A Variational Bayesian Algorithm for Inverse Problem of Computed Tomography

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Content

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- VB for inverse problems
- Gauss-Markov-Potts prior moedels for images
- VB with Gauss-Markov-Potts prior moedels
- Application in Computed Tomography
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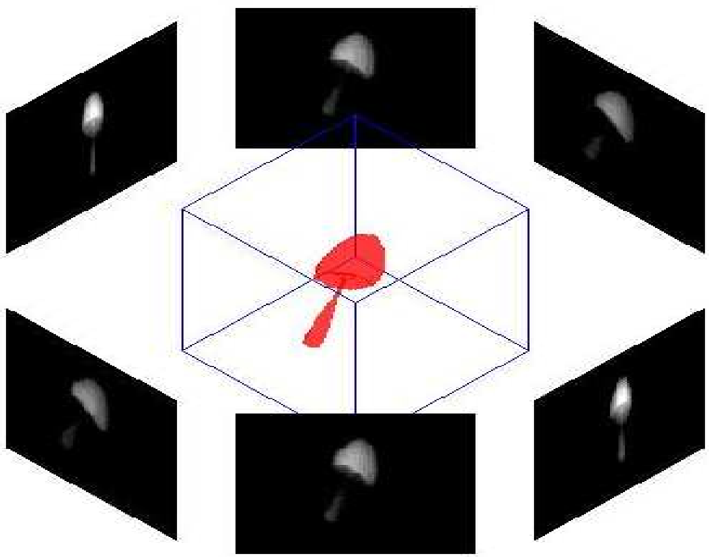
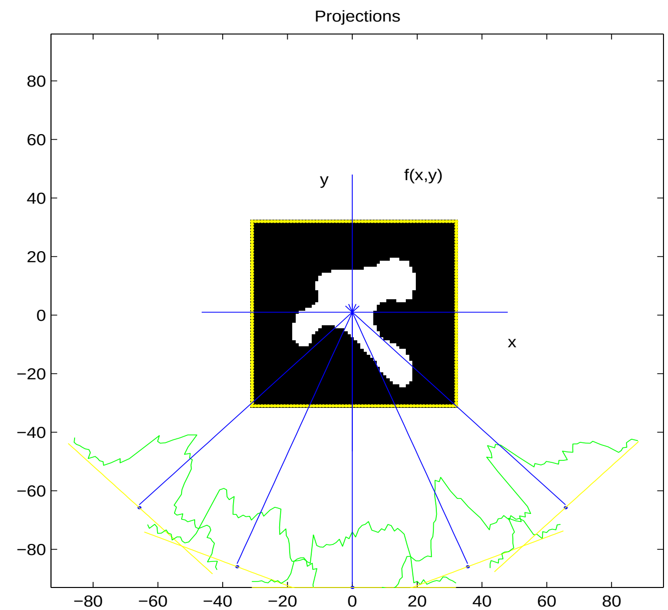
2D and 3D Computed Tomography

3D



2D

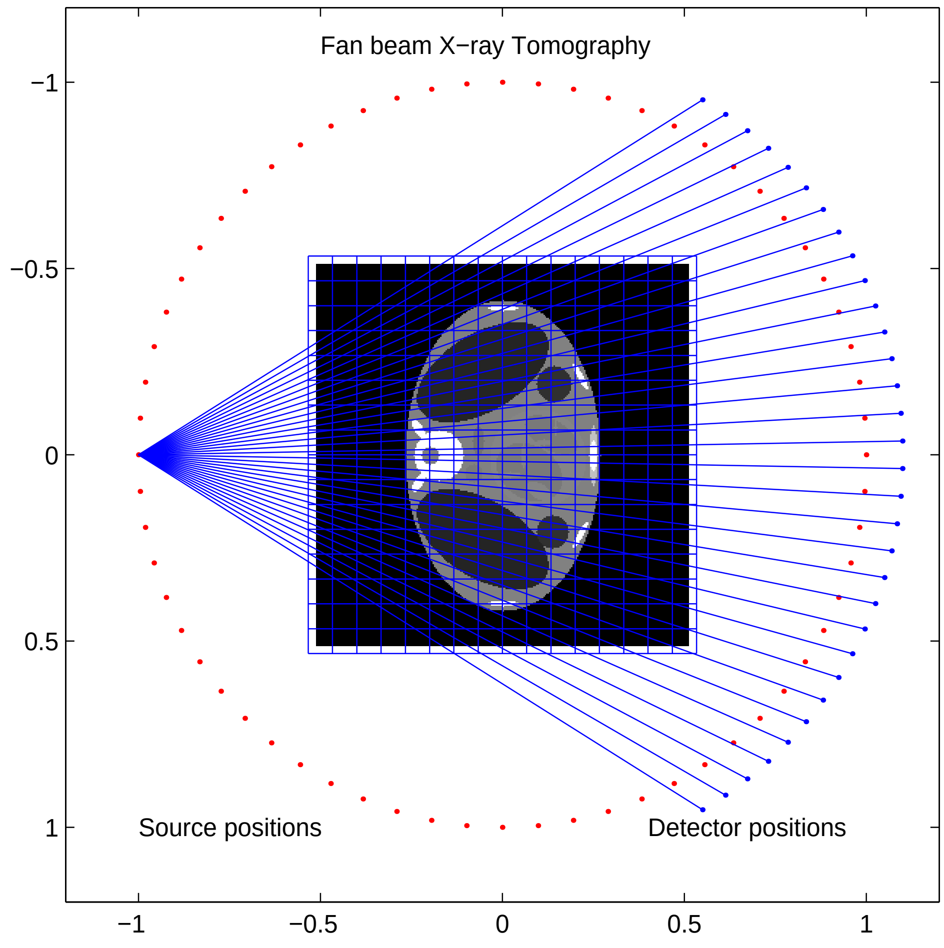


$$g_\phi(r_1, r_2) = \int_{\mathcal{L}_{r_1,r_2,\phi}} f(x,y,z) \, \mathrm{d}l \qquad g_\phi(r) = \int_{\mathcal{L}_{r,\phi}} f(x,y) \, \mathrm{d}l$$

Forward probelm: $f(x,y)$ or $f(x,y,z) \longrightarrow g_\phi(r)$ or $g_\phi(r_1,r_2)$

Inverse problem: $g_\phi(r)$ or $g_\phi(r_1,r_2) \longrightarrow f(x,y)$ or $f(x,y,z)$

CT as a linear inverse problem



$$g(\boldsymbol{s}_i) = \int_{L_i} f(\boldsymbol{r}) \; \mathrm{d}l_i \longrightarrow \text{Discretization} \longrightarrow \boldsymbol{g} = \boldsymbol{H}\,\boldsymbol{f} + \boldsymbol{\epsilon}$$

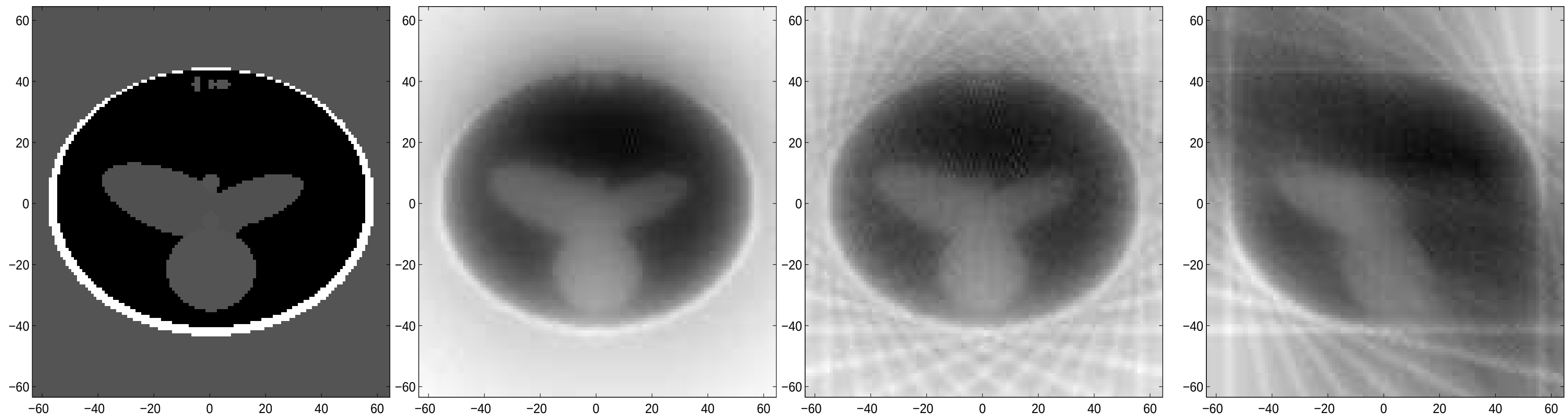
Classical methods in CT

$$g(\boldsymbol{s}_i) = \int_{L_i} f(\boldsymbol{r}) \; \mathrm{d}l_i \longrightarrow \text{Discretization} \longrightarrow \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- $\boldsymbol{H}$ is a huge dimensional matrix of line integrals;
- $\boldsymbol{H}\boldsymbol{f}$ is the forward or projection operation;
- $\boldsymbol{H}^t\boldsymbol{g}$ is the backward or backprojection operation;
- $(\boldsymbol{H}^t\boldsymbol{H})^{-1}\boldsymbol{H}^t\boldsymbol{g}$ is the filtered backprojection;
- $\widehat{\boldsymbol{f}}^{(k+1)} = \widehat{\boldsymbol{f}}^{(k)} + \alpha^{(k)}\boldsymbol{H}^t\left(\boldsymbol{g} - \boldsymbol{H}\widehat{\boldsymbol{f}}^{(k)}\right)$

  is the Least squares iterative reconstruction method minimizing $\|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2$.
- Regularization: $J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{D}\boldsymbol{f}\|^2$.

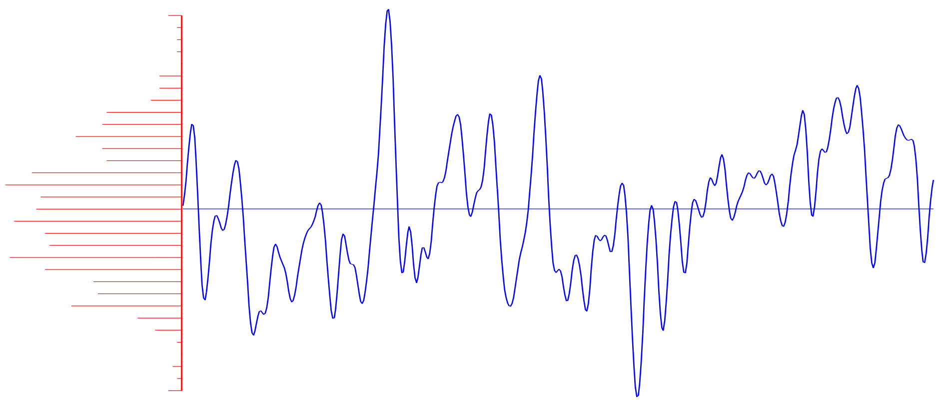
Limited angle or noisy data:
Need for prior information



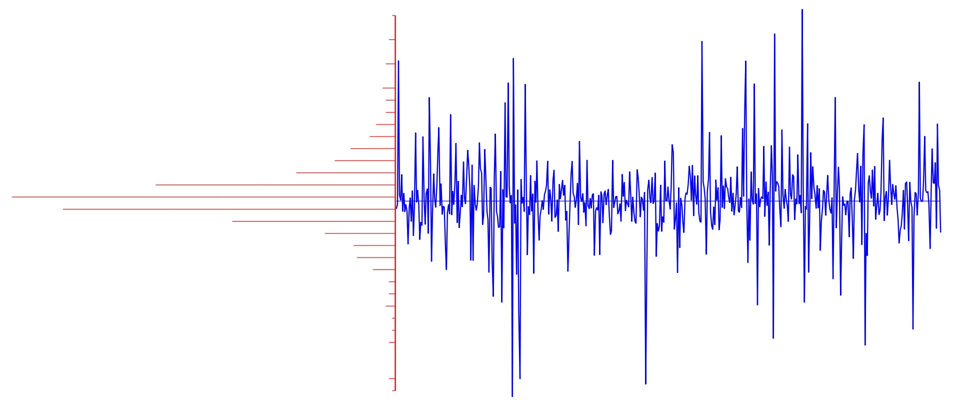
Original 64 proj. 16 proj. 8 proj. $[0, \pi/2]$

- Regularization: Continuity
  $J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{D}\boldsymbol{f}\|^2.$
- Piecewize continuity, Positivity, Binary, Homogeneity in regions, .. $\longrightarrow$ Bayesian approach

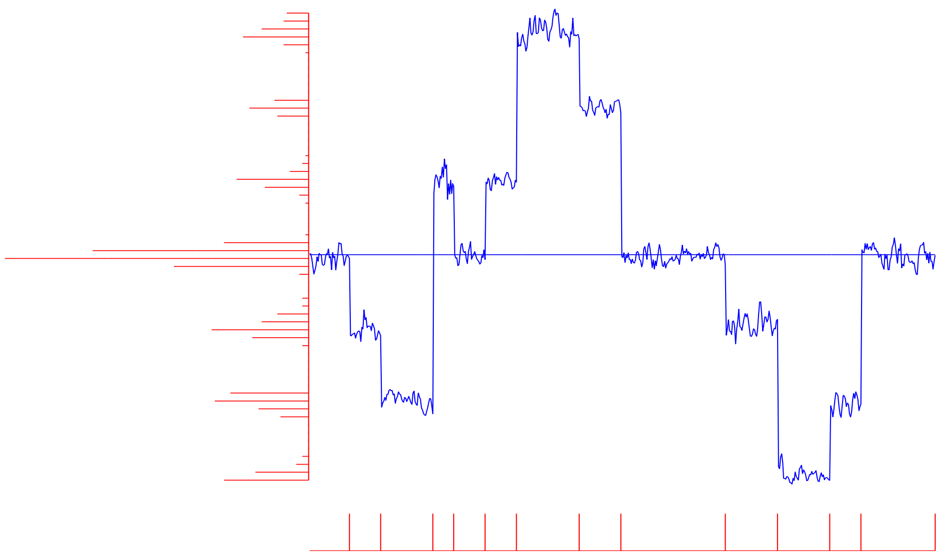
Which image I am looking for?



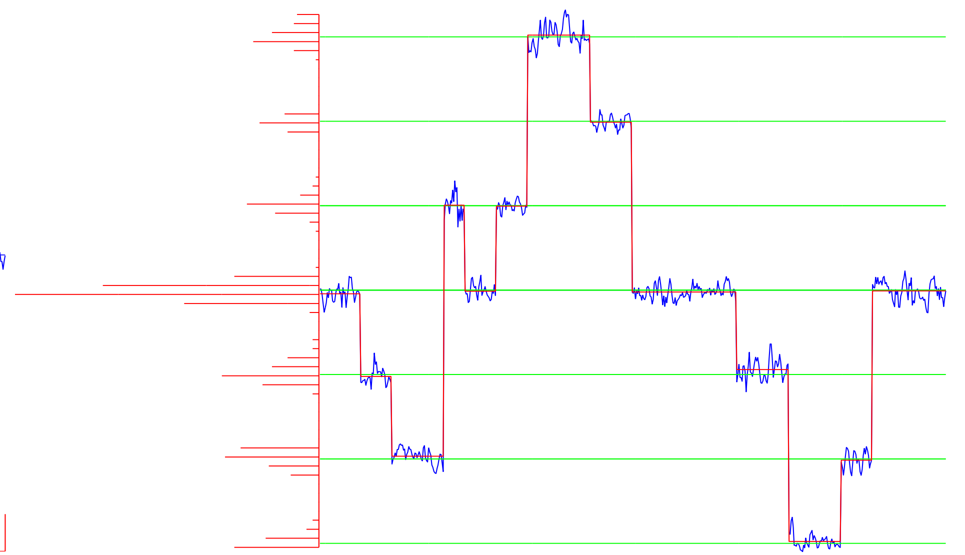
Gauss-Markov



Generalized GM



Piecewize Gaussian



Mixture of GM

Bayesian approach for invers problems

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Forward & errors model $\longrightarrow p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta};\mathcal{M})$
- Prior model $\longrightarrow p(\boldsymbol{f}|\boldsymbol{\theta};\mathcal{M})$
- Bayes: $\longrightarrow p(\boldsymbol{f}|\boldsymbol{\theta},\boldsymbol{g};\mathcal{M}) = \frac{p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta};\mathcal{M})\, p(\boldsymbol{f}|\boldsymbol{\theta};\mathcal{M})}{p(\boldsymbol{g}|\boldsymbol{\theta};\mathcal{M})}$
- MAP or PM solutions:

$$\begin{aligned}\widehat{\boldsymbol{f}}_{MAP} &= \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{g},\boldsymbol{\theta};\mathcal{M})\}\\ &= \arg\min_{\boldsymbol{f}} \{-\ln p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta};\mathcal{M}) - \ln p(\boldsymbol{f}|\boldsymbol{\theta};\mathcal{M})\}\end{aligned}$$

Link with Regularization

$$\widehat{\boldsymbol{f}}_{PM} = \int \boldsymbol{f}\; p(\boldsymbol{f}|\boldsymbol{g},\boldsymbol{\theta};\mathcal{M})\, \mathrm{d}\boldsymbol{f}$$

Full Bayesian approach

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Forward & errors model: $\longrightarrow p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}; \mathcal{M})$
- Prior models $\longrightarrow p(\boldsymbol{f}|\boldsymbol{\theta}; \mathcal{M})$ and $p(\boldsymbol{\theta}|\mathcal{M})$
- Bayes: $\longrightarrow p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M}) = \frac{p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta};\mathcal{M})\, p(\boldsymbol{f}|\boldsymbol{\theta};\mathcal{M})\, p(\boldsymbol{\theta}|\mathcal{M})}{p(\boldsymbol{g}|\mathcal{M})}$
- Joint MAP: $(\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\theta}}) = \arg\max_{(\boldsymbol{f},\boldsymbol{\theta})} \{p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\}$
- Posterior means: $\begin{cases} \widehat{\boldsymbol{f}} &= \int \boldsymbol{f}\, p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\, \mathrm{d}\boldsymbol{f}\, \mathrm{d}\boldsymbol{\theta} \\ \widehat{\boldsymbol{\theta}} &= \int \boldsymbol{\theta}\, p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})\, \mathrm{d}\boldsymbol{f}\, \mathrm{d}\boldsymbol{\theta} \end{cases}$
- Evidence of the model:

$$p(\boldsymbol{g}|\mathcal{M}) = \int p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}; \mathcal{M}) p(\boldsymbol{f}|\boldsymbol{\theta}; \mathcal{M}) p(\boldsymbol{\theta}|\mathcal{M})\, \mathrm{d}\boldsymbol{f}\, \mathrm{d}\boldsymbol{\theta}$$

Bayesian Computation

- Direct computation and use of $p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})$ is too complex
- Approximations :
  - Gauss-Laplace (Gaussian approximation)
  - Exploration (Sampling) using MCMC methods
  - Separable approximation (Variational techniques)
- Main idea:

Approximate $p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})$ by $q(\boldsymbol{f}, \boldsymbol{\theta}) = q_1(\boldsymbol{f})\ q_2(\boldsymbol{\theta})$

  - Choice of approximation criterion
  - Choice of appropriate families of probability laws for $q_1(\boldsymbol{f})$ and $q_2(\boldsymbol{\theta})$

Approximation : Choice of criterion

$$
\begin{aligned}
\ln p(\boldsymbol{g}|\mathcal{M}) &= \ln \iint q(\boldsymbol{f},\boldsymbol{\theta}) \frac{p(\boldsymbol{g},\boldsymbol{f},\boldsymbol{\theta}|\mathcal{M})}{q(\boldsymbol{f},\boldsymbol{\theta})} \,\mathrm{d}\boldsymbol{f}\,\mathrm{d}\boldsymbol{\theta} \\
&\geq \iint q(\boldsymbol{f},\boldsymbol{\theta}) \ln \frac{p(\boldsymbol{g},\boldsymbol{f},\boldsymbol{\theta}|\mathcal{M})}{q(\boldsymbol{f},\boldsymbol{\theta})} \,\mathrm{d}\boldsymbol{f}\,\mathrm{d}\boldsymbol{\theta}.
\end{aligned}
$$

$$
\mathcal{F}(q) = \iint q(\boldsymbol{f},\boldsymbol{\theta}) \ln \frac{p(\boldsymbol{g},\boldsymbol{f},\boldsymbol{\theta}|\mathcal{M})}{q(\boldsymbol{f},\boldsymbol{\theta})} \,\mathrm{d}\boldsymbol{f}\,\mathrm{d}\boldsymbol{\theta}
$$

$$
\mathrm{KL}(q:p) = -\iint q(\boldsymbol{f},\boldsymbol{\theta}) \ln \frac{p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g};\mathcal{M})}{q(\boldsymbol{f},\boldsymbol{\theta})} \,\mathrm{d}\boldsymbol{f}\,\mathrm{d}\boldsymbol{\theta}
$$

$$
\ln p(\boldsymbol{g}|\mathcal{M}) = \mathcal{F}(q) + \mathrm{KL}(q:p)
$$

Approximation : Separable Approximation

$$\ln p(\boldsymbol{g}|\mathcal{M}) = \mathcal{F}(q) + \mathrm{KL}(q:p)$$

$$q(\boldsymbol{f},\boldsymbol{\theta}) = q_1(\boldsymbol{f})\; q_2(\boldsymbol{\theta})$$

$$(\widehat{q}_1,\widehat{q}_2) = \arg\min_{(q_1,q_2)} \{\mathrm{KL}(q_1 q_2 : p)\} = \arg\max_{(q_1,q_2)} \{\mathcal{F}(q_1 q_2)\}$$

$\mathrm{KL}(q_1 q_2 : p)$ is convex in q_1 for a given q_2 and vise versa:

$$\left\{\begin{array}{ll} \widehat{q}_1 & = \arg\min_{q_1} \{\mathrm{KL}(q_1 \widehat{q}_2 : p)\} = \arg\max_{q_1} \{\mathcal{F}(q_1 \widehat{q}_2)\} \\ \widehat{q}_2 & = \arg\min_{q_2} \{\mathrm{KL}(\widehat{q}_1 q_2 : p)\} = \arg\max_{q_2} \{\mathcal{F}(\widehat{q}_1 q_2)\} \end{array}\right.$$

$$\left\{\begin{array}{ll} \widehat{q}_1(\boldsymbol{f}) & \propto \exp\left\{\langle \ln p(\boldsymbol{g},\boldsymbol{f},\boldsymbol{\theta};\mathcal{M})\rangle_{\widehat{q}_2(\boldsymbol{\theta})}\right\} \\ \widehat{q}_2(\boldsymbol{\theta}) & \propto \exp\left\{\langle \ln p(\boldsymbol{g},\boldsymbol{f},\boldsymbol{\theta};\mathcal{M})\rangle_{\widehat{q}_1(\boldsymbol{f})}\right\} \end{array}\right.$$

Separable Approximation :
Choice of appropriate families for q_1 and q_2

- Degenerate case 1 : $\longrightarrow$ Joint MAP

$$\left\{\begin{array}{ll}\widehat{q}_1(\boldsymbol{f}|\widetilde{\boldsymbol{f}}) & =\delta(\boldsymbol{f}-\widetilde{\boldsymbol{f}}) \\ \widehat{q}_2(\boldsymbol{\theta}|\widetilde{\boldsymbol{\theta}}) & =\delta(\boldsymbol{\theta}-\widetilde{\boldsymbol{\theta}})\end{array}\right. \longrightarrow \left\{\begin{array}{l}\widetilde{\boldsymbol{f}}=\arg\max_{\boldsymbol{f}}\left\{p(\boldsymbol{f},\widetilde{\boldsymbol{\theta}}|\boldsymbol{g};\mathcal{M})\right\} \\ \widetilde{\boldsymbol{\theta}}=\arg\max_{\boldsymbol{\theta}}\left\{p(\widetilde{\boldsymbol{f}},\boldsymbol{\theta}|\boldsymbol{g};\mathcal{M})\right\}\end{array}\right.$$

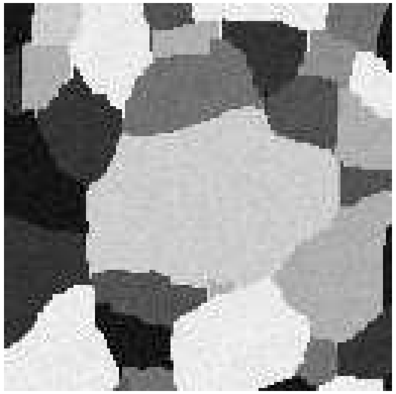
- Degenerate case 2 : $\longrightarrow$ Expectation-Maximisation

$$\left\{\begin{array}{ll}\widehat{q}_1(\boldsymbol{f}) & \propto p(\boldsymbol{f}|\boldsymbol{\theta},\boldsymbol{g}) \\ \widehat{q}_2(\boldsymbol{\theta}|\widetilde{\boldsymbol{\theta}}) & =\delta(\boldsymbol{\theta}-\widetilde{\boldsymbol{\theta}})\end{array}\right. \longrightarrow \left\{\begin{array}{ll}Q(\boldsymbol{\theta},\widetilde{\boldsymbol{\theta}})= & \left\langle\ln p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g};\mathcal{M})\right\rangle_{q_1(\boldsymbol{f}|\widetilde{\boldsymbol{\theta}})} \\ \widetilde{\boldsymbol{\theta}} & =\arg\max_{\boldsymbol{\theta}}\left\{Q(\boldsymbol{\theta},\widetilde{\boldsymbol{\theta}})\right\}\end{array}\right.$$

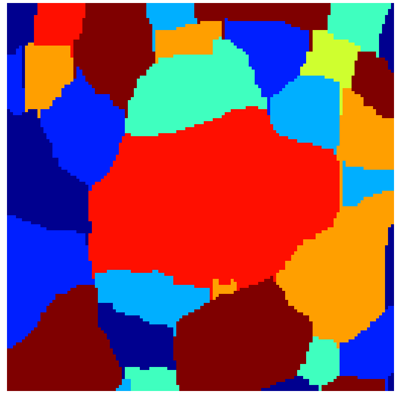
- Appropriate choice for inverse problems

$$\left\{\begin{array}{ll}\widehat{q}_1(\boldsymbol{f}) & \propto p(\boldsymbol{f}|\widetilde{\boldsymbol{\theta}},\boldsymbol{g};\mathcal{M}) \\ \widehat{q}_2(\boldsymbol{\theta}) & \propto p(\boldsymbol{\theta}|\boldsymbol{f},\boldsymbol{g};\mathcal{M})\end{array}\right. \longrightarrow \left\{\begin{array}{l}\text{Accounting for uncertainties} \\ \text{of } \widehat{\boldsymbol{\theta}} \text{ in } \widehat{\boldsymbol{f}} \text{ and vise versa.}\end{array}\right.$$

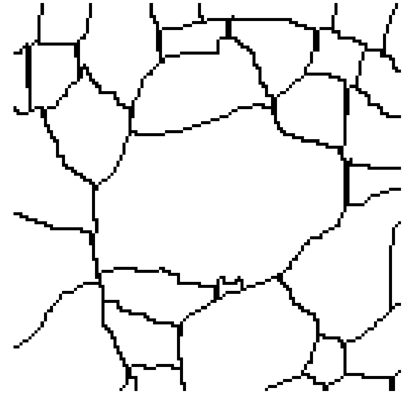
Gauss-Markov-Potts prior models for images



$f(\boldsymbol{r})$



$z(\boldsymbol{r})$



$c(\boldsymbol{r}) = 1 - \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))$

$$p(f(\boldsymbol{r})|z(\boldsymbol{r}) = k, m_k, v_k) = \mathcal{N}(m_k, v_k)$$

$$p(f) = \sum_k P(z = k)\, \mathcal{N}(m_k, v_k) \quad \text{Mixture of Gaussians}$$

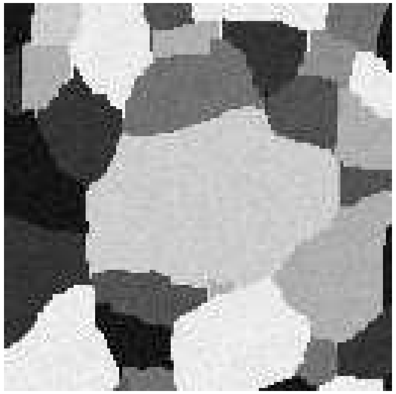
Separable iid hidden variables: $p(\boldsymbol{z}) = \prod_{\boldsymbol{r}} p(z(\boldsymbol{r}))$

Markovian hidden variables: $p(\boldsymbol{z})$ Potts-Markov

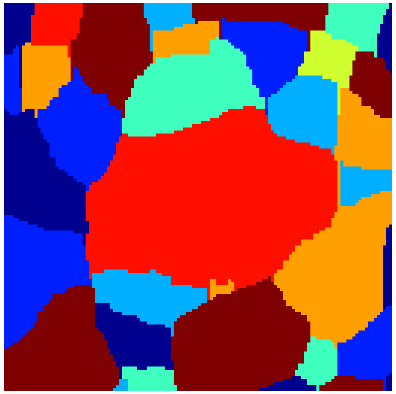
$$p(z(\boldsymbol{r})|z(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) \propto \exp\left\{ \gamma \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}')) \right\}$$

Four different cases

- $\boldsymbol{f}|\boldsymbol{z}$ iid, $\quad \boldsymbol{z}$ iid :
  Classical case of
  Mixture of Gaussians
- $\boldsymbol{f}|\boldsymbol{z}$ Markov, $\quad \boldsymbol{z}$ iid :
  (Markov composite)
  Mixture of Gauss-Markov
- $\boldsymbol{f}|\boldsymbol{z}$ iid, $\quad \boldsymbol{z}$ Markov :
  (Hidden Potts-Markov)
  Gauss-Potts
- $\boldsymbol{f}|\boldsymbol{z}$ Markov, $\quad \boldsymbol{z}$ Markov :
  (Gauss-Markov-Potts)



$f(\boldsymbol{r})$



$z(\boldsymbol{r})$

Case 1: $f|z$ iid, z iid

Independent Mixture of Independent Gaussiens (IMIG):

$p(f(\boldsymbol{r})|z(\boldsymbol{r})=k)=\mathcal{N}(m_k,v_k), \forall \boldsymbol{r}\in\mathcal{R}$

$p(f(\boldsymbol{r}))=\sum_{k=1}^K \alpha_k \mathcal{N}(m_k,v_k)$, with $\sum_k \alpha_k=1$.

$p(\boldsymbol{z})=\prod_{\boldsymbol{r}} p(z(\boldsymbol{r})=k)=\prod_{\boldsymbol{r}}\alpha_k=\prod_k \alpha_k^{n_k}$

Noting

$$m_z(\boldsymbol{r})=m_k, v_z(\boldsymbol{r})=v_k, \alpha_z(\boldsymbol{r})=\alpha_k, \forall \boldsymbol{r}\in\mathcal{R}_k$$

we have:

$$p(\boldsymbol{f}|\boldsymbol{z}) = \prod_{\boldsymbol{r}\in\mathcal{R}} \mathcal{N}(m_z(\boldsymbol{r}), v_z(\boldsymbol{r}))$$

$$p(\boldsymbol{z}) = \prod_{\boldsymbol{r}} \alpha_z(\boldsymbol{r}) = \prod_k \alpha_k^{\sum_{\boldsymbol{r}\in\mathcal{R}}\delta(z(\boldsymbol{r})-k)} = \prod_k \alpha_k^{n_k}$$

Case 2: $f|z$ Gauss-Markov, z iid

Independent Mixture of Gauss-Markov (IMGM):

$$p(f(\boldsymbol{r})|z(\boldsymbol{r}), z(\boldsymbol{r}'), f(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) = \mathcal{N}(\mu_z(\boldsymbol{r}), v_z(\boldsymbol{r})), \forall \boldsymbol{r} \in \mathcal{R}$$

$$\begin{aligned} \mu_z(\boldsymbol{r}) &= \frac{1}{|\mathcal{V}(\boldsymbol{r})|} \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \mu_z^*(\boldsymbol{r}') \\ \mu_z^*(\boldsymbol{r}') &= \delta(z(\boldsymbol{r}') - z(\boldsymbol{r}))\, f(\boldsymbol{r}') + (1 - \delta(z(\boldsymbol{r}') - z(\boldsymbol{r}))\, m_z(\boldsymbol{r}') \\ &= \quad (1 - c(\boldsymbol{r}')) \quad f(\boldsymbol{r}') \; + \qquad c(\boldsymbol{r}') \qquad m_z(\boldsymbol{r}') \end{aligned}$$

$$\begin{aligned} p(\boldsymbol{f}|\boldsymbol{z}) &\propto \prod_{\boldsymbol{r}} \mathcal{N}(\mu_z(\boldsymbol{r}), v_z(\boldsymbol{r})) &\propto \prod_k \alpha_k \mathcal{N}(m_k 1, \boldsymbol{\Sigma}_k) \\ p(\boldsymbol{z}) &= \prod_{\boldsymbol{r}} v_z(\boldsymbol{r}) &= \prod_k \alpha_k^{n_k} \end{aligned}$$

with $1_k = 1, \forall \boldsymbol{r} \in \mathcal{R}_k$ and $\boldsymbol{\Sigma}_k$ a covariance matrix $(n_k \times n_k)$.

Case 3: $\boldsymbol{f}|\boldsymbol{z}$ Gauss iid, $\boldsymbol{z}$ Potts

Gauss iid as in Case 1:

$$p(\boldsymbol{f}|\boldsymbol{z}) = \prod_{\boldsymbol{r}\in\mathcal{R}} \mathcal{N}(m_z(\boldsymbol{r}), v_z(\boldsymbol{r})) = \prod_{k} \prod_{\boldsymbol{r}\in\mathcal{R}_k} \mathcal{N}(m_k, v_k)$$

Potts-Markov

$$p(z(\boldsymbol{r})|z(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) \propto \exp\left\{\gamma \sum_{\boldsymbol{r}'\in\mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

$$p(\boldsymbol{z}) \propto \exp\left\{\gamma \sum_{\boldsymbol{r}\in\mathcal{R}} \sum_{\boldsymbol{r}'\in\mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

Case 4: $f|z$ Gauss-Markov, z Potts

Gauss-Markov as in Case 2:

$$p(f(\boldsymbol{r})|z(\boldsymbol{r}), z(\boldsymbol{r}'), f(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) = \mathcal{N}(\mu_z(\boldsymbol{r}), v_z(\boldsymbol{r})), \forall \boldsymbol{r} \in \mathcal{R}$$

$$\begin{aligned} \mu_z(\boldsymbol{r}) &= \frac{1}{|\mathcal{V}(\boldsymbol{r})|} \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \mu_z^*(\boldsymbol{r}') \\ \mu_z^*(\boldsymbol{r}') &= \delta(z(\boldsymbol{r}') - z(\boldsymbol{r}))\, f(\boldsymbol{r}') + (1 - \delta(z(\boldsymbol{r}') - z(\boldsymbol{r}))\, m_z(\boldsymbol{r}') \end{aligned}$$

$$p(\boldsymbol{f}|\boldsymbol{z}) \propto \prod_{\boldsymbol{r}} \mathcal{N}(\mu_z(\boldsymbol{r}), v_z(\boldsymbol{r})) \propto \prod_k \alpha_k \mathcal{N}(m_k 1, \boldsymbol{\Sigma}_k)$$

Potts-Markov as in Case 3:

$$p(\boldsymbol{z}) \propto \exp\left\{ \gamma \sum_{\boldsymbol{r} \in \mathcal{R}} \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}')) \right\}$$

Bayesian computation with Gauss-Markov-Potts prior models

$$p(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\boldsymbol{g})=\frac{p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta})\;p(\boldsymbol{f}|\boldsymbol{z},\boldsymbol{\theta})\;p(\boldsymbol{z})}{p(\boldsymbol{g}|\boldsymbol{\theta})}$$

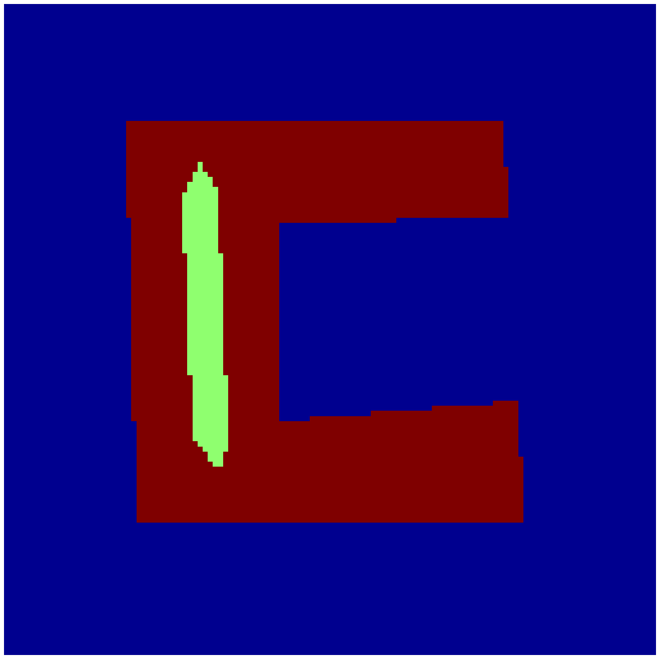
Approximations :

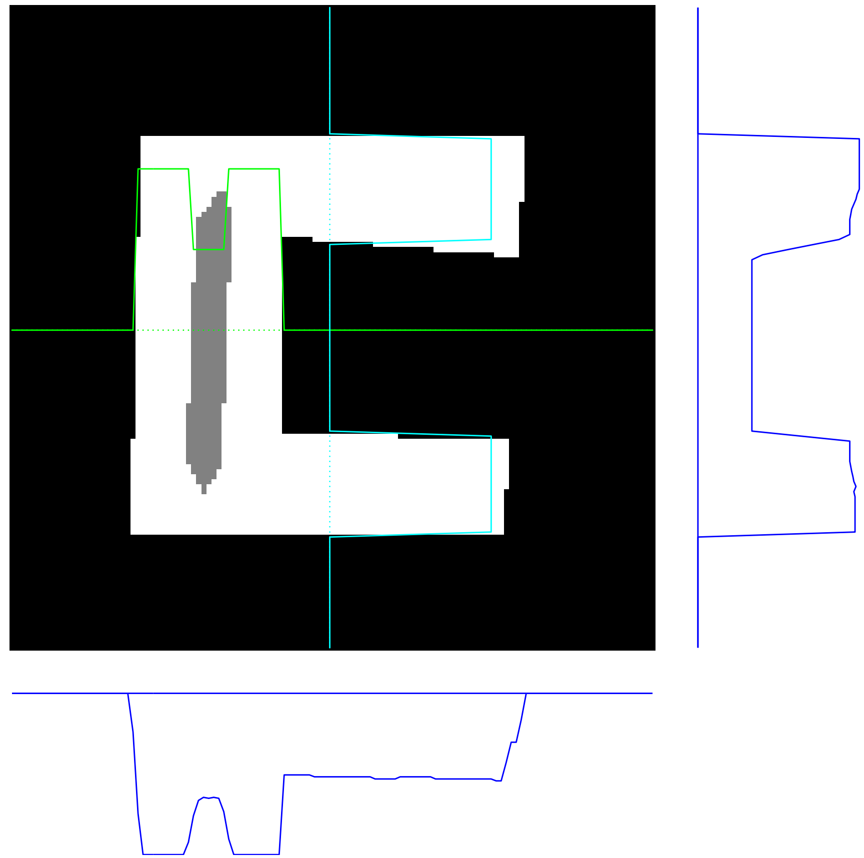
- $\boldsymbol{f}|\boldsymbol{z}$ iid, $\boldsymbol{z}$ iid :
  $q(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\boldsymbol{g})=q_1(\boldsymbol{f}|\boldsymbol{z})\;q_2(\boldsymbol{z})\;q_3(\boldsymbol{\theta})$.
- $\boldsymbol{f}|\boldsymbol{z}$ iid, $\boldsymbol{z}$ Markov :
  $q(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\boldsymbol{g})=q_1(\boldsymbol{f}|\boldsymbol{z})\;q_{2w}(\boldsymbol{z}_w)\;q_{2b}(\boldsymbol{z}_b)\;q_3(\boldsymbol{\theta})$.
- $\boldsymbol{f}|\boldsymbol{z}$ Markov, $\boldsymbol{z}$ iid :
  $q(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\boldsymbol{g})=q_{1w}(\boldsymbol{f}_w|\boldsymbol{z})\;q_{1b}(\boldsymbol{f}_b|\boldsymbol{z})\;q_2(\boldsymbol{z})\;q_3(\boldsymbol{\theta})$.
- $\boldsymbol{f}|\boldsymbol{z}$ Markov, $\boldsymbol{z}$ iid :

$$q(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\boldsymbol{g})=q_{1w}(\boldsymbol{f}_w|\boldsymbol{z})\;q_{1b}(\boldsymbol{f}_b|\boldsymbol{z})\;q_{2w}(\boldsymbol{z}_w)\;q_{2b}(\boldsymbol{z}_b)\;q_3(\boldsymbol{\theta})$$

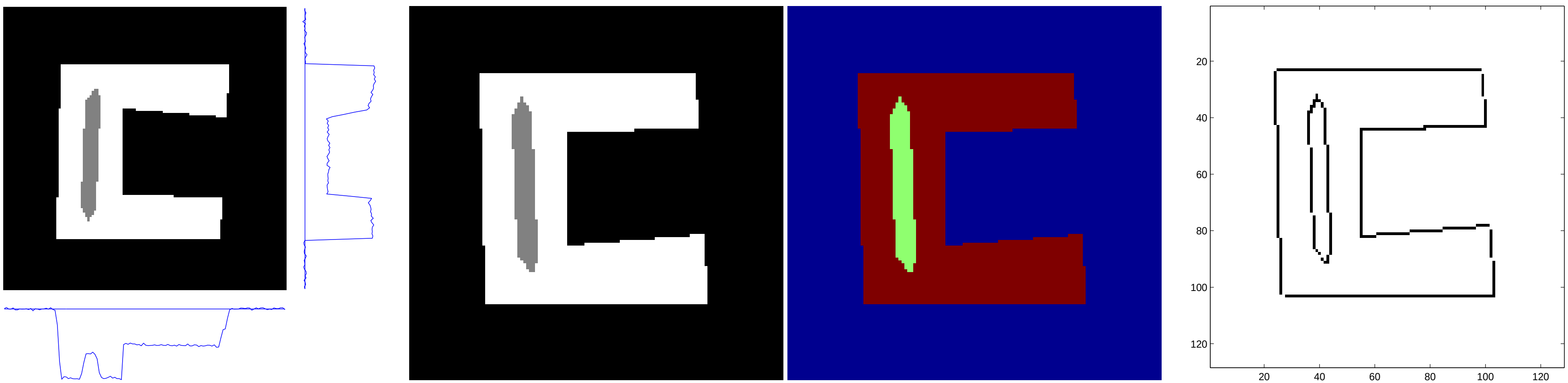
Application of CT in NDT

Reconstruction from only 2 projections





Application in CT



$\boldsymbol{g}\vert\boldsymbol{f}$	$\boldsymbol{f}\vert\boldsymbol{z}$	$\boldsymbol{z}$	$\boldsymbol{c}$
$\boldsymbol{g}=\boldsymbol{H}\boldsymbol{f}+\boldsymbol{\epsilon}$	iid Gaussian	iid	$c(\boldsymbol{r})\in\{0,1\}$
$\boldsymbol{g}\vert\boldsymbol{f}\sim\mathcal{N}(\boldsymbol{H}\boldsymbol{f},v_\epsilon\boldsymbol{I})$	or	or	$1-\delta(z(\boldsymbol{r})-z(\boldsymbol{r}'))$
Gaussian	Gauss-Markov	Potts	binary

Proposed algorithm

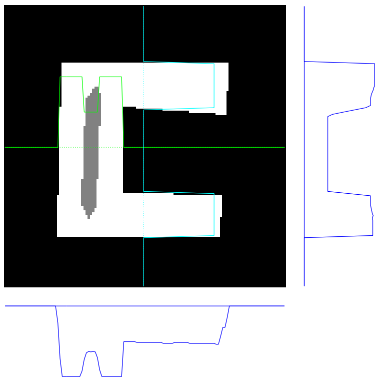
$$p(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta})\, p(\boldsymbol{f}|\boldsymbol{z},\boldsymbol{\theta})\, p(\boldsymbol{\theta})$$

General scheme:

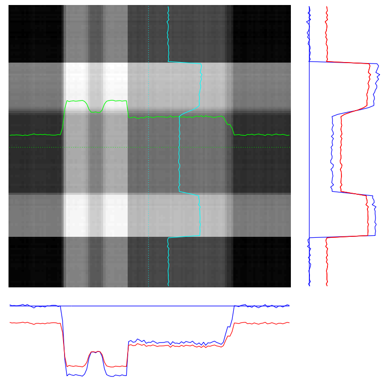
$$\widehat{\boldsymbol{f}} \sim p(\boldsymbol{f}|\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\theta}},\boldsymbol{g}) \longrightarrow \widehat{\boldsymbol{z}} \sim p(\boldsymbol{z}|\widehat{\boldsymbol{f}},\widehat{\boldsymbol{\theta}},\boldsymbol{g}) \longrightarrow \widehat{\boldsymbol{\theta}} \sim (\boldsymbol{\theta}|\widehat{\boldsymbol{f}},\widehat{\boldsymbol{z}},\boldsymbol{g})$$

- Estimate $\boldsymbol{f}$ using $p(\boldsymbol{f}|\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\theta}},\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta})\, p(\boldsymbol{f}|\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\theta}})$
  Needs optimisation of a quadratic criterion.
- Estimate $\boldsymbol{z}$ using $p(\boldsymbol{z}|\widehat{\boldsymbol{f}},\widehat{\boldsymbol{\theta}},\boldsymbol{g}) \propto p(\boldsymbol{g}|\widehat{\boldsymbol{f}},\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\theta}})\, p(\boldsymbol{z})$
  Needs sampling of a Potts Markov field.
- Estimate $\boldsymbol{\theta}$ using
  $p(\boldsymbol{\theta}|\widehat{\boldsymbol{f}},\widehat{\boldsymbol{z}},\boldsymbol{g}) \propto p(\boldsymbol{g}|\widehat{\boldsymbol{f}},\sigma_\epsilon^2\boldsymbol{I})\, p(\widehat{\boldsymbol{f}}|\widehat{\boldsymbol{z}},(m_k,v_k))\, p(\boldsymbol{\theta})$
  Conjugate priors $\longrightarrow$ analytical expressions.

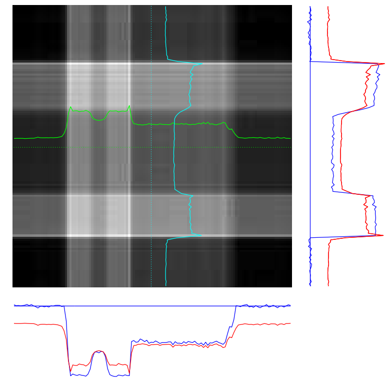
Results



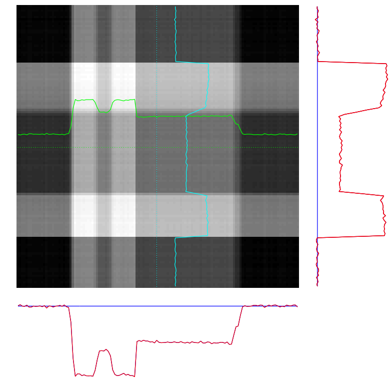
Original



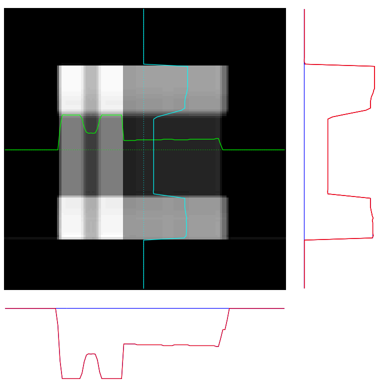
Backprojection



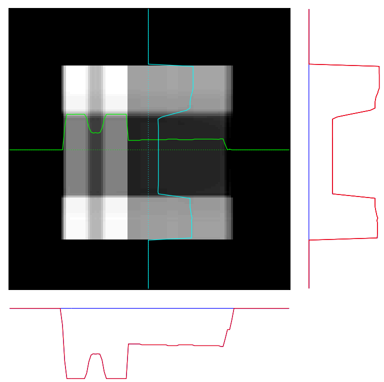
Filtered BP



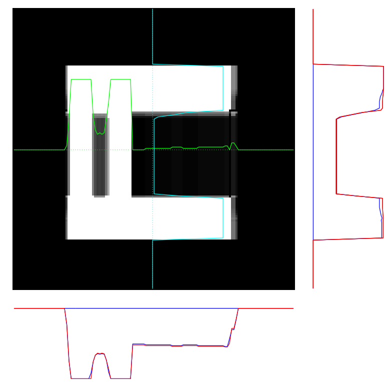
LS



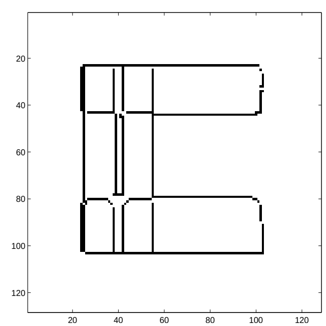
Gauss-Markov+pos



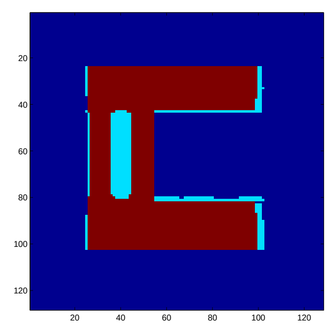
GM+Line process



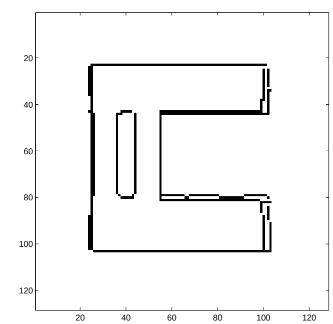
GM+Label process



c



z



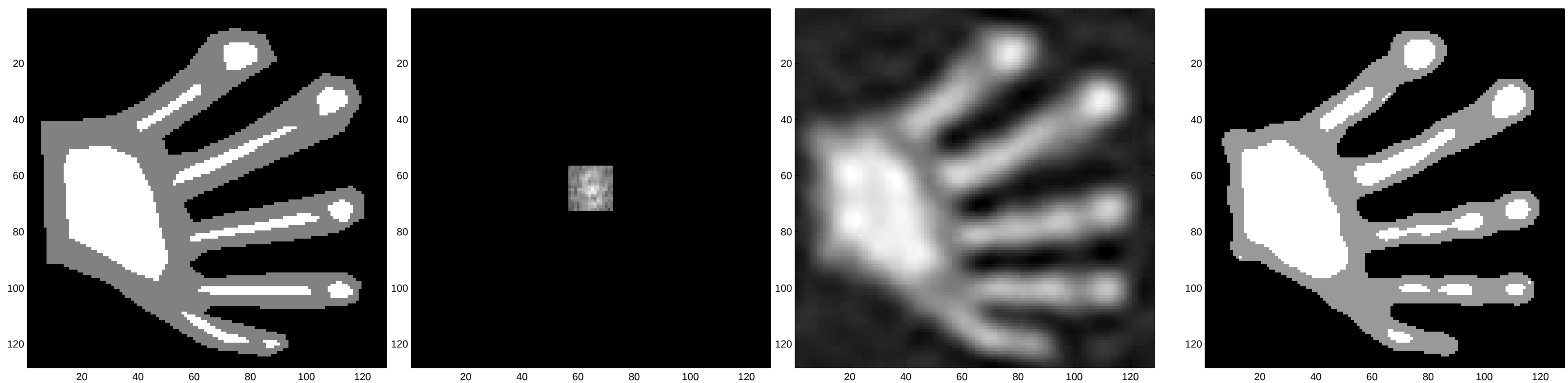
c

Application in Microwave imaging

$$g(\boldsymbol{\omega}) = \int f(\boldsymbol{r}) \exp\{-j(\boldsymbol{\omega}.\boldsymbol{r})\} \; \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{\omega})$$

$$g(u,v) = \int f(x,y) \exp\{-j(ux+vy)\} \; \mathrm{d}x \, \mathrm{d}y + \epsilon(u,v)$$

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$



$f(x,y)$ $g(u,v)$ $\widehat{\boldsymbol{f}}$ IFT $\widehat{\boldsymbol{f}}$ Proposed method

Conclusions

- Bayesian Inference for inverse problems
- Approximations (Laplace, MCMC, Variational)
- Gauss-Markov-Potts are useful prior models for images incorporating regions and contours
- Separable approximations for Joint posterior with Gauss-Markov-Potts priors
- Application in different CT (X ray, US, Microwaves, PET, SPECT)

Perspectives :

- Efficient implementation in 2D and 3D cases
- Evaluation of performances and comparison with MCMC

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Questions and Discussions

- Thanks for your attentions
- ...
- ...
- Questions ?
- Discussions ?
- ...
- ...