Homogenization in problems of viscoplasticity and Steklov regularization

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Mathematical model: Unknown variables

Notation and unknowns:

- $\mathcal{S}^3$: symmetric 3×3–matrices
- $\Omega \subseteq \mathbb{R}^3$: bounded open set with smooth boundary
  Ω represents an inelastic solid body at time $t = 0$
- $u(x,t) \in \mathbb{R}^3$: displacement of the material point $x \in \Omega$ at time t
- $T(x,t) \in \mathcal{S}^3$ Cauchy stress tensor
- $z(x,t) \in \mathbb{R}^N$: vector of internal variables with N components
  $z =$ (components of the plastic strain $\varepsilon_p(x,t) \in \mathcal{S}^3, \ldots$)

Mathematical model for inelastic materials

$$
\begin{aligned}
-\operatorname{div}_x T &= b(x,t), \\
T(x,t) &= F(\varepsilon(\nabla_x u(x,t)), z(x,t)), \\
z_t(x,t) &\in f(\varepsilon(\nabla_x u(x,t)), z(x,t)), \\
u(x,t) &= \gamma(x,t), \quad x \in \partial\Omega \\
z(x,0) &= z^{(0)}(x), \quad x \in \Omega
\end{aligned}
$$

Given: $b(x,t)$ volume force, $\gamma(x,t)$ boundary displacement, $z^{(0)}(x)$ initial data

Second law of thermodynamics

Restrictions for the choice of F, f: The **Clausius-Duhem inequality** $\frac{\partial}{\partial t}\psi + \mathrm{div}_x q \leq b \cdot u_t$ requires

$$T = \nabla_\varepsilon \psi(\varepsilon, z),$$
$$\nabla_z \psi(\varepsilon, z) \cdot f(\varepsilon, z) \leq 0.$$

(ψ = free energy, q = flux) Fulfilled if

$$\begin{aligned} F(\varepsilon, z) &= \nabla_\varepsilon \psi(\varepsilon, z), \\ f(\varepsilon, z) &= g\big(-\nabla_z \psi(\varepsilon, z)\big), \end{aligned}$$

with $g : \mathbb{R}^n \to 2^{\mathbb{R}^N}$ satisfying $g(z) \cdot z \geq 0$ for all $z \in \mathbb{R}^N$.

Satisfied, if g is monotone with $0 \in g(0)$.

qsIBV-problem of monotone type (I)

Hypothesis: All displacements $u(x,t)$ are small $\Longrightarrow$ the free energy can be chosen as positive semi-definite quadratic form:

$$\rho\psi(\varepsilon, z) = \frac{1}{2}[D(\varepsilon - Bz)] \cdot (\varepsilon - Bz) + \frac{1}{2}(Lz) \cdot z\,,$$

where

$D : \mathcal{S}^3 \to \mathcal{S}^3$	linear, symmetric, positive definite (elasticity tensor)
$B : R^N \to \mathcal{S}^3$	linear $(\,Bz = \varepsilon_p\,)$
$L : \mathbb{R}^N \to \mathbb{R}^N$	linear, symmetric, positive semi-definite

qsIBV-problem of monotone type (II)

$$
\begin{aligned}
-\mathrm{div}_x T &= b(x,t), \\
T &= D\big(\varepsilon(\nabla_x u) - Bz\big), \\
z_t &\in g(B^T T - Lz), \\
u(x,t) &= \gamma(x,t), \quad x \in \partial\Omega \\
z(x,0) &= z^{(0)}(x)
\end{aligned}
$$

$g : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ monotone with $0 \in g(0)$: **Initial-boundary value problem of monotone type**.

Special cases: $g = \nabla\varphi$, φ convex : Generalized standard material

$g = \partial I_K$ subdifferential of indicator function I_K of elastic range K: Rate independent generalized standard material

Existence of solutions

Assumptions:

- g maximal monotone, $0 \in g(0)$
- $\psi(\varepsilon, z)$ positive definite ($\Longleftrightarrow L$ positive definite)
- $b \in W^{i,\infty}(0,\infty; H^{-1}(\Omega, \mathbb{R}^3))$, $\gamma \in W^{i,\infty}(0,\infty; H^{1}(\Omega, \mathbb{R}^3))$, for $i = 1$ or $i = 2$
- $z^{(0)} \in L^2(\Omega, \mathbb{R}^N)$, $B^T T^{(0)} - L z^{(0)} \in \Delta(g)$

Then there exists a unique solution of the qsIBVP

$i = 2$: $(u, T, z) \in W^{1,\infty}(0,\infty; H^1 \times L^2 \times L^2)$ (strong solution)

$i = 1$: weak solution

Regularity

Assumptions as previously, and

- $b \in W^{2,\infty}\big(0,\infty; L^2(\Omega,\mathbb{R}^3)\big),\ \gamma \in W^{2,\infty}\big(0,\infty; H^2(\Omega,\mathbb{R}^3)\big)$
- $z^{(0)} \in H^1(\Omega,\mathbb{R}^N)$

Then the solution satisfies

- $(u,T,z) \in L^\infty\big(0,\infty; H^2_{\mathrm{loc}} \times H^1_{\mathrm{loc}} \times H^1_{\mathrm{loc}}\big)$
- $(u,T,z) \in L^\infty\big(0,\infty; B^{1+s,2}_\theta \times B^{s,2}_\theta \times B^{s,2}_\theta\big)$, for all $s < \frac{1}{3}$, $1 \le \theta \le \infty$

where $B^{s,2}_\theta(\Omega)$ **is the Besov space,** $B^{s,2}_2(\Omega) = W^{s,2}(\Omega)$

Framework of proofs (I)

For $\hat{\varepsilon}_p \in L^2(\Omega)$ let $(\tilde{u}, \tilde{\sigma})$ and (v, σ) solve

$$-\mathrm{div}\tilde{\sigma} = 0, \qquad -\mathrm{div}\sigma = b(x,t),$$
$$\tilde{\sigma} = D(\varepsilon(\nabla\tilde{v}) - \hat{\varepsilon}_p)\,, \qquad \sigma = D(\varepsilon(\nabla v))\,,$$
$$\tilde{v} = 0, \quad x \in \partial\Omega, \qquad v = \gamma(x,t), \quad x \in \partial\Omega.$$

Define the projector

$$\hat{\varepsilon}_p \to Q\hat{\varepsilon}_p = \hat{\varepsilon}_p - \varepsilon(\nabla\tilde{v})) : L^2(\Omega) \to \{\tau \in L^2 \mid \mathrm{div}(D\tau) = 0\}$$

Then $T = -DQBz + \sigma$ and

$$z_t \in g(B^T T - Lz) = g\Big(-(B^T DQB + L)z + B^T\sigma(t)\Big).$$

Framework of proofs (II)

Let $M = B^T DQB + L$ (linear, selfadjoint, positive definite on $L^2(\Omega)$). Then

$$z_t + A(t)z \ni 0, \qquad \text{with } A(t)z = -g\big(-Mz + B^T\sigma(t)\big).$$

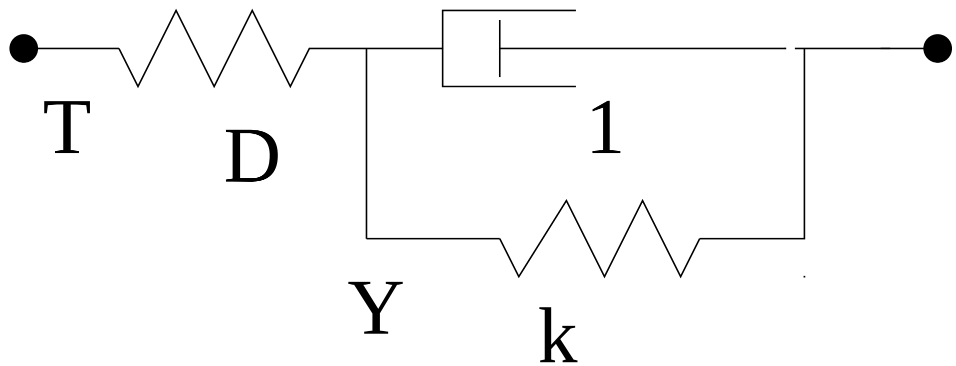
This **evolution equation is not autonomous**, since $A(t)$ depends on t via $\sigma(t)$. Set $h = -Mz + B^T\sigma$. Then

$$h_t(t) + \mathcal{C}h(t) \ni B^T\sigma_t(t), \quad \text{with } \mathcal{C}h = Mg(h).$$

$\mathcal{C}$ is **maximal monotone** on $L^2(\Omega, \mathbb{R}^N)$ with the scalar product

$$[z, \hat{z}]_\Omega = \big(M^{-1}z, \hat{z}\big)_\Omega .$$

Example for positive definite ψ: Linear hardening



$$\begin{aligned} T &= D(\varepsilon - \varepsilon_p) \\ \partial_t \varepsilon_p &= c|T-Y|^r \frac{T-Y}{|T-Y|} \\ Y &= k\varepsilon_p\,, \qquad c, k, r > 0 \end{aligned}$$

For this constitutive model the free energy is positive definite and the assumptions of the theorem are satisfied.

Homogenization: Microscopic qsIBV

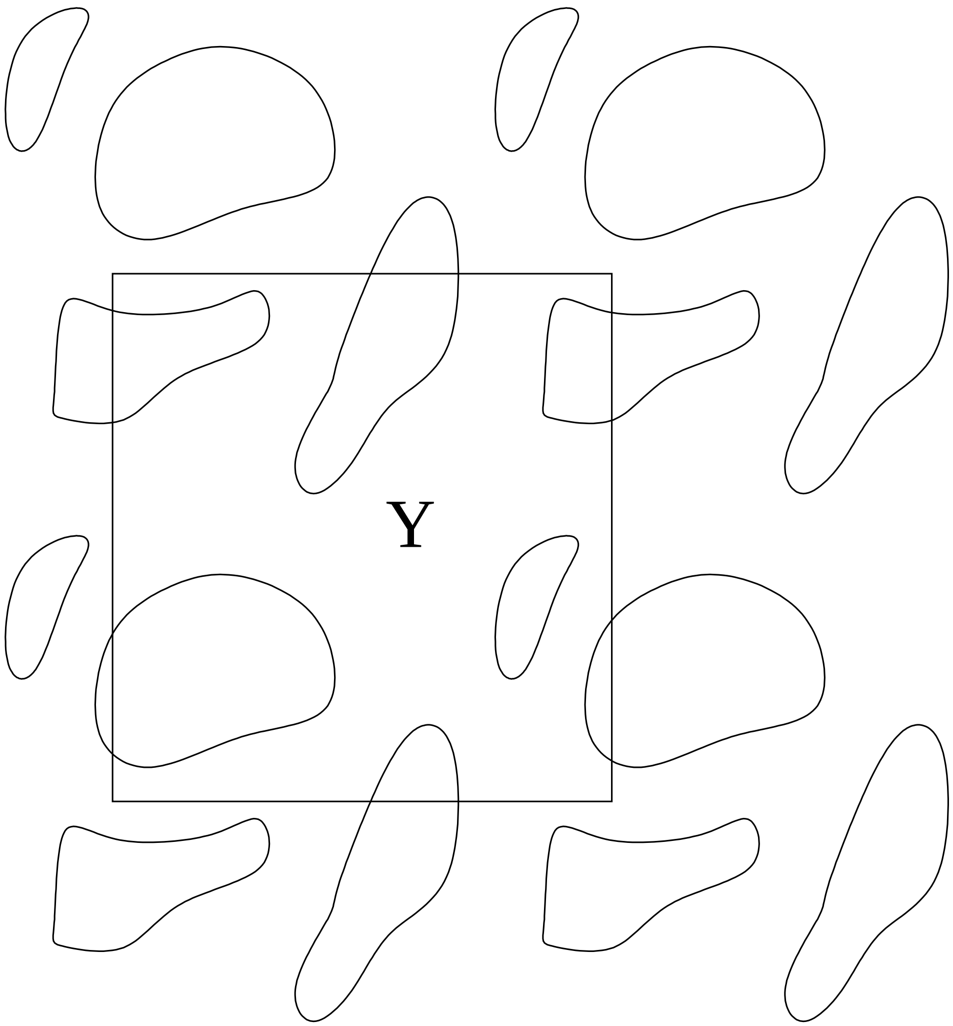
qsIBV for a solid body with periodic microstructure:

$$
\begin{aligned}
-\mathrm{div}_x T_\eta(x,t) &= b(x,t) \\
T_\eta(x,t) &= D\big(\frac{x}{\eta}\big)\Big(\varepsilon\big(\nabla_x u_\eta(x,t)\big) - Bz_\eta(x,t)\Big) \\
z_{\eta;t}(x,t) &\in g\big(\frac{x}{\eta},\, B^T T_\eta(x,t) - Lz_\eta(x,t)\big) \\
z_\eta(x,0) &= z_0^{(0)}(x,\frac{x}{\eta}) \\
u_\eta(x,t) &= \gamma(x,t), \quad (x,t) \in \partial\Omega \times [0,\infty)
\end{aligned}
$$

$y \mapsto D(y)$, $y \mapsto g(y,z)$, $y \mapsto z_0^{(0)}(x,y)$: periodic on $\mathbb{R}^3$ with periodicity cell $Y \subseteq \mathbb{R}^3$

$\eta > 0$: scaling parameter of the microstructure

Homogenization: Periodicity cell Y



Periodic structure of the material.

$Y \subset \mathbb{R}^3$: Periodicity cell

Homogenization: Quasiperiodic ansatz

For $\eta \to 0$ one expects that $(u_\eta, T_\eta, z_\eta) \sim (\hat{u}_\eta, \hat{T}_\eta, \hat{z}_\eta)$ with $\hat{u}_\eta, \hat{T}_\eta, \hat{z}_\eta$ given by the ansatz

$$
\begin{aligned}
\hat{u}_\eta(x,t) &= u_0(x,t) + \eta u_1(x, \frac{x}{\eta}, t) \\
\hat{T}_\eta(x,t) &= T_0(x, \frac{x}{\eta}, t) \\
\hat{z}_\eta(x,t) &= z_0(x, \frac{x}{\eta}, t),
\end{aligned}
$$

where $(x,y,t) \mapsto (u_1, T_0, z_0)(x,y,t)$ is periodic with respect to y; the periodicity cell is Y.

Formal calculation $\Longrightarrow u_0, u_1, T_0, z_0$ and the overall stress T_∞ must satisfy the following

Homogenized initial-boundary value problem

$$
\begin{aligned}
-\mathrm{div}_x\, T_\infty(x,t) &= b(x,t) \\
T_\infty(x,t) &= \frac{1}{|Y|}\int_Y T_0(x,y,t)dy \\
-\mathrm{div}_y\, T_0(x,y,t) &= 0 \\
T_0(x,y,t) &= D(y)\Big(\varepsilon\big(\nabla_y u_1(x,y,t)\big) - Bz_0(x,y,t) \\
&\qquad + \varepsilon\big(\nabla_x u_0(x,t)\big)\Big) \\
\frac{\partial}{\partial t} z_0(x,y,t) &\in g\big(y, B^T T_0(x,y,t) - Lz_0(x,y,t)\big) \\
z_0(x,y,0) &= z_0^{(0)}(x,y), \quad u_0|_{x\in\partial\Omega} = \gamma|_{x\in\partial\Omega}
\end{aligned}
$$

Homogenization: Existence

- **If** $z \mapsto g(y,z)$ is maximal monotone, $0 \in g(y,0)$
- L is positive definite
- $b \in W^{2,\infty}(0,\infty; L^2(\Omega,\mathbb{R}^3)),\ \gamma \in W^{2,\infty}(0,\infty; H^1(\Omega,\mathbb{R}^3))$
- $z_0^{(0)} \in L^2(\Omega \times Y, \mathbb{R}^N),\ B^T T^{(0)} - L z_0^{(0)} \in \Delta(g)$,

then the homogenized problem has solutions satisfying

$$\begin{aligned}(u_0, T_\infty) &\in W^{1,\infty}\big(0,\infty; H^1(\Omega) \times L^2(\Omega)\big)\\ (u_1, T_0, z_0) &\in W^{1,\infty}\big(0,\infty; H^1(\Omega \times Y) \times (L^2(\Omega \times Y))^2\big)\end{aligned}$$

Justification: straightforward approach

With this solution **define**

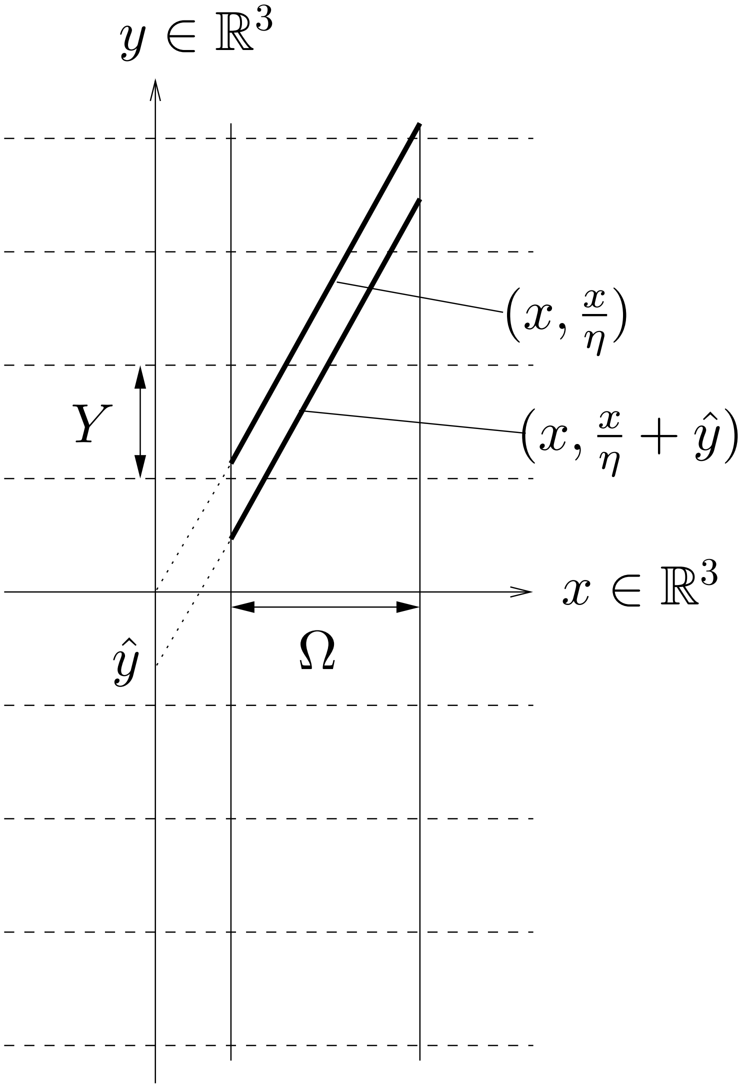
$$\hat{T}_\eta(x,t) = T_0(x, \frac{x}{\eta}, t),$$
$$\hat{z}_\eta(x,t) = z_0(x, \frac{x}{\eta}, t),$$

and **try to show** for the solution (u_η, T_η, z_η) of micro-qsIBV that

$$\lim_{\eta \to 0} \Big[\|u_\eta(t) - u_0(t)\|_{L^2(\Omega)} + \|T_\eta(t) - \hat{T}_\eta(t)\|_{L^2(\Omega)} + \|z_\eta(t) - \hat{z}_\eta(t)\|_{L^2(\Omega)} \Big] = 0, \quad 0 \le t < \infty\,.$$

Problem: Regularity of T_0 and z_0 is too low for this definition!

Homogenization: Low regularity of the solution of the homogenized IBV



$$(x, y) \mapsto T_0(x, y, t) \in L^{2,\mathrm{loc}}(\overline{\Omega \times \mathbb{R}^3})$$

$$\Downarrow$$

no trace on the 3–dimensional

submanifold $\{(x, \frac{x}{\eta}) \mid x \in \Omega\}$

of the 6–dimensional set $\Omega \times \mathbb{R}^3$

$$\Downarrow$$

$x \mapsto T_0(x, \frac{x}{\eta}, t)$ is not defined!

Justification: Steklov regularization

Theorem. With solution (u_0, T_0, z_0) of homogenized problem define

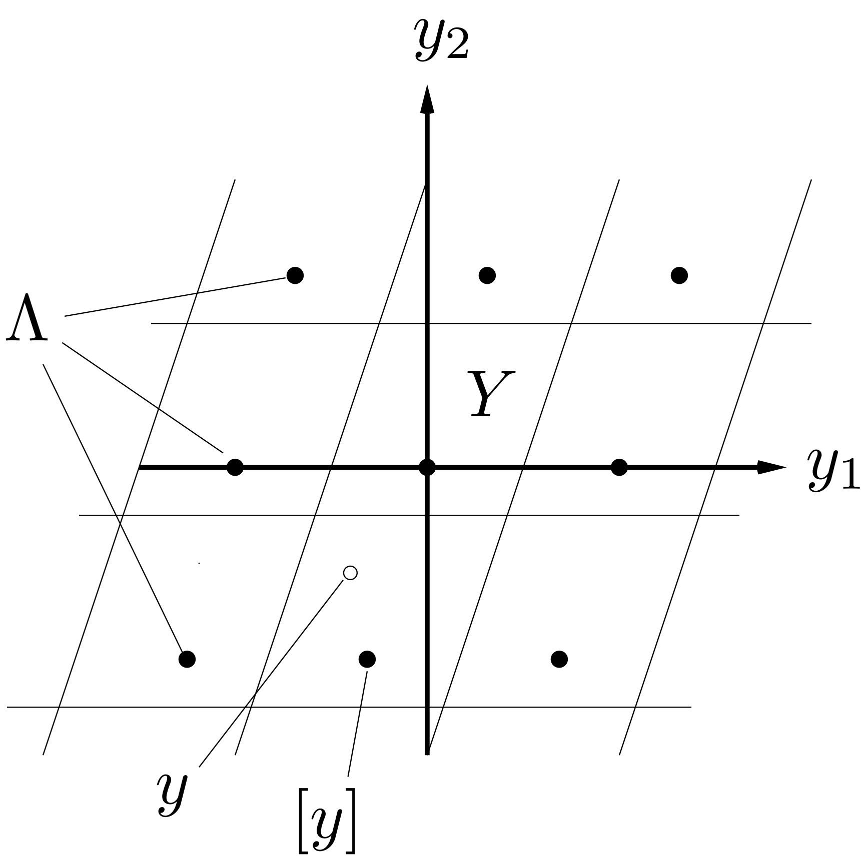
$$T_\eta^*(x,t) = \frac{1}{|Y|}\int_Y T_0(x - \eta y, \tfrac{x}{\eta}, t)dy,$$
$$z_\eta^*(x,t) = \frac{1}{|Y|}\int_Y z_0(x - \eta y, \tfrac{x}{\eta}, t)dy.$$

Then for solution (u_η, T_η, z_η) of microscopic problem

$$\lim_{\eta\to 0} \Big[\|u_\eta(t) - u_0(t)\|_{L^2(\Omega)} + \|T_\eta(t) - T_\eta^*(t)\|_{L^2(\Omega)} + \|z_\eta(t) - z_\eta^*(t)\|_{L^2(\Omega)}\Big] = 0, \quad 0 \le t < \infty\,.$$

(Notion introduced by Zhikov and coworkers)

Justification I: Lattice



Let $0 \in Y$ and let $\Lambda \subseteq \mathbb{R}^3$ be a set of lattice points such that

$$\mathbb{R}^3 = \bigcup_{\lambda \in \Lambda} (Y + \lambda)$$

is a disjoint union.

For $y \in \mathbb{R}^3$ let $[y] \in \Lambda$ be such that

$$y \in (Y + [y])$$

Justification I: Strong 2s–convergence

Theorem (A. Mielke & A. Timofte):

Let (u_η, T_η, z_η) solve micro-IBV, (u_0, T_0, z_0) solve homo-IBV. **Then**

$$\begin{aligned} \|u_\eta(t) - u_0(t)\|_{L^2(\Omega)} &\to 0, \\ T_\eta(t) &\stackrel{s2}{\to} T_0, \\ z_\eta(t) &\stackrel{s2}{\to} z_0\,, \end{aligned}$$

for $\eta \to 0$, where $v_\eta(t) \stackrel{s2}{\to} v_0$ (strong two scale convergence) means

$$\int_{\Omega\times Y} |v_\eta(\eta\,[\tfrac{x}{\eta}] + \eta y) - v_0(x,y)|^2 d(x,y) \to 0,$$

with $v_\eta : \Omega \to \mathbb{R}^m$ extended to $\mathbb{R}^3$ by zero.

Justification II: Microscopic qsIBV with phase shift

If $\big(u_0 + \eta u_1, T_0, z_0\big)(x, \frac{x}{\eta}, t)$ solves micro-qsIBV asymptotically, then

$$(x,t) \mapsto \big(u_0 + \eta u_1, T_0, z_0\big)(x, \tfrac{x}{\eta} + y, t)$$

with $y \in \mathbb{R}^3$ is asymtotic sol'n of

$$\begin{aligned}
-\operatorname{div}_x \tilde{T}_\eta(x,y,t) &= b(x,t) \\
\tilde{T}_\eta(x,y,t) &= D\Big(\frac{x}{\eta} + y\Big)\Big(\varepsilon\Big(\nabla_x \tilde{u}_\eta(x,y,t)\Big) - B\tilde{z}_\eta(x,y,t)\Big) \\
\frac{\partial}{\partial t}\tilde{z}_\eta(x,y,t) &\in g\Big(\frac{x}{\eta} + y\,,\, B^T\tilde{T}_\eta(x,y,t) - L\tilde{z}_\eta(x,y,t)\Big) \\
\tilde{z}_\eta(x,y,0) &= z_0^{(0)}\Big(x, \frac{x}{\eta} + y\Big) \\
\tilde{u}_\eta(x,y,t) &= \gamma(x,t), \quad (x,y,t) \in \partial\Omega \times Y \times [0,\infty)
\end{aligned}$$

(S. Nesenenko, A.)

Justification II: Phase-shift-convergence

Theorem (S. Nesenenko):

- For a solution $(u_0, u_1, T_\infty, T_0, z_0)$ of the homogenized IBV set

$$\hat{T}_\eta(x,y,t) = T_0(x, \frac{x}{\eta} + y, t)$$
$$\hat{z}_\eta(x,y,t) = z_0(x, \frac{x}{\eta} + y, t)$$

- Then the solution $(\tilde{u}_\eta, \tilde{T}_\eta, \tilde{z}_\eta)$ of the microscopic problem with phase shift satisfies

$$\lim_{\eta \to 0} \Big[\|\tilde{u}_\eta(t) - u_0(t)\|_{2,\Omega\times Y} + \|\tilde{T}_\eta(t) - \hat{T}_\eta(t)\|_{2,\Omega\times Y} + \|\tilde{z}_\eta(t) - \hat{z}_\eta(t)\|_{2,\Omega\times Y} \Big] = 0, \quad 0 \le t \le \infty$$

Shifted solution solves phase-shift-IBV

Theorem:

- **Let** $(\tilde{u}_\eta, \tilde{T}_\eta, \tilde{z}_\eta)$ solve micro-IBV with phase-shift
- **Let** (u_η, T_η, z_η) solve micro-IBV without phase-shift (original problem), extended from $\Omega \times [0, \infty]$ to $\mathbb{R}^3 \times [0, \infty]$ by 0
- **Then**, uniformly with respect to $y \in Y$,

$$\lim_{\eta \to 0} \|(\tilde{u}_\eta, \tilde{T}_\eta, \tilde{z}_\eta)(\cdot, y, t) - (u_\eta, T_\eta, z_\eta)(\cdot + \eta y, t)\|_{L^2(\Omega)} = 0,$$

(Shifted sol'n approximates sol'n to coefficients with shifted phase)

Strong 2sa–convergence

Combine this and Nesenenko's result by the triangle ineq'lity $\Longrightarrow$

Corollary:

Let (u_η, T_η, z_η) solve micro-IBV, (u_0, T_0, z_0) solve homo-IBV. **Then**

$$\begin{aligned} \|u_\eta(t) - u_0(t)\|_{L^2(\Omega)} &\to 0, \\ T_\eta(t) &\stackrel{s2a}{\to} T_0, \\ z_\eta(t) &\stackrel{s2a}{\to} z_0 \,, \end{aligned}$$

for $\eta \to 0$, where $v_\eta(t) \stackrel{s2a}{\to} v_0$ means

$$\lim_{\eta\to 0} \int_{\Omega\times Y} |v_\eta(x+\eta y) - v_0(x, \tfrac{x}{\eta} + y)|^2 d(x,y) = 0.$$

s2–convergence versus s2a–convergence

$y \mapsto v_0, (x.y)$ has periodicity cell Y, hence $v_0(x, [\frac{x}{\eta}] + y) = v_0(x, y)$

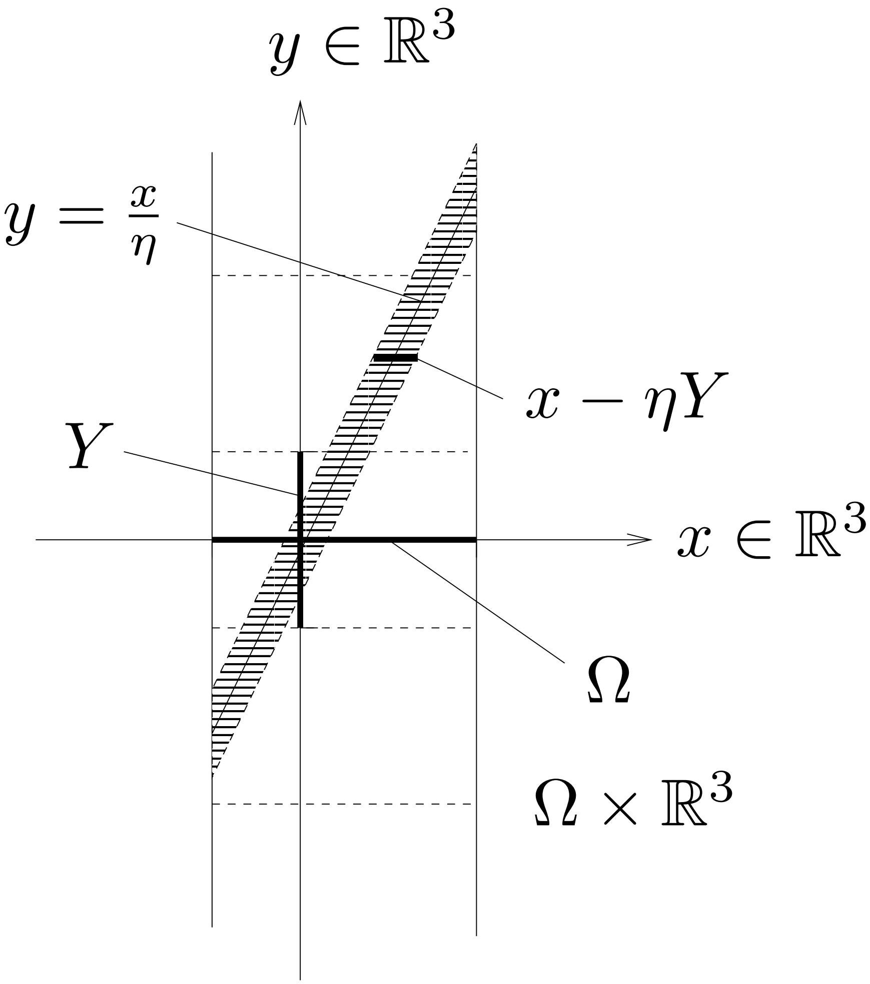
$$\Downarrow$$

$$v_\eta \overset{s2}{\longrightarrow} v_0 \quad \Leftrightarrow \quad \int_{\Omega \times Y} |v_\eta(\eta\left[\tfrac{x}{\eta}\right] + \eta y) - v_0(x, \frac{1}{\eta}\eta\left[\tfrac{x}{\eta}\right] + y)|^2 d(x, y) \longrightarrow 0$$

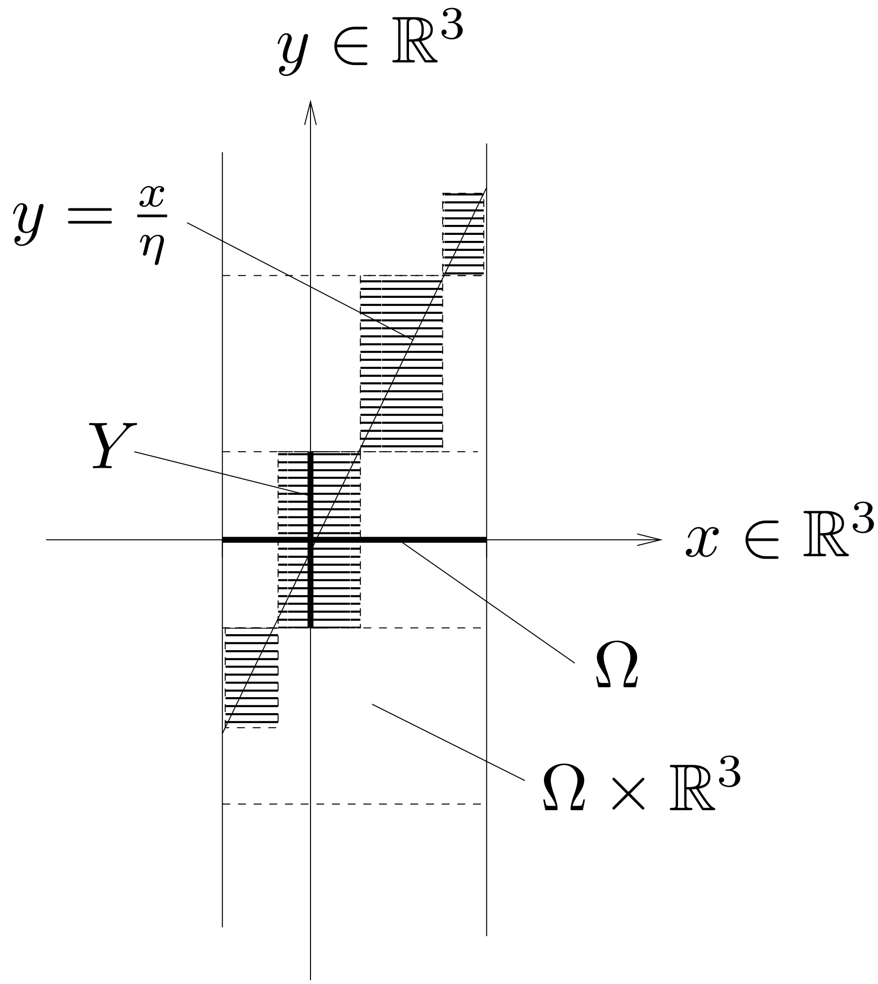
$$v_\eta \overset{s2a}{\longrightarrow} v_0 \quad \Leftrightarrow \quad \int_{\Omega \times Y} |v_\eta(x + \eta y) - v_0(x, \tfrac{1}{\eta}x + y)|^2 d(x, y) \longrightarrow 0$$

(replace x by $\eta[\frac{x}{\eta}]$!)

Regions of averaging



$s2a$–convergence



$s2$–convergence

Justification of Steklov regularization

Write six-dimensional integral from 2sa–convergence as iterated integral:

$$
\begin{aligned}
0 &\leftarrow \int_{\Omega\times Y} |T_\eta(x+\eta y,t) - T_0(x,\tfrac{x}{\eta}+y,t)|^2 d(x,y) \\
&= \int_{\Omega_\eta}\int_{Y_{x,\eta}} |T_\eta(x,t) - T_0(x-\eta y,\tfrac{x}{\eta},t)|^2 dy dx \\
&\geq \frac{1}{|Y_{x,\eta}|}\int_{\Omega_\eta} \left| T_\eta(x,t)|Y_{x,\eta}| - \int_{Y_{x,\eta}} T_0(x-\eta y,\tfrac{x}{\eta},t) dy \right|^2 dx \\
&\approx |Y| \int_{\Omega_\eta} |T_\eta(x,t) - T^*(x,t)|^2 dx.
\end{aligned}
$$