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Homogenization of Visco-Elastic and Plastic Processes

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Analogical Models

A large class of mathematical models are built by coupling

- a (universal) balance law,

e.g., the dynamical equation, the Maxwell system, the energy balance, and so on,

- a set of constitutive relations (that characterize the specific material),
- appropriate initial- and boundary-conditions.

In continuum mechanics, electricity, magnetism, and so on
constitutive behaviours are often represented via so-called *analogical models*, namely

networks of elementary components arranged in series and / or in parallel.

If each element fulfils a constitutive law, a global law is then derived for each network.

Rheological Models

ε: deformation tensor, σ: stress tensor.

— For a discrete family of elements $\{A_j : j = 1, ..., M\}$

(i) *Combination in Series:* $\sigma = \sigma_1 = \sigma_2, \quad \varepsilon = \varepsilon_1 + \varepsilon_2;$

(ii) *Combination in Parallel:* $\varepsilon = \varepsilon_1 = \varepsilon_2, \quad \sigma = \sigma_1 + \sigma_2.$

E.g., for a parallel arrangement

$$\sigma_j = B_j : \varepsilon_j \quad \forall j, \quad \Rightarrow \quad \sigma = \sum_{j=1}^{M} B_j : \varepsilon.$$

— For a continuous distribution of elements $\{A(y) : y \in Y\}$ $(Y := [0, 1[^3)$:

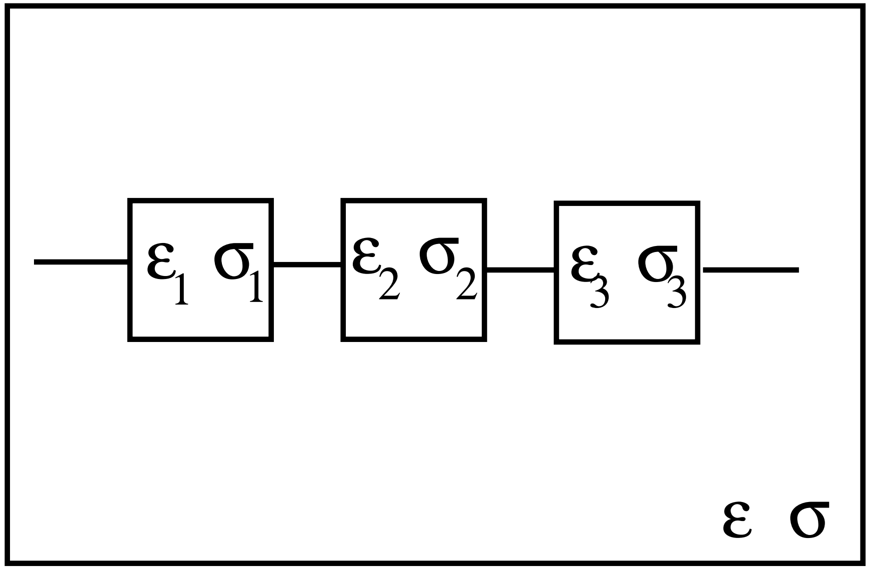
(i) *Combination in Series*: $\sigma(y) = \text{constant}, \quad \varepsilon = \int_Y \varepsilon(y)\, dy;$

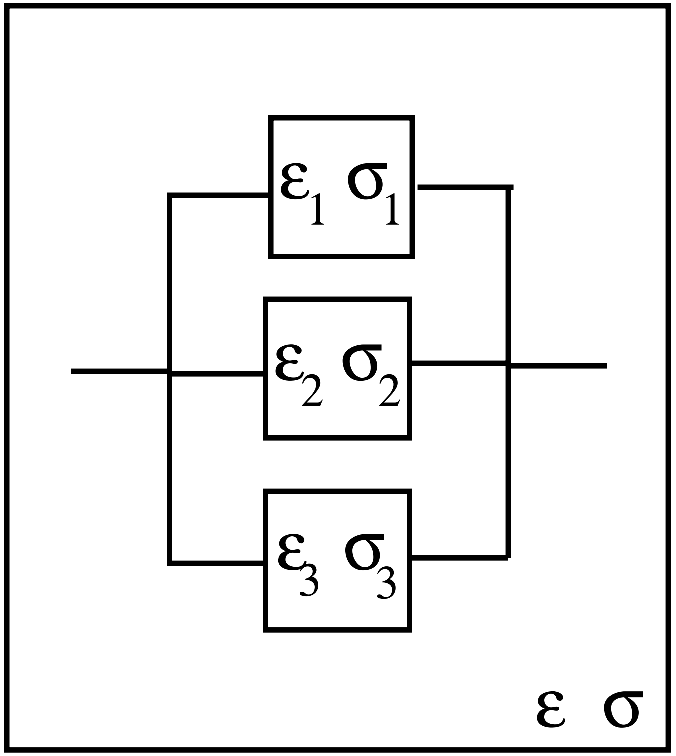
(ii) *Combination in Parallel*: $\varepsilon(y) = \text{constant}, \quad \sigma = \int_Y \sigma(y)\, dy.$

E.g., for a parallel arrangement

$$\sigma(y) = B(y) : \varepsilon(y) \quad \text{for a.e. } y \quad \Rightarrow \quad \sigma = \int_Y B(y)\, dy : \varepsilon.$$

Schemes of Series and Parallel Arrangements





Series:

$$\begin{cases} \varepsilon = \sum_j \varepsilon_j \\ \sigma = \sigma_j \qquad \forall j. \end{cases}$$

Parallel:

$$\begin{cases} \sigma = \sum_j \sigma_j \\ \varepsilon = \varepsilon_j \qquad \forall j. \end{cases}$$

Examples of Basic Components

Classically linear elasticity is assumed for the spheric components: $\sigma_{(s)} = a\varepsilon_{(s)}$, whereas several relations are considered for the deviatoric components, e.g.:

(i) *Linear Elasticity:* $\sigma_{(d)} = A : \varepsilon_{(d)}$ $(A = A_{ijk\ell})$.

(ii) *Nonlinear Viscosity:* $\dot{\varepsilon}_{(d)} \in \partial\varphi(\sigma_{(d)})$, with φ l.s.c. and convex.

(iii) *Rigid Perfect Plasticity:* as above for $\varphi = I_K$, K being the yield criterion.

Examples of Composed Model

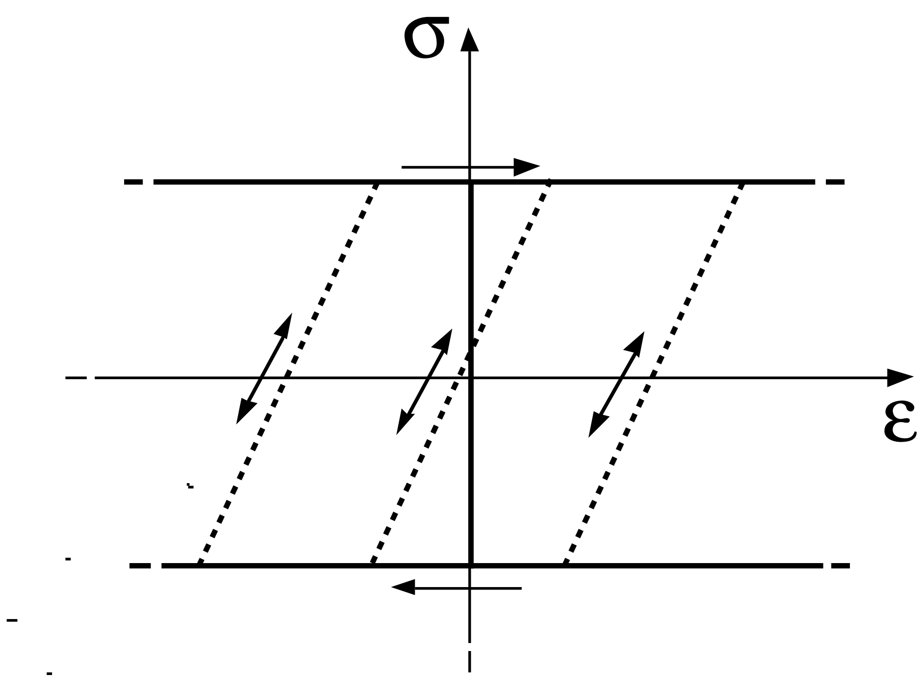
(i) *Maxwell model:* series arrangement of linear elasticity and nonlinear viscosity:

$$B : \dot{\sigma} + \partial\varphi(\sigma) \ni \dot{\varepsilon} \qquad \text{whence} \qquad \sigma = \mathcal{G}(\varepsilon).$$

(ii) *Generalized Maxwell model:* parallel arrangement of Maxwell models:

$$\sigma = \sum_j \mathcal{G}_j(\varepsilon) \qquad \text{or} \qquad \sigma = \int_Y \mathcal{G}(\varepsilon, y)\, dy.$$

Two Mechanical Models with Hysteresis



(iii) *Prandtl-Reuss Model* (or *Stop):* as in the Maxwell model, with $\varphi = I_K$:

$$\sigma = \mathcal{G}(\varepsilon) \qquad (\mathcal{G}: \text{ hysteresis operator}).$$

(iv) *Prandtl-Ishlinskiĭ Model of Stop-Type:* parallel arrangement of stops:

$$\sigma = \sum_j \mathcal{G}_j(\varepsilon) \qquad \text{or} \qquad \sigma = \int_Y \mathcal{G}(\varepsilon, y)\, dy.$$

Some References on Rheological Models

W. Flügge: *Viscoelasticity.* Springer, Berlin 1975

B. Halphen, Nguyen Quoc Son: *Sur les matériaux standard généralisés.* J. de Méchanique **14** (1975), 39–63

W. Han, B.D. Reddy: *Plasticity.* Springer, New York 1999

M.J. Leitman, G.M.C. Fisher: *The linear theory of viscoelasticity.* In: Handbuch der Physik (S. Flügge, ed.), vol. VIa/3. Springer, Berlin 1973, pp. 1–123

J. Lemaitre, J.-L. Chaboche: *Mechanics of Solid Materials.* Cambridge Univ. Press, Cambridge 1990

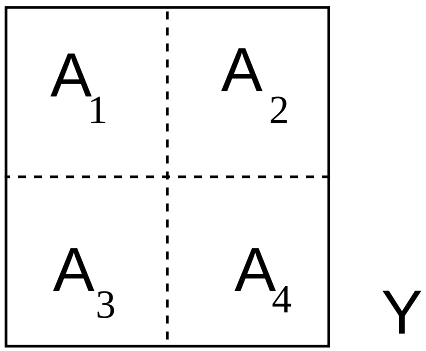
M. Reiner: *Rheology.* In: Handbuch der Physik (S. Flügge, ed.), vol. VI. Springer, Berlin 1958, pp. 434–550

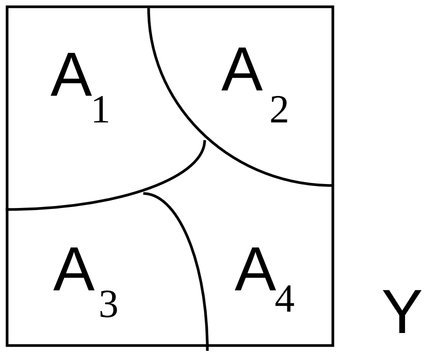
What is the significance of analogical models ?

May networks of series / parallel arrangements represent composites?

May the corresponding constitutive relations be then retrieved via homogenization?

Which models do arise by assembling (either elementary or composite) models?





The answer depends upon the coupled PDEs and the space-dimension:

$$\varepsilon := \frac{\partial u}{\partial x}, \qquad \rho \frac{\partial^2 u}{\partial t^2} - \frac{\partial \sigma}{\partial x} = f \qquad \text{in 1 space-dimension,}$$

$$\varepsilon := \nabla^s \vec{u}, \qquad \rho \frac{\partial^2 \vec{u}}{\partial t^2} - \nabla \cdot \sigma = \vec{f} \qquad \text{in 3 space-dimensions.}$$

A Model of Elasto-Visco-Plasticity

σ: stress tensor, ε: linearized strain tensor,

$B(x)$: compliance tensor, $\varphi(\cdot, x) : \mathbf{R}^9_s \to \mathbf{R} \cup \{+\infty\}$ convex l.s.c.

$$\frac{\partial \varepsilon}{\partial t} - B(x) : \frac{\partial \sigma}{\partial t} \in \partial \varphi(\sigma, x), \qquad \text{namely} \tag{1}$$

$$\left(\frac{\partial \varepsilon}{\partial t} - B(x) : \frac{\partial \sigma}{\partial t}\right) : (\sigma - v) \geq \varphi(\sigma, x) - \varphi(v, x) \qquad \forall v \in \mathbf{R}^9_s. \tag{1$'$}$$

This relation accounts for elasto-visco-plasticity, including
the nonlinear Maxwell model, and
the Prandtl-Reuss model.
(The latter is a weak formulation of the evolution of the elasto-plastic interface...)

(1) is assumed pointwise and is coupled with the dynamical equation

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} - \nabla \cdot \sigma = \vec{f} \qquad \text{in } \Omega_T := \Omega \times]0, T[. \tag{2}$$

Program for Two- and Single-Scale Homogenization

1. Model of a Macroscopically Inhomogeneous Material. Here the fields only depend on the coarse-scale variable x (besides time). A single-scale initial- and boundary-value problem P_1 is formulated and solved.

2. Model of a Mesoscopically Inhomogeneous Material. The constitutive data B and φ are assumed to depend periodically on a fine-scale variable $y := x/\eta$ (η being a the ratio between the two space-scales). The problem P_1 is then relabelled as $P_{1\eta}$.

3. Two-Scale Homogenization. As $\eta \to 0$ a subsequence of solutions of $P_{1\eta}$ weakly *two-scale converges* to a solution of a two-scale problem, P_2, in which the fields depend on both the coarse- and fine-scale variables x and y (besides time).

4. Scale-Transformation of the Two-Scale Problem ("Upscaling"). A single-scale problem P_3 is derived from the two-scale problem P_2, by averaging the mesoscopic fields over the reference set $\mathcal{Y}$ and by homogenizing the constitutive relation.

5. Inversion of the Scale-Transformation ("Downscaling"). Conversely any solution of P_3 is represented as the $\mathcal{Y}$-average of a solution of problem P_2.

We may thus represent processes in our composite by means of four different models:

— (i) a single-scale model that can be represented via an analogical model, and rests on an (apparently unjustified) mean-field-type hypothesis;

— (ii) an approximate single-scale model, that is characterized by a *small* but finite parameter η; this might also be regarded as intermediate between a single-scale and a two-scale model;

— (iii) a detailed representation via a two-scale problem, in which the fields depend on both the coarse- and fine-scale variables x and y;

— (iv) a more synthetic but equivalent formulation, via a single-scale homogenized model in which the fields only depend on the coarse-scale variable x.

The models (iii) and (iv) contain the same amount of information, although this is fully displayed just in (iii).

In general the single-scale models (i) and (iv) need not be equivalent, for apparently there is no reason why either the stress or the strain should be mesoscopically uniform.

Two-Scale Convergence

After Nguetseng and Allaire, denoting by $\mathcal{Y}$ the N-dim. unit torus,

$$u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } L^2(\mathbf{R}^N \times \mathcal{Y}) \quad \underset{\text{def}}{\Longleftrightarrow} \quad \|u_\varepsilon\|_{L^2(\mathbf{R}^N)} \le C \quad \text{and}$$

$$\int_{\mathbf{R}^N} u_\varepsilon(x)\, \psi\Big(x, \frac{x}{\varepsilon}\Big)\, dx \to \iint_{\mathbf{R}^N \times \mathcal{Y}} u(x,y)\, \psi(x,y)\, dx dy \quad \forall \psi \in \mathcal{D}(\mathbf{R}^N \times \mathcal{Y}).$$

Example. For any $\psi \in \mathcal{D}(\mathbf{R}^N \times \mathcal{Y})$, $\quad \psi(x, x/\varepsilon) \underset{2}{\rightharpoonup} \psi(x,y)$ in $L^2(]0,1[\times\mathcal{Y})$. E.g.:

$$x \sin(2\pi x/\varepsilon) \underset{2}{\rightharpoonup} x \sin(2\pi y) \qquad \text{in } L^2(]0,1[\times \mathcal{Y}).$$

G. Allaire: *Homogenization and two-scale convergence.* S.I.A.M. J. Math. Anal. **23** (1992) 1482–1518

G. Nguetseng: *A general convergence result for a functional related to the theory of homogenization.* S.I.A.M. J. Math. Anal. **20** (1989) 608–623

Theorem . *If*

$$u_\varepsilon \rightharpoonup u \qquad \text{in } H^1(\Omega), \tag{1}$$

then there exists $w \in L^2\big(\Omega; H^1(\mathcal{Y})\big)$ *such that* $\int_{\mathcal{Y}} w(\cdot, y)dy = 0$ *a.e. in* Ω*, and such that, as* $\varepsilon \to 0$ *along a suitable subsequence,*

$$\nabla u_\varepsilon \underset{2}{\rightharpoonup} \nabla u + \nabla_y w \qquad \text{in } L^2(\Omega \times \mathcal{Y})^3. \tag{2}$$

Example.

$$u_\varepsilon(x) := \varepsilon x \sin(2\pi x/\varepsilon) \rightharpoonup 0 =: u(x) \qquad \text{in } H^1(0,1), \tag{3}$$

$$\begin{aligned} D_x u_\varepsilon(x) &= \varepsilon \sin(2\pi x/\varepsilon) + 2\pi x \cos(2\pi x/\varepsilon) \\ &\underset{2}{\rightharpoonup} 2\pi x \cos(2\pi y) = D_x u(x) + D_y w(x,y) \quad \text{in } L^2(]0,1[\times\mathcal{Y}), \end{aligned} \tag{4}$$

where $w(x,y) = x \sin(2\pi y)$.

Some References for Homogenization

G. Bensoussan, J.L. Lions, G. Papanicolaou: *Asymptotic Analysis for Periodic Structures.* North-Holland, Amsterdam 1978

A. Braides, A. Defranceschi: *Homogenization of Multiple Integrals.* Oxford University Press, Oxford 1998

D. Cioranescu, P. Donato: *An Introduction to Homogenization.* Oxford Univ. Press, New York 1999

V.V. Jikov, S.M. Kozlov, O.A. Oleinik: *Homogenization of Differential Operators and Integral Functionals.* Springer, Berlin 1994

The homogenization of the *Prandtl-Ishlinskiĭ models of play-type* was studied for uniaxial systems via hysteresis operators in

J. Franců, P. Krejčí: *Homogenization of scalar wave equations with hysteresis.* Continuum Mech. Thermodyn. **11** (1999) 371–390

That analysis might easily be extended to the nonlinear Kelvin-Voigt model.

1. Model of a Macroscopically Inhomogeneous Material

Here the fields only depend on the coarse-scale variable x (besides time).

Problem 1. *Find* $(\vec{u}, \sigma)$ *such that, setting* $\varepsilon := \nabla^s \vec{u}$,

$$\vec{u} \in W^{2,\infty}\big(0,T; L^2(\Omega)^3\big) \cap W^{1,\infty}\big(0,T; W_0^{1,q}(\Omega)^3\big) \tag{1}$$

$$\sigma \in W^{1,\infty}\big(0,T; L^2(\Omega)_s^9\big), \qquad \nabla\cdot\sigma \in L^\infty\big(0,T; L^2(\Omega)^3\big) \tag{2}$$

$$\frac{\partial \varepsilon}{\partial t} - B(x) : \frac{\partial \sigma}{\partial t} \in \partial\varphi(\sigma, x) \qquad \textit{a.e. in } \Omega_T \tag{3}$$

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} - \nabla\cdot\sigma = \vec{f} \qquad \textit{in } \mathcal{D}'(\Omega_T). \tag{4}$$

This problem is well-posed.

2. Model of a Mesoscopically Inhomogeneous Material

Just replace x by x/η, η being a (small) positive parameter.

3. Two-Scale Model

Problem 2. *Find* $\vec{u} = \vec{u}(x,t),\ \varepsilon = \varepsilon(x,y,t),\ \sigma = \sigma(x,y,t)$ *such that*

$$\vec{u} \in W^{2,\infty}\big(0,T;L^2(\Omega)^3\big) \cap W^{1,\infty}\big(0,T;W_0^{1,q}(\Omega)^3\big) \tag{1}$$

$$\sigma \in W^{1,\infty}\big(0,T;L^2(\Omega\times\mathcal{Y})_s^9\big), \qquad \nabla\cdot\int_{\mathcal{Y}}\sigma\,dy \in L^\infty\big(0,T;L^2(\Omega)^3\big) \tag{2}$$

$$\exists\vec{u}_{(1)} \in L^q\big(\Omega_T;W^{1,q}(\mathcal{Y})^3\big): \qquad \varepsilon = \nabla^s\vec{u} + \nabla_y\vec{u}_{(1)} \quad \textit{a.e. in } \Omega_T\times\mathcal{Y} \tag{3}$$

$$\frac{\partial\varepsilon}{\partial t} - B(y):\frac{\partial\sigma}{\partial t} \in \partial\varphi(\sigma,y) \qquad \textit{a.e. in } \Omega_T\times\mathcal{Y} \tag{4}$$

$$\rho\frac{\partial^2\vec{u}}{\partial t^2} - \nabla\cdot\int_{\mathcal{Y}}\sigma\,dy = \vec{f} \qquad \textit{in } \mathcal{D}'(\Omega_T) \tag{5}$$

$$\nabla_y\cdot\sigma = \vec{0} \qquad \textit{in } \mathcal{D}'(\mathcal{Y})^3\textit{, a.e. in } \Omega_T. \tag{6}$$

This is retrieved by passing to the two-scale limit as $\eta \to 0$ in Problem 1_η.

4. Single-Scale Homogenization of the Constitutive Law

Basic scale decomposition: we define the average and fluctuating components:

$$\hat{v} := \int_{\mathcal{Y}} v(y)\, dy, \qquad \tilde{v} := v - \hat{v} \qquad \forall v \in L^1(\mathcal{Y}). \tag{1}$$

Henceforth we take $p = q = 2$. We define the spaces

$$W := \{\eta \in L^2(\mathcal{Y})^9 : \hat{\eta} = 0, \nabla \cdot \eta = \vec{0} \text{ in } \mathcal{D}'(\mathcal{Y})^3\} \tag{2}$$

$$Z := \{\zeta \in L^2(\mathcal{Y})^9 : \hat{\zeta} = 0, \zeta = \nabla^s \vec{v} \text{ a.e. in } \mathcal{Y}, \text{ for some } \vec{v} \in H^1(\mathcal{Y})^3\} \tag{3}$$

and notice the obvious orthogonality properties

$$\int_{\mathcal{Y}} \zeta(y) : \eta(y)\, dy = 0 \qquad \forall \zeta \in Z, \forall \eta \in W \tag{4}$$

$$\int_{\mathcal{Y}} \hat{\zeta} : \tilde{\eta}(y)\, dy = 0 \qquad \forall \zeta, \eta \in L^2(\mathcal{Y})^9. \tag{5}$$

The Fenchel Properties

$$\forall u, w, \qquad F(u) + F^*(w) \geq w \cdot u \qquad \text{(Fenchel inequality)}$$
$$w \in \partial F(u) \quad \Leftrightarrow \quad F(u) + F^*(w) = w \cdot u \qquad \text{(Fenchel property – I).} \tag{1}$$

The latter statement then also reads

$$w \in \partial F(u) \quad \Leftrightarrow \quad F(u) + F^*(w) \leq w \cdot u \qquad \text{(Fenchel property – II).} \tag{2}$$

Trivial example : $F(v) = |v|^2/2$, whence $\partial F(u) = u$

$$\forall u, w, \qquad \frac{|u|^2}{2} + \frac{|w|^2}{2} \geq w \cdot u \qquad \text{(Fenchel inequality)}$$
$$w = u \quad \Leftrightarrow \quad \frac{|u|^2}{2} + \frac{|w|^2}{2} = w \cdot u \qquad \text{(Fenchel property – I)} \tag{3}$$

$$w = u \quad \Leftrightarrow \quad \frac{|u|^2}{2} + \frac{|w|^2}{2} \leq w \cdot u \qquad \text{(Fenchel property – II).} \tag{4}$$

By the Fenchel properties, $\frac{\partial\varepsilon}{\partial t} - B(x) : \frac{\partial\sigma}{\partial t} \in \partial\varphi(\sigma, x)$ a.e. in Ω_T is equivalent to

$$\varphi(\sigma, x) + \varphi^*\left(\frac{\partial\varepsilon}{\partial t} - B(x) : \frac{\partial\sigma}{\partial t}, x\right) = \sigma : \left(\frac{\partial\varepsilon}{\partial t} - B(x) : \frac{\partial\sigma}{\partial t}\right), \tag{1}$$

namely

$$\begin{aligned}&\iiint_{\Omega_\tau\times\mathcal{Y}}\left[\varphi(\sigma, x) + \varphi^*\left(\frac{\partial\varepsilon}{\partial t} - B(x) : \frac{\partial\sigma}{\partial t}, x\right)\right] dxdydt \\ &+ \frac{1}{2}\iint_{\Omega\times\mathcal{Y}}(\sigma : B(x) : \sigma)\Big|_{t=0}^{t=\tau} dxdy = \iiint_{\Omega_\tau\times\mathcal{Y}} \sigma : \frac{\partial\varepsilon}{\partial t}\, dxdydt \qquad \forall\tau \in\,]0, T].\end{aligned} \tag{2}$$

After a further integration in time and using the above orthogonality properties, we get an equation of the form

$$A(\sigma, \varepsilon) = \iiint_{\Omega_T\times\mathcal{Y}}(T - t)\sigma : \frac{\partial\varepsilon}{\partial t}\, dxdydt = \iint_{\Omega_T}(T - t)\hat{\sigma} : \frac{\partial\hat{\varepsilon}}{\partial t}\, dxdt. \tag{3}$$

Setting $\Lambda(\hat{\sigma}, \hat{\varepsilon}) := \inf\left\{A(\hat{\sigma} + \tilde{\sigma}, \hat{\varepsilon} + \tilde{\varepsilon}) : (\tilde{\sigma}, \tilde{\varepsilon}) \in L^2(\Omega_T; W \times Z)\right\}$, we then get (by the Fenchel properties...)

$$\Lambda(\hat{\sigma}, \hat{\varepsilon}) = \iint_{\Omega_T}(T - t)\hat{\sigma} : \frac{\partial\hat{\varepsilon}}{\partial t}\, dxdt. \tag{4}$$

4. Homogenized Single-Scale Model

Problem 3. *Find* $(\vec{u}, \overline{\varepsilon}, \overline{\sigma})$ *such that*

$$\vec{u} \in W^{2,\infty}\big(0,T;L^2(\Omega)^3\big) \cap W^{1,\infty}\big(0,T;H_0^1(\Omega)^3\big) \tag{1}$$

$$\overline{\sigma} \in W^{1,\infty}\big(0,T;L^2(\Omega)_s^9\big), \quad \nabla\cdot\overline{\sigma} \in L^\infty\big(0,T;L^2(\Omega)^3\big) \tag{2}$$

$$\Lambda(\overline{\sigma},\overline{\varepsilon}) = \iint_{\Omega_T} (T-t)\overline{\sigma} : \frac{\partial\overline{\varepsilon}}{\partial t}\, dxdt \tag{3}$$

$$\rho\frac{\partial^2\vec{u}}{\partial t^2} - \nabla\cdot\overline{\sigma} = \vec{f} \qquad \textit{in } \mathcal{D}'(\Omega_T). \tag{4}$$

A.V. : *Homogenization of the nonlinear Kelvin-Voigt model of visco-elasticity and of the Prager model of plasticity.* Continuum Mech. Thermodyn. **18** (2006) 223-252

A.V. : *Homogenization of the nonlinear Maxwell model of visco-elasticity and of the Prandtl-Reuss model of elasto-plasticity.* (in preparation)