Mouvements of measures for optimal transportation problems

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What presented here is in the paper

*"An optimization problem
for mass transportation
with congested dynamics"*

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The goal of this research is to provide a dynamical formulation of mass transportation problems with possible concentration or congestion effects.

• Concentration effects for instance occur in several models of branching transportation (roots of trees, roads, delta of rivers, blood vessels, . . .)

• Congestion effects may be used to simulate traffic flows with high density and movement of crowds under panic effects.

The main tool is a good comprehension of lower semicontinuous functionals defined on the space of measures.

A complete analysis (lower semicontinuity, relaxation, integral representation) for this kind of functionals was made in a series of paper by Bouchitté-Buttazzo:
Nonlinear Anal., **15** (1990), 679–692
Ann.IHP Anal.NonLin., **9** (1992), 101–117
Ann.IHP Anal.NonLin., **10** (1993), 345–361

Example 1 - Lebesgue For L^p measures $\mu = u\,dx$ define

$$F(\mu) = \int_\Omega |u|^p\,dx \qquad p > 1.$$

Example 2 - Dirac For discrete measures $\mu = \sum m_k \delta_{x_k}$ define

$$F(\mu) = \sum_k |m_k|^\alpha = \int_\Omega |\mu(x)|^\alpha\,d\#(x) \qquad \alpha < 1.$$

Example 3 - Mumford-Shah For measures with no Cantor part $\mu = u\,dx + \sum m_k \delta_{x_k}$ define

$$F(\mu) = \int_\Omega |u|^p\,dx + \int_\Omega |\mu(x)|^\alpha\,d\#(x) \quad p > 1,\ \alpha < 1.$$

A full classification of all weakly* l.s.c. functionals on $\mathcal{M}(\Omega)$ (translation invariant for simplicity) is the following

$$
\begin{aligned}
F(\mu) = \; & \int_\Omega f(\mu^a)\, dm(x) && \text{Lebesgue part} \\
& + \int_\Omega f^\infty(\mu^c) && \text{Cantor part} \\
& + \int_\Omega g(\mu(x))\, d\#(x) && \text{Dirac part}
\end{aligned}
$$

where f is convex, f^∞ is its recession function, g is subadditive, and the compatibility condition $f^\infty = g^0$ holds.

In Example 1 $f(z) = |z|^p$, $g(z) \equiv +\infty$;
In Example 2 $f(z) \equiv +\infty$, $g(z) = |z|^\alpha$;
In Example 3 $f(z) = |z|^p$, $g(z) = |z|^\alpha$.

Previous attempts have been made to model concentration/congestion effects:

- Q. Xia (2003) through the minimization of a suitable functional defined on currents;
- V. Caselles, J. M. Morel, S. Solimini, ... (2002) through a kind of analogy of fluid flow in thin tubes;
- A. Brancolini, G. Buttazzo, F. Santambrogio (2006) through geodesic curves in the space of measures.

The idea in this last paper was to study the evolution of densities as a curve in the spa-ce of probabilities $\mathcal{P}(\Omega)$ endowed with the Wasserstein distance which minimize a kind of length functional:

$$\mathcal{L}(\mu) = \int_0^1 J(\mu(t))|\mu'(t)|_W \, dt.$$

Here $|\mu'|_W$ is the metric derivative in the Wasserstein space. In a general (X, d) space the definition of the metric derivative is

$$|x'(t)|_X = \lim_{\varepsilon \to 0} \frac{d\Big(x(t+\varepsilon), x(t)\Big)}{\varepsilon}.$$

Theorem *Let X be a compact metric space (or closed bounded subsets of X are compact), let $x_0, x_1 \in X$ and consider*

$$\mathcal{L}(\phi) = \int_0^1 J(\phi(t))|\phi'(t)|_W \, dt.$$

Assume that
i) J is lower semicontinuous in X;
ii) $J \geq c$ with $c > 0$;
iii) $\mathcal{L}(\phi) < +\infty$ for at least one curve ϕ joining x_0 to x_1.
Then there exists an optimal path for the problem

$$\min \Big\{ \mathcal{L}(\phi) \ : \ \phi(0) = x_0, \ \phi(1) = x_1 \Big\}.$$

Take now X the Wasserstein space of probabilities on Ω (a compact subset of $\mathbf{R}^N$).

In the concentration case (J of Dirac type with $\alpha < 1$):

• Two discrete measures μ_0, μ_1 can be joined by a path $\phi(t)$ of finite minimal cost $\mathcal{L}$.

• If $\alpha > 1 - 1/N$ every μ_0, μ_1 can be joined by a path $\phi(t)$ of finite minimal cost $\mathcal{L}$, with counterexamples if $\alpha \leq 1 - 1/N$).

In the diffusion case (J of Lebesgue type with $p > 1$):

• Two measures μ_0, μ_1 with L^p densities can be joined by a path $\phi(t)$ of finite minimal cost $\mathcal{L}$.

• If $p < 1 + 1/N$ every μ_0, μ_1 can be joined by a path $\phi(t)$ of finite minimal cost $\mathcal{L}$, with counterexamples if $p \geq 1 + 1/N$).

A coefficient J of Lebesgue type then provides a congestion functional, while J of Dirac type gives a model for describing concentrations.

In this presentation however we adopt a different point of view, introduced by Brenier to give a dynamic formulation of mass transportation problems.

$$\min\left\{\int_0^1\int_\Omega \rho|v|^2\,dx\,dt \ : \ \rho_t + \mathrm{div}_x(\rho v) = 0\right\}$$

under the initial/terminal conditions $\rho|_{t=0} = \rho_0$ and $\rho|_{t=1} = \rho_1$.

Setting $\rho v = E$ the continuity equation becomes linear:

$$\rho_t + \mathrm{div}_x\, E = 0$$

and the cost functional (representing the kinetic energy) becomes convex:

$$\int_0^1 \int_\Omega \frac{|E|^2}{\rho}\, dx\, dt.$$

To be precise, the correct meaning has to be given in terms of measures:

$$\int_0^1 \int_\Omega \left|\frac{dE}{d\rho}\right|^2 d\rho(x)\, dt.$$

Setting $Q = [0, T] \times \Omega$, $\sigma = (\rho, E)$, and $f = \delta_T(t) \otimes \rho_1(x) - \delta_0(t) \otimes \rho_0(x)$ the problem above can be written in the form

$$\min \left\{ \Psi(\sigma) \ : \ -\operatorname{div} \sigma = f \text{ in } Q,\ \sigma \cdot \nu = 0 \text{ on } \partial Q \right\}$$

where $\Psi(\sigma)$ is a functional defined on $\mathcal{M}(Q)$.

Theorem *If Ψ is a weakly* l.s.c. functional on $\mathcal{M}(Q)$ and $f \in \mathcal{M}(Q)$, then the minimum problem*

$$\min \left\{ \Psi(\sigma) \ : \ -\operatorname{div} \sigma = f \textit{ in } Q,\ \sigma \cdot \nu = 0 \textit{ on } \partial Q \right\}$$

has a solution, provided $\int_Q df = 0$ and Ψ is coercive, i.e. $\Psi(\sigma) \geq c|\sigma| - c_1$.

The functionals Ψ we have in mind are of the form

$$\Psi(\sigma) = \int_0^T J\Big(\sigma(t)\Big)\,dt$$

and again J of Lebesgue type would provide congestion models, while J of Dirac type would provide concentration models.

From now on we limit ourselves to the case of congestion, where the function J is convex. Similar arguments for the other non-convex cases have not yet been developed.

Dual formulation:

$$\sup \left\{ \langle f, \phi \rangle - \Psi^*(D\phi) \ : \ \phi \in C^1(Q) \right\}.$$

Primal-dual relation:

$$\Psi(\sigma_{opt}) + \Psi^*(D\phi_{opt}) = \langle \sigma_{opt}, D\phi_{opt} \rangle.$$

The point is that the maximizer in the dual formulation is not of class C^1 in general. A relaxation formula is then needed for Ψ^* to extend it to its natural space.

The natural spaces for functionals like Ψ^* are the Sobolev spaces $W_\mu^{1,p}$ with respect to a measure μ, defined by relaxation of the energies

$$\int |Du|^p \, d\mu.$$

All the usual properties known for the standard Sobolev spaces continue to hold, provided the gradient is replaced by the tangential gradient $D_\mu u$ suitably defined.

We do not enter in the details of this rather delicate theory, referring to Bouchitté-Buttazzo-Seppecher (Calc.Var. 1997).

The numerical approximation has been performed following the scheme used in Benamou-Brenier, which is an augmented Lagrangian algorithm. This consists in solving, instead of

$$\min\left\{\Psi(\sigma)\ :\ -\operatorname{div}\sigma = f \text{ in } Q,\ \sigma\cdot\nu = 0 \text{ on } \partial Q\right\}$$

the min-max problem

$$\min_{\sigma}\ \max_{\varphi\in\mathcal{C}(Q)} L(\sigma,\varphi)$$

where $L(\sigma,\varphi)$ is the Lagrangian:

$$L(\sigma,\varphi) = \Psi(\sigma) - \langle D\varphi,\sigma\rangle + \langle\varphi, f\rangle.$$

Using the primal-dual relation this is in turn equivalent to solve the max-min problem

$$\max_{\sigma} \min_{\sigma^*, \varphi} L_r(\sigma, \sigma^*, \varphi)$$

where L_r is the augmented Lagrangian

$$\begin{aligned} L_r(\sigma, \sigma^*, \varphi) := \; & \Psi^*(\sigma^*) + \langle D\varphi - \sigma^*, \sigma \rangle - \langle \varphi, f \rangle \\ & + \frac{r}{2} \int |D\varphi - \sigma^*|^2 \, dy \end{aligned}$$

for $r > 0$ fixed.

This is the iterative process we used (algorithm ALG2, Fortin-Glowinski):

- let $(\sigma_n, \sigma^*_{n-1}, \varphi_{n-1})$ be given;
- Step A: find φ_n solving (freeFEM3D by Del Pino-Pironneau)

$$\min \Big\{ L_r(\sigma_n, \sigma^*_{n-1}, \varphi) \ : \ \varphi \in \mathcal{C}^1(Q) \Big\}$$

- Step B: find σ^*_n solving

$$\min \Big\{ L_r(\sigma_n, \sigma^*, \varphi_n) \ : \ \sigma^* \in \mathcal{C}^1(Q, \mathbf{R}^{d+1}) \Big\}$$

- Step C: set $\sigma_{n+1} = \sigma_n + r(D\varphi_n - \sigma^*_n)$;
- go back to Step A.

The following animations deal with a domain Ω not convex (a kind of subway gate) and with the cases:

- $J(\rho, E) = \frac{|E|^2}{\rho}$ in which the transportation simply follows the Wasserstein geodesics.

- $J(\rho, E) = \frac{|E|^2}{\rho} + c\rho^2$ in which the Wasserstein transportation is perturbed by the addition of a diffusion term (panic effect).

- $J(\rho, E) = \frac{|E|^2}{\rho} + \chi_{\{\rho \leq M\}}$ in which there is the additional constraint that two different individual do not want to stay too close.