Relaxation of Non-Convex Pointwise Gradient Constrained Energies and Applications to Differential Inclusions and Hamilton-Jacobi Equations

Riccardo De Arcangelis

University of Naples Federico II

Dipartimento di Matematica e Applicazioni Renato Caccioppoli

Rate-Independence, Homogenization and Multiscaling

Centro di Ricerca Matematica Ennio De Giorgi

Pisa

November 17, 2007

Relaxation of homogeneous energies

$f\colon \mathbf{R}^n \to [0, +\infty[$ Borel,

Ω a smooth bounded open subset of $\mathbf{R}^n$.

Relaxation problems for the integral energy

$$F\colon u \in U \mapsto \int_\Omega f(\nabla u)\,\mathrm{d}\mathcal{L}^n,$$

where $U = W^{1,\infty}(\Omega)$ (Neumann problem), $U = u_0 + W_0^{1,\infty}(\Omega)$ (Dirichlet problem).

Relaxation process has been widely developed in the last decades in various frameworks, and under different assumptions on f.

Books:

Morrey, Ekeland-Temam, Attouch, Buttazzo, Dacorogna, Dal Maso, Carbone-De Arcangelis, Fonseca-Leoni, ...

Topology $= L^1(\Omega)$.

Two main $L^1(\Omega)$-lower semicontinuous envelope of F corresponding to Neumann and Dirichlet problems

$$\overline{F}\colon u \in L^1(\Omega) \mapsto \min \left\{ \liminf_{h \to +\infty} \int_\Omega f(\nabla u_h) \, \mathrm{d}\mathcal{L}^n : \{u_h\} \subseteq W^{1,\infty}(\Omega), \ u_h \to u \text{ in } L^1(\Omega) \right\},$$

$$\overline{F_0}\colon u \in L^1(\Omega) \mapsto \min \left\{ \liminf_{h \to +\infty} \int_\Omega f(\nabla u_h) \, \mathrm{d}\mathcal{L}^n : \{u_h\} \subseteq u_0 + W_0^{1,\infty}(\Omega), \ u_h \to u \text{ in } L^1(\Omega) \right\}.$$

Integral representation properties of $\overline{F}$ and $\overline{F_0}$

Under **convexity** assumptions on f, case of the Neumann problem in $BV(\Omega)$

U a Sobolev space, or a space of smooth functions,

$$\overline{F}(u) = \int_\Omega f(\nabla u)\,\mathrm{d}\mathcal{L}^n + \int_\Omega f^\infty\left(\frac{\mathrm{d}D^{\mathrm{s}}u}{\mathrm{d}|D^{\mathrm{s}}u|}\right)\mathrm{d}|D^{\mathrm{s}}u| \text{ for every } u \in BV(\Omega),$$

where (z_0 any point in $\mathbf{R}^n$)

$$f^\infty\colon z \in \mathbf{R}^n \mapsto \lim_{t\to+\infty} \frac{f(z_0+tz)-f(z_0)}{t}.$$

The case of the Dirichlet problem in $BV(\Omega)$

$$\overline{F_0}(u) = \int_\Omega f(\nabla u)\,\mathrm{d}\mathcal{L}^n + \int_\Omega f^\infty\left(\frac{\mathrm{d}D^{\mathrm{s}}u}{\mathrm{d}|D^{\mathrm{s}}u|}\right)\mathrm{d}|D^{\mathrm{s}}u| + \int_{\partial\Omega} f^\infty((u_{z_0}-u)\mathbf{n}_\Omega)\,\mathrm{d}\mathcal{H}^{n-1}$$
$$\text{for every } u \in BV(\Omega).$$

When f is **not convex**, it turns out that

$$\overline{F}(u) = \int_\Omega \mathrm{co}f(\nabla u)\, \mathrm{d}\mathcal{L}^n \text{ for every } u \in W^{1,\infty}(\Omega),$$

$$\overline{F_0}(u) = \int_\Omega \mathrm{co}f(\nabla u)\, \mathrm{d}\mathcal{L}^n \text{ for every } u \in W_0^{1,\infty}(\Omega),$$

where $\mathrm{co}f$ is the **convex envelope** of f defined as

$$\mathrm{co}f \colon z \in \mathbf{R}^n \mapsto \sup\{\phi(z) : \phi \colon \mathbf{R}^n \to [0,+\infty] \text{ convex, } \phi(\zeta) \leq f(\zeta) \text{ for every } \zeta \in \mathbf{R}^n\}.$$

Results in the same spirit hold also in different settings. For example, when f is defined on the set of the $n \times m$ matrices and the elements of U are $\mathbf{R}^m$-valued, the above formulas still holds provided $\mathrm{co}f$ is replaced by the **quasiconvex envelope** of f.

In the above results the gradients of the elements of U are allowed to lie in the whole of $\mathbf{R}^n$ without any restriction. When this does not occur, namely when a condition like

$$\nabla u(x) \in E \text{ for a.e. } x \in \Omega,$$

must be fulfilled by the elements of U for some given subset E of $\mathbf{R}^n$, the corresponding relaxation processes become **pointwise gradient constrained**. The treatment of this case can be handled by allowing the value $+\infty$ in the target space of f. Indeed, in this case the only elements of U that play a role are those that satisfy the following pointwise gradient constraint

$$\nabla u(x) \in \mathrm{dom} f \text{ for a.e. } x \in \Omega,$$

where $\mathrm{dom} f = \{z \in \mathbf{R}^n : f(z) < +\infty\}$.

Constraint conditions can be very **restrictive**, entailing **serious technical difficulties** and hindering the development of a wide range of results like those described in the unconstrained case.

Gradient constrained convex homogenization processes

Carbone L., Corbo Esposito A., De Arcangelis R.: *Homogenization of Neumann Problems for Unbounded Functionals*; Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8), **2**-B, (1999), 463–491,

Carbone L., Cioranescu D., De Arcangelis R., Gaudiello A.: *Homogenization of Unbounded Functionals and Nonlinear Elastomers. The Case of the Fixed Constraints Set*; ESAIM Control Optim. Calc. Var. **10**, 1, (2004), 53–83,

Cioranescu D., Damlamian A., De Arcangelis R.: *Homogenization of Integrals with Pointwise Gradient Constraints via the Periodic Unfolding Method*; Ric. Mat. **55**, (2006), 31–53.

Few results exist in literature on pointwise **gradient constrained relaxation** for **non-convex** f with **convex** domain (in some cases just a ball).

Ekeland I., Temam R.: "Convex Analysis and Variational Problems"; Stud. Math. Appl. **1**, North-Holland, Amsterdam (1976),

Marcellini P., Sbordone C.: *Semicontinuity Problems in the Calculus of Variations*; Nonlinear Anal. **4**, (1980), 241–257,

Carbone L., De Arcangelis R.: *On the Relaxation of Some Classes of Unbounded Integral Functionals*; Matematiche **51**, (1996), 221–256,

Carbone L., De Arcangelis R.: *On the Relaxation of Dirichlet Minimum Problems for Some Classes of Unbounded Integral Functionals*; Ricerche Mat. **48**-Suppl., (1999), 347–372,

Carbone L., De Arcangelis R.: *On a Non-Standard Convex Regularization and the Relaxation of Unbounded Functionals of the Calculus of Variations*; J. Convex Anal. **6**, (1999), 141–162,

Carbone L., De Arcangelis R.: "Unbounded Functionals in the Calculus of Variations. Representation, Relaxation, and Homogenization"; Chapman & Hall/CRC Monogr. Surv. Pure Appl. Math. **125**, Chapman & Hall/CRC, Boca Raton, FL (2001).

Representation formulas

$f\colon \mathbf{R}^n \to [0,+\infty]$ Borel, Ω convex bounded open subset of $\mathbf{R}^n$,

$$\overline{F}(u) = \int_\Omega f^{**}(\nabla u)\,\mathrm{d}\mathcal{L}^n + \int_\Omega (f^{**})^\infty \left(\frac{\mathrm{d}D^{\mathrm{s}}u}{\mathrm{d}|D^{\mathrm{s}}u|}\right) \mathrm{d}|D^{\mathrm{s}}u| \text{ for every } u \in BV(\Omega),$$

$$\overline{F_0}(u) = \int_\Omega f^{**}(\nabla u)\,\mathrm{d}\mathcal{L}^n + \int_\Omega (f^{**})^\infty \left(\frac{\mathrm{d}D^{\mathrm{s}}u}{\mathrm{d}|D^{\mathrm{s}}u|}\right) \mathrm{d}|D^{\mathrm{s}}u| + \int_{\partial\Omega} (f^{**})^\infty((u_{z_0} - u)\mathbf{n}_\Omega)\,\mathrm{d}\mathcal{H}^{n-1}$$

$$\text{for every } u \in BV(\Omega),$$

where f^{**} is the **convex lower semicontinuous envelope** of f defined as

$$f^{**}\colon z \in \mathbf{R}^n \mapsto \sup\{\phi(z) : \phi\colon \mathbf{R}^n \to [0,+\infty] \text{ convex and lower semicontinuous},$$

$$\phi(\zeta) \le f(\zeta) \text{ for every } \zeta \in \mathbf{R}^n\}.$$

Some cases in which $\mathrm{dom} f$ has empty interior have been treated.

Remark

$$f^{**}(z) = \mathrm{sc}^-(\mathrm{co}f)(z).$$

$$\inf\left\{\liminf_{h\to+\infty}\int_\Omega f(\nabla u_h)\,\mathrm{d}\mathcal{L}^n : \{u_h\}\subseteq W^{1,\infty}(\Omega),\ u_h\to u \text{ in weak*-}W^{1,\infty}(\Omega),\right.$$
$$\left.\nabla u_h(x)\in \mathrm{dom}f \text{ for a.e. } x\in\Omega\right\} = \int_\Omega \mathrm{co}(\mathrm{sc}^- f)(\nabla u)\,\mathrm{d}\mathcal{L}^n.$$

$$\mathrm{co}(\mathrm{sc}^- f) \neq \mathrm{sc}^-(\mathrm{co}f).$$

This feature does not occur if f is only real valued.

Sufficient conditions for identity between $\mathrm{co}(\mathrm{sc}^- f)$ and $\mathrm{sc}^-(\mathrm{co}f)$.

If $\lim_{z\to\infty} f(z)/|z| = +\infty$, then $\mathrm{co}(\mathrm{sc}^- f) = \mathrm{sc}^-(\mathrm{co}f)$.

All these papers assume the structure assumption

(Cd) $\qquad$ $\mathrm{dom}f$ is **convex**.

In particular, the treatment of the case in which the gradients of the admissible configurations lie in disconnected or finite sets cannot be approached in this context.

Pointwise gradient constrained relaxation processes when assumption (Cd) is dropped.

De Arcangelis R.: *On the Relaxation of Some Classes of Pointwise Gradient Constrained Energies*; Ann. Inst. H. Poincaré Anal. Non Linéaire **24**, (2007), 113–137.

Very little is known on this problem, **the measure theoretic techniques developed in the above mentioned papers seem not to be well suited for this case**.

New approach allows us to treat both the cases of Neumann and Dirichlet problems.

Gradient constrained Neumann problems

$U = W^{1,\infty}(\Omega)$,

Theorem. *Let* $f\colon \mathbf{R}^n \to [0,+\infty]$ *be Borel. Then*

$$\overline{F}(u) = \int_\Omega f^{**}(\nabla u)\,\mathrm{d}\mathcal{L}^n + \int_\Omega (f^{**})^\infty \left(\frac{\mathrm{d}D^{\mathrm{s}}u}{\mathrm{d}|D^{\mathrm{s}}u|}\right) \mathrm{d}|D^{\mathrm{s}}u| \text{ for every } u \in BV(\Omega)$$

for every convex bounded open set Ω, $u \in BV(\Omega)$.

Of course, the above formula agrees with the above recalled one established under (Cd).

No need to assume any topological or geometrical condition on $\mathrm{dom} f$.

The constraint condition involved in the relaxed problem, at least on Sobolev functions, is given by

$$\nabla u(x) \in \overline{\mathrm{co}(\mathrm{dom} f)} \text{ for a.e. } x \in \Omega.$$

Corollary. *Let* $g\colon \mathbf{R}^n \to [0,+\infty[$ *be Borel, and let* E *be a Borel subset of* $\mathbf{R}^n$*. Then*

$$\inf\left\{ \liminf_{h\to+\infty} \int_\Omega g(\nabla u_h)\,\mathrm{d}\mathcal{L}^n : \{u_h\} \subseteq W^{1,\infty}(\Omega),\ \nabla u_h(x) \in E \textit{ for every } h \in \mathbf{N} \textit{ and a.e.}\right.$$

$$\left. x \in \Omega,\ u_h \to u \textit{ in } L^1(\Omega) \right\} = \int_\Omega (g+I_E)^{**}(\nabla u)\,\mathrm{d}\mathcal{L}^n + \int_\Omega ((g+I_E)^{**})^\infty \left(\frac{\mathrm{d}D^{\mathrm{s}}u}{\mathrm{d}|D^{\mathrm{s}}u|}\right)\mathrm{d}|D^{\mathrm{s}}u|$$

$$\textit{for every } \Omega \in \mathcal{A}_0(\mathbf{R}^n) \textit{ convex},\ u \in BV(\Omega).$$

Gradient constrained Dirichlet problems

$$\mathrm{int}(\mathrm{co}(\mathrm{dom} f)) \neq \emptyset$$

Proposition. *Let* $f\colon \mathbf{R}^n \to [0,+\infty]$ *be Borel with* $\mathrm{int}(\mathrm{co}(\mathrm{dom} f)) = \emptyset$. *Let* Ω *be a bounded open set,* $u_0 \in W^{1,\infty}_{\mathrm{loc}}(\mathbf{R}^n)$ *satisfy* $\nabla u_0(x) \in \mathrm{aff}(\mathrm{dom} f)$ *for a.e.* $x \in \Omega$. *Then, for every* $u \in L^1(\Omega)$,

$$\overline{F_0}(\Omega, u) = \begin{cases} \int_\Omega f(\nabla u)\,\mathrm{d}\mathcal{L}^n & \text{if } u = u_0 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem. *Let* $f\colon \mathbf{R}^n \to [0,+\infty]$ *be Borel with* $\mathrm{int}(\mathrm{co}(\mathrm{dom} f)) \neq \emptyset$. *Let* $z_0 \in \mathrm{int}(\mathrm{co}(\mathrm{dom} f))$. *Then*

$$\overline{F_0}(\Omega, u) = \int_\Omega f^{**}(\nabla u)\,\mathrm{d}\mathcal{L}^n + \int_\Omega (f^{**})^\infty\left(\frac{\mathrm{d}D^{\mathrm{s}}u}{\mathrm{d}|D^{\mathrm{s}}u|}\right)\mathrm{d}|D^{\mathrm{s}}u| + \int_{\partial\Omega} (f^{**})^\infty((u_{z_0} - u)\mathbf{n}_\Omega)\,\mathrm{d}\mathcal{H}^{n-1}$$

for every convex bounded open set Ω, $u \in BV(\Omega)$.

Corollary. *Let* $g\colon \mathbf{R}^n \to [0, +\infty[$ *be Borel, and let* E *be a Borel subset of* $\mathbf{R}^n$ *with* $\mathrm{int}(\mathrm{co}(E)) \neq \emptyset$. *Let* $z_0 \in \mathrm{int}(\mathrm{co}(E))$. *Then*

$$\inf\Big\{ \liminf_{h\to+\infty} \int_\Omega g(\nabla u_h)\,\mathrm{d}\mathcal{L}^n : \{u_h\} \subseteq u_{z_0} + W_0^{1,\infty}(\Omega),$$

$$\nabla u_h(x) \in E \text{ for every } h \in \mathbf{N} \text{ and a.e. } x \in \Omega,\ u_h \to u \text{ in } L^1(\Omega)\Big\} =$$

$$= \int_\Omega (g + I_E)^{**}(\nabla u)\,\mathrm{d}\mathcal{L}^n + \int_\Omega ((g + I_E)^{**})^\infty \left(\frac{\mathrm{d}D^{\mathrm{s}}u}{\mathrm{d}|D^{\mathrm{s}}u|}\right) \mathrm{d}|D^{\mathrm{s}}u| +$$

$$+ \int_{\partial\Omega} ((g + I_E)^{**})^\infty ((u_{z_0} - u)\mathbf{n}_\Omega)\,\mathrm{d}\mathcal{H}^{n-1}$$

$$\text{for every } \Omega \in \mathcal{A}_0(\mathbf{R}^n) \text{ convex},\ u \in BV(\Omega).$$

Pointwise gradient constrained relaxation problems are related to **first order differential inclusions** and **Hamilton-Jacobi equations**.

Ω open subset of $\mathbf{R}^n$, E any subset of $\mathbf{R}^n$, $H\colon \mathbf{R}^n \to \mathbf{R}$.

$$\nabla u(x) \in E \text{ for a.e. } x \in \Omega,$$

$$H(\nabla u(x)) = 0 \text{ for a.e. } x \in \Omega \text{ (eikonal-type Hamilton-Jacobi equation)},$$

the two frameworks are equivalent provided $E = \{z \in \mathbf{R}^n : H(z) = 0\}$.

$$f = I_E,$$

where E is a Borel subset of $\mathbf{R}^n$, and

$$I_E(z) = \begin{cases} 0 & \text{if } z \in E \\ +\infty & \text{if } z \in \mathbf{R}^n \setminus E. \end{cases}$$

Literature on existence in the scalar and vector-valued case by A. Bressan, F. Flores, A. Cellina, M.G. Crandall, L.C. Evans, P.-L. Lions, I. Capuzzo-Dolcetta, G. Friesecke, B. Kirchheim, S. Müller, V. Šverák, M.A. Sychev, B. Dacorogna, P. Marcellini, . . .

Existence but not uniqueness since, in general, the set of the solutions can even be so large to verify a density property.

Criteria to select among the solutions.

De Arcangelis R.: *First Order Differential Inclusions and Hamilton-Jacobi Equations with Applications to Selection Criteria of the Solutions*; to appear on Boll. Unione Mat. Ital. (2008)

A selection criterium based on a mass optimization type principle is proposed.

The criterium selects those solutions, possibly coupled with suitable boundary conditions, that minimize (or maximize) a given integral functional G, for example of the type

$$G(u) = \int_\Omega g(x, u(x))\, \mathrm{d}x,$$

where g is a Carathéodory integrand.

$$Sol(\Omega, E) = \{u \in W^{1,\infty}(\Omega) : \nabla u(x) \in E \text{ for a.e. } x \in \Omega\},$$

$$Sol(u_0, \Omega, E) = \{u \in u_0 + W_0^{1,\infty}(\Omega) : \nabla u(x) \in E \text{ for a.e. } x \in \Omega\}.$$

Minimization problems of G in $Sol(\Omega, E)$ or in $Sol(u_0, \Omega, E)$ may have no solution.

One is led to use relaxation methods in calculus of variations to get existence of at least relaxed solutions. These methods naturally require to close the above sets in suitable topological spaces in which such closures turn out to be compact, and where G enjoys good continuity, or lower semicontinuity, properties. Then, one obtains relaxed solutions of minimization problems of G in $Sol(\Omega, E)$ or in $Sol(u_0, \Omega, E)$ as elements of such closures.

In this setting, uniqueness of relaxed solutions can be obtained as well provided suitable assumptions on g are fulfilled.

Problem: Representation of the closures of $Sol(\Omega, E)$ and $Sol(u_0, \Omega, E)$.

The elements of such closures can be seen as generalized or weak solutions of the differential inclusion $\nabla u(x) \in E$ for a.e. $x \in \Omega$.

If E is bounded, the gradients of the elements of $Sol(\Omega, E)$ are uniformly bounded. Consequently, $Sol(\Omega, E)$ enjoys natural compactness properties in the space of continuous functions. Because of this, a natural closure of $Sol(\Omega, E)$ to take into account is the one in the topology of uniform convergence. In this case, the closure of $Sol(\Omega, E)$ remains a subset of $W^{1,\infty}(\Omega)$, and its description follows directly from the previous relaxation results.

When E is not bounded, for example the one where E is the graph of a function defined on $\mathbf{R}^k$ for some $k < n$, such kind of compactness does not hold anymore, and a natural way to retrieve a similar property is to assume growth conditions on g, for example of the type

$$|u|^p \leq g(x, u) \text{ for a.e. } x \in \Omega \text{ and every } u \in \mathbf{R}$$

for some $p > 1$, that provide the weak compactness in $L^p(\Omega)$ of the sublevel sets of G in $Sol(\Omega, E)$. Consequently, the closure of $Sol(\Omega, E)$ to take into account is the one in such topology.

When applied to $f = I_E$, the above relaxation reaults provides the following description of the intersection of the closure in $L^1(\Omega)$ of $Sol(\Omega, E)$ with $BV(\Omega)$, provided E is Borel and Ω is convex and bounded,

$$BV(\Omega) \cap \mathrm{cl}(L^1(\Omega))Sol(\Omega, E) =$$

$$= \left\{ u \in BV(\Omega) : \nabla u \in \overline{\mathrm{co}(E)} \text{ for a.e. } x \in \Omega \text{ and } \frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|}(x) \in \left(\overline{\mathrm{co}(E)}\right)^\infty \text{ for } |D^s u|\text{-a.e. } x \in \Omega \right\}.$$

Extension to any subset E of $\mathbf{R}^n$ by dropping the assumption that E is Borel.

Such new result, coupled with suitable approximation techniques, is used to describe the whole closure of $Sol(\Omega, E)$ in various classes of topological spaces.

$$E \subseteq \mathbf{R}^n,\ \Omega \text{ convex bounded open set}$$

$$\text{cl(weak*-}\mathcal{D}'(\Omega))Sol(\Omega, E) =$$
$$= \left\{ u \in \mathcal{D}'(\Omega) : -\langle u, \nabla\varphi \rangle \in \overline{\text{co}(E)} \text{ for every } \varphi \in C_0^\infty(\Omega) \text{ with } \varphi \geq 0 \text{ and } \int_\Omega \varphi \, \mathrm{d}x = 1 \right\},$$

$$\text{cl(weak*-}\mathcal{M}_{\text{loc}}(\Omega))Sol(\Omega, E) =$$
$$= \left\{ u \in \mathcal{M}_{\text{loc}}(\Omega) : -\int_\Omega \nabla\varphi \, \mathrm{d}u \in \overline{\text{co}(E)} \text{ for every } \varphi \in C_0^\infty(\Omega) \text{ with } \varphi \geq 0 \text{ and } \int_\Omega \varphi \, \mathrm{d}x = 1 \right\},$$

and, provided $p \in [1, +\infty[$ (when $p = +\infty$ the weak*-$L^\infty(\Omega)$-topology must be considered),

$$\text{cl}(L^p(\Omega))Sol(\Omega, E) =$$
$$= \left\{ u \in L^p(\Omega) : -\int_\Omega u \nabla\varphi \, \mathrm{d}x \in \overline{\text{co}(E)} \text{ for every } \varphi \in C_0^\infty(\Omega) \text{ with } \varphi \geq 0 \text{ and } \int_\Omega \varphi \, \mathrm{d}x = 1 \right\}.$$

Again when Ω is bounded and convex, the same representation problems are approached for $Sol(u_0, \Omega, E)$ when u_0 is affine and $\nabla u_0 \in \mathrm{int}(\mathrm{co}(E))$.

$$\mathrm{cl}(\text{weak*-}\mathcal{D}'(\mathbf{R}^n))Sol(u_0, \Omega, E) = \Big\{u \in \mathcal{D}'(\mathbf{R}^n) :$$

$$\mathrm{spt}(u - u_0) \subseteq \overline{\Omega},\ -\langle u, \nabla\varphi\rangle \in \overline{\mathrm{co}(E)} \text{ for every } \varphi \in C_0^\infty(\mathbf{R}^n) \text{ with } \varphi \geq 0 \text{ and } \int_{\mathbf{R}^n} \varphi \, \mathrm{d}x = 1\Big\},$$

$$\mathrm{cl}(\text{weak*-}\mathcal{M}(\overline{\Omega}))Sol(u_0, \Omega, E) = \Big\{u \in \mathcal{M}(\overline{\Omega}) :$$

$$-\int_{\overline{\Omega}} \nabla\varphi \, \mathrm{d}u + \int_{\partial\Omega} \varphi u_0 \mathbf{n}_\Omega \, \mathrm{d}\mathcal{H}^{n-1} \in \overline{\mathrm{co}(E)} \text{ for every } \varphi \in C_0^\infty(\mathbf{R}^n) \text{ with } \varphi \geq 0 \text{ and } \int_{\Omega} \varphi \, \mathrm{d}x = 1\Big\},$$

and, if $p \in [1, +\infty[$ (when $p = +\infty$ the weak*-$L^\infty(\Omega)$-topology must be considered),

$$\mathrm{cl}(L^p(\Omega))Sol(u_0, \Omega, E) = \Big\{u \in L^p(\Omega) :$$

$$-\int_{\Omega} u\nabla\varphi \, \mathrm{d}x + \int_{\partial\Omega} \varphi u_0 \mathbf{n}_\Omega \, \mathrm{d}\mathcal{H}^{n-1} \in \overline{\mathrm{co}(E)} \text{ for every } \varphi \in C_0^\infty(\mathbf{R}^n) \text{ with } \varphi \geq 0 \text{ and } \int_{\Omega} \varphi \, \mathrm{d}x = 1\Big\}.$$

If $p = 1$, representation result on the whole $L^1(\Omega)$ for the $L^1(\Omega)$-lower semicontinuous envelope of the non-coercive functional $I_{Sol(\Omega,E)}$.

$$\inf\left\{\liminf_{h\to+\infty} I_{Sol(\Omega,E)}(u_h) : \{u_h\} \subseteq L^1(\Omega),\ u_h \to u \text{ in } L^1(\Omega)\right\} =$$

$$= I_{\{v\in L^1(\Omega): -\int_\Omega v\nabla\varphi\,\mathrm{d}x \in \overline{\mathrm{co}(E)} \text{ for every } \varphi\in C_0^\infty(\Omega) \text{ with } \varphi\geq 0 \text{ and } \int_\Omega \varphi\,\mathrm{d}x=1\}}(u)$$

$$\text{for every } u \in L^1(\Omega).$$

Analogously, all the above formulas can be interpreted as relaxed forms, in various spaces, of differential inclusions and Hamilton-Jacobi equations. Roughly speaking, they establish relaxed formulations in a distributional sense in spaces of distributions, measures, and not necessarily smooth functions, obtained by means of the convexification of the constraint set E.

As application, optimization based selection criterium.

Theorem. *Let* Ω *be a convex bounded open subset of* $\mathbf{R}^n$, *let* E *be a subset of* $\mathbf{R}^n$ *with* $\mathrm{int}(\mathrm{co}(E)) \neq \emptyset$, *and let* $z_0 \in \mathrm{int}(\mathrm{co}(E))$. *Let* $g\colon \Omega \times \mathbf{R} \to [0, +\infty[$ *be a Carathéodory integrand such that* $g(x, \cdot)$ *is convex for a.e.* $x \in \Omega$, *and assume that*

$$|u|^p \leq g(x,u) \leq \Lambda(a(x) + |u|^p) \text{ for a.e. } x \in \Omega \text{ and every } u \in \mathbf{R},$$

for some $p \in]1, +\infty[$, $\Lambda \geq 0$ *and* $a \in L^1(\Omega)$. *Then, if* u_0 *is an affine function on* $\mathbf{R}^n$ *having* z_0 *as gradient,*

$$\inf\left\{\int_\Omega g(x,u(x))\,\mathrm{d}x : u \in Sol(u_0, \Omega, E)\right\} =$$

$$= \min\left\{\int_\Omega g(x,u(x))\,\mathrm{d}x : u \in L^p(\Omega),\ -\int_\Omega u\nabla\varphi\,\mathrm{d}x + \int_{\partial\Omega} \varphi u_0 \mathbf{n}_\Omega\,\mathrm{d}\mathcal{H}^{n-1} \in \overline{\mathrm{co}(E)}\right.$$

$$\left.\text{for every } \varphi \in C_0^\infty(\mathbf{R}^n) \text{ with } \varphi \geq 0 \text{ and } \int_\Omega \varphi\,\mathrm{d}x = 1\right\}.$$

Moreover, the minimizing sequences of the above left-hand side have weakly $L^p(\Omega)$*-converging subsequences that converge to solutions of the right-hand side. In particular, if* g *is strictly convex in the* u *variable for a.e.* $x \in \Omega$, *then the minimizing sequences of the left-hand side weakly converge in* $L^p(\Omega)$ *to the unique minimizer of the right-hand side.*

Example. $n = 2$, $\Omega =]-1, 1[^2$,

$$\theta \colon (x, y) \in \Omega \mapsto \begin{cases} 1/2 & \textit{if } xy \geq 0 \\ -1/2 & \textit{if } xy < 0. \end{cases}$$

Find among the solutions of the Dirichlet problem ($u_0 = 0$)

$$\begin{cases} \nabla_y u(x, y) = \sin(\nabla_x u(x, y)) \text{ for a.e. } (x, y) \in \Omega \\ u \in W_0^{1,\infty}(\Omega), \end{cases}$$

or equivalently

$$\begin{cases} (\nabla_x u(x, y), \nabla_y u(x, y)) \in E = \{(z_1, z_2) \in \mathbf{R}^2 : z_2 = \sin z_1\} \text{ for a.e. } (x, y) \in \Omega \\ u \in W_0^{1,\infty}(\Omega), \end{cases}$$

those that minimize the integral

$$\int_\Omega |u(x, y) - \theta(x, y)|^2 \, \mathrm{d}x \, \mathrm{d}y.$$

$$\inf\left\{ \int_{\Omega} |u(x,y) - \theta(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y : u \in W_0^{1,\infty}(\Omega),\ \nabla_y u(x,y) = \sin(\nabla_x u(x,y)) \text{ for a.e. } (x,y) \in \Omega \right\}$$

This minimization problem has no solution in $W_0^{1,\infty}(\Omega)$.

Relaxed minimization problem

$$\overline{\mathrm{co}(E)} = \mathbf{R} \times [-1, 1]$$

$$(0, 0) \in \mathrm{int}(\mathrm{co}(E))$$

$$\min \left\{ \int_\Omega |u(x,y) - \theta(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y : u \in L^2(\Omega), \right.$$

$$\left. - \int_\Omega u(x,y) \nabla \varphi(x,y) \, \mathrm{d}x \, \mathrm{d}y \in \mathbf{R} \times [-1, 1] \text{ for every } \varphi \in C_0^\infty(\mathbf{R}^2) \text{ with } \varphi \geq 0 \text{ and } \int_\Omega \varphi \, \mathrm{d}x = 1 \right\} =$$

$$= \min \left\{ \int_\Omega |u(x,y) - \theta(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y : u \in L^2(\Omega), \right.$$

$$\left. u(x, \cdot) \in W_0^{1,\infty}(]-1,1[) \text{ for a.e. } x \in]-1,1[, \ |\nabla_y u(x,y)| \leq 1 \text{ for a.e. } y \in]-1,1[\right\}.$$

$$u_\infty(x,y) \in \Omega \mapsto \begin{cases} -x(1+y)/|x| & \text{if } -1 < y < -1/2 \\ xy/|x| & \text{if } -1/2 \le y \le 1/2 \\ x(1-y)/|x| & \text{if } 1/2 < y < 1 \end{cases}$$

is the only (non smooth) solution of the relaxed minimization problem.

Each minimizing sequence of the infimum problem weakly converges in $L^2(\Omega)$ to u_∞.