and $\bar{\theta}_\tau \to \theta$ strongly (Aubin-Lions theorem).

$\Rightarrow$ $\lim_{\tau\downarrow 0}\int_Q \zeta_1(\frac{\partial z_\tau}{\partial t})\,\mathrm{d}x\mathrm{d}t = \mathrm{Var}_K(z;0,T)$
$\Rightarrow$ $\zeta_1(\frac{\partial z_\tau}{\partial t}) \overset{*}{\rightharpoonup} \zeta_1(\frac{\partial z}{\partial t})$ in measures.

and also

$\Rightarrow$ $\lim_{\tau\downarrow 0}\int_Q \zeta_2(e(\frac{\partial u_\tau}{\partial t}),\nabla e(\frac{\partial u_\tau}{\partial t}))\,\mathrm{d}x\mathrm{d}t = \int_Q \zeta_2(e(\frac{\partial u}{\partial t}),\nabla e(\frac{\partial u}{\partial t}))\,\mathrm{d}x\mathrm{d}t$
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Example :

Thermoplasticity with isotropic thermal expansion and hardening:
Free energy:

$$\psi(e,\pi,\eta,\theta) = \frac{\mathbb{C}}{2}\big(e-\pi-\alpha\theta\mathbb{I}\big) : \big(e-\pi-\alpha\theta\mathbb{I}\big) - \frac{\alpha^2\mathbb{C}\mathbb{I}{:}\mathbb{I}}{2}\theta^2 + b\eta^2 - c_{\rm v}\theta\ln(\theta)$$

where α is an thermal expansion coefficient,
π plastic strain,
η hardening.
The homogeneous degree-1 dissipation potential is

$$\zeta_1(\dot\pi,\dot\eta) = \delta^*_P(\dot\pi) + \delta_K(\dot\pi,\dot\eta).$$

The ansatz with the coupling term $\theta\phi(e)$ is satisfied because $\alpha\theta\mathbb{I} : \pi = 0$ since $\mathrm{tr}(\pi) = 0$.
Here, higher gradients in ζ_2 should be still added (or, alternatively, nonlinear viscosity the a "p-growth/coercivity", $p > 1 + n/2$).

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nonlinear viscosity the a "p-growth/coercivity", $p > 1 + n/2$).

Example: combination of thermally expansive material

- $\mathbb{C}_2$ elastic-moduli tensor,
- $\mathbb{E}$ matrix of thermal-expansion coefficients,

with a material ungergoing damage.

- $\mathbb{C}_1$ elastic-moduli tensor,
- a_0 specific energy dissipated into change of structure,
- a_1 specific energy dissipated into heat.

Free energy:

$$\psi(e, z, \nabla z, \theta) := z\mathbb{C}_1 e{:}e + \frac{1}{2}\mathbb{C}_2(e+\theta\mathbb{E}){:}(e+\theta\mathbb{E}) - \frac{\mathbb{C}_2\mathbb{E}{:}\mathbb{E}}{2}\theta^2 + \frac{\kappa}{2}|\nabla z|^2 - c_{\rm v}\theta\ln(\theta) - a_0 z + \delta_{[0,+\infty)}(z),$$

i.e. $\phi(e) = \mathbb{C}_2\mathbb{E}{:}e$.
The dissipation energy: $\zeta_1(\dot{z}) = \delta^*_K(\dot{z}) = \delta_{(-\infty,0]}(\dot{z}) - a_1\dot{z}$.

Thanks a lot for your attention.

Gracie per la vostra gentile attenzione

Vielen dank für Ihre Beachtung.

Merci bien pour votre atention.

Děkuji za pozornost.